On Critical Graphs for Opsut's Conjecture

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Abstract

A graph $G = (V(G), E(G))$ is the competition graph of an acyclic digraph $D = (V(D), A(D))$ if $V(G) = V(D)$ and there is an edge in G between vertices $x, y \in V(G)$ if and only if there is some $v \in V(D)$ such that $xv, yv \in A(D)$. The competition number $k(G)$ of a graph G is the minimum number of isolated vertices needed to add to G to obtain a competition graph of an acyclic digraph. Opsut conjectured in 1982 that if $\theta(N(v)) \leq 2$ for all $v \in V(G)$, then the competition number $k(G)$ of G is at most 2, with equality if and only if $\theta(N(v)) = 2$ for all $v \in V(G)$. (Here, $\theta(H)$ is the smallest number of cliques covering the vertices of H.) Though Opsut (1982) proved that the conjecture is true for line graphs and recently Kim and Roberts (1989) proved a variant of it, the original conjecture is still open. In this paper, we introduce a class of so called critical graphs. We reduce the question of settling Opsut's conjecture to the study of critical graphs by proving that Opsut's conjecture is true for all graphs which are disjoint unions of connected non-critical graphs. All K_4-free critical graphs are characterized. It is proved that Opsut's conjecture is true for critical graphs which are K_4-free or are K_4-free after contracting vertices of the same closed neighborhood. Some structural properties of critical graphs are discussed.

1 Introduction

A graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$ is the *competition graph* of an acyclic digraph $D = (V(D), A(D))$ if $V(G) = V(D)$ and there is an edge in G between vertices $x, y \in V(G)$ if and only if there is vertex $v \in V(D)$ such that $xv, yv \in A(D)$. The

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competition graph of an acyclic digraph D is usually denoted by $C(D)$. Competition graphs were introduced by Cohen in 1968 [1] when he studied the ecological phase spaces of food webs. Since then, a large amount of literature has been devoted to them. For recent surveys, see Kim [3] and Lundgren [5]. In this paper, we discuss a conjecture in competition graph theory proposed by Opsut in 1982 [7]. Opsut proved [7] that his conjecture is true for line graphs. Kim and Roberts [4] proved a variant of Opsut's conjecture which generalizes Opsut's result. Here, we introduce a class of so called critical graphs. The question of settling Opsut's conjecture is reduced to the study of critical graphs.

In section 2, we show that Opsut's conjecture is true for graphs which are disjoint unions of connected non-critical graphs. Hence Opsut's conjecture is true if and only if it is true among critical graphs. This generalizes the results of Opsut [7] and Kim and Roberts [4]. In section 3, K_4-free critical graphs are characterized and it is proved that Opsut's conjecture is true for all K_4-free critical graphs. In section 4, we study the approach of characterizing critical graphs by minimal critical graphs under the process of taking induced subgraphs and by critical lift operations. It is proved that Opsut's conjecture is true among what are called K_4-free reducible graphs. Finally, further problems are discussed in section 5.

To end this section, we define some terms and notation.

All graphs considered here are simple graphs, i.e., without loops and parallel edges. Given a graph $G = (V(G), E(G))$, we denote by $x \sim y$ that $xy \in E(G)$, and by $x \not\sim y$ that $xy \notin E$. $N_G(v) = \{x|xv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ (or $N(v)$ and $N[v]$ when there is no ambiguity) are the *open neighborhood* and *closed neighborhood* of vertex $v \in V(G)$, respectively. The *degree* d_v of $v \in V(G)$ is $|N(v)|$, the cardinality of $N(v)$. v is a *simplicial* vertex if $\theta(N(v)) = 1$.

Let $S \subseteq V(G)$. Let $G(S)$ be the subgraph of G induced by S. For convenience, we also use S instead of $G(S)$ when there is no ambiguity. Let $G - S$ be the graph with vertex set $V(G) - S$, edge set $E(G) - \{xy|x \in S\}$. When $S = \{v\}$, an one vertex set, we denote $G - S$ by $G - v$ for convenience.

A *clique* of a graph G is a complete subgraph (not necessarily maximal). $\theta(N(v))$ is the minimum number of cliques needed to cover the vertices of the subgraph induced by $N(v)$. Following Kim and

Roberts [4], for $v \in V(G)$ we say that $\theta^*(N(v)) = 2$ if $\theta(N(v)) = 2$ and there are two cliques C_1 and C_2 covering vertices of $N[v]$, both containing v, so that for all $w \in C_1$, $N(w) - C_1$ is either empty or a clique of G. We say that $\theta^*(N(v)) \leq 2$ if $\theta(N(v)) \leq 1$ or $\theta^*(N(v)) = 2$.

Other terminologies not defined here are defined in [8].

2 Opsut's Conjecture and Critical Graphs

In 1978, Roberts [9] proved that given a graph G, by adding sufficiently many isolated vertices, one can obtain a competition graph of some acyclic digraph. The *competition number* of a graph is then defined as the minimum number of isolated vertices needed to add to G to obtain a competition graph. The competition number of a graph G is usually denoted by $k(G)$. Opsut [7] proved that computing competition number is an NP-complete problem. Therefore, the competition number of a graph is difficult to compute for general graphs. Nevertheless, Roberts [9] showed that the competition number of a triangulated graph is at most 1, and the competition number of a connected triangle free graph is $|E(G)| - |V(G)| + 2$. Opsut [7] also proved that if G is a line graph, then $k(G)$ is at most 2 and $k(G)$ can be efficiently determined. Some more results on competition number can be found in [3, 5, 7, 9] and in references cited there. In looking for a generalization of the result on line graphs, Opsut conjectured the following (which he proved for line graphs):

Conjecture 2.1 *(Opsut [7]) If $\theta(N(v)) \leq 2$ for all $v \in V(G)$, then $k(G) \leq 2$, and $k(G) = 2$ if and only if $\theta(N(v)) = 2$ for all $v \in V(G)$.*

Recently, after introducing the parameter θ^*, Kim and Roberts generalized the proof in [7] to obtain the following theorem, which proves a variation of Conjecture 2.1:

Theorem 2.2 *(Kim and Roberts [4]) If $\theta^*(N(v)) \leq 2$ for all $v \in V(G)$, then $k(G) \leq 2$, and $k(G) = 2$ if and only if $\theta^*(N(v)) = 2$ for all $v \in V(G)$.*

It is easy to show that line graphs satisfy the condition in Theorem 2.2. There are also graphs which are not line graphs satisfying

Theorem 2.2. Therefore Theorem 2.2 generalizes Opsut's result on line graphs. On the other hand, the graph in Figure 2.1 satisfies the condition in Conjecture 2.1 but does not satisfy the condition in Theorem 2.2. Nevertheless, it is not a counterexample of Conjecture 2.1. The digraph in Figure 2.2 shows that its competition number is at most 2. Using the following Lemma 2.3, we have that its competition number is 2. Hence Conjecture 2.1 remains open.

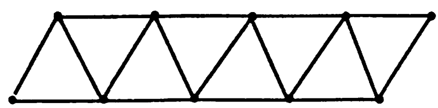

Figure 2.1

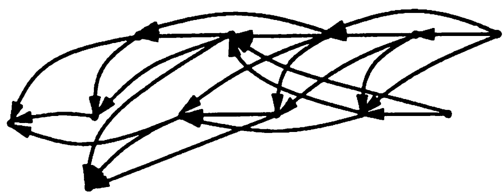

Figure 2.2

The following is a useful lemma due to Opsut.

Lemma 2.3 *(Opsut [7]) For any graph G,* $k(G) \geq min_v \theta(N(v))$.

Now let G_1, G_2 be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ is the disjoint union of G_1 and G_2.

Lemma 2.4 *Suppose that* $V(G_1) \cap V(G_2) = \emptyset$ *and* $k(G_2) \leq 2$. *If* $|V(G_1)| \geq 2$, *then* $k(G_1 \cup G_2) \leq k(G_1)$. *If* $|V(G_1)| = 1$, *then* $k(G_1) \cup G_2) \leq 1$.

Proof. Let D_1, D_2 be two acyclic digraphs such that $C(D_1) = G_1 \cup I_{k(G_1)}$, $C(D_2) = G_2 \cup I_{k(G_2)}$ and such that there is no vertex in these two digraphs having only one incoming arc. (If any vertex has only one incoming arc, eliminating that arc will not change the competition graph.) Then because D_1 is acyclic and $|V(G_1)| \geq 2$, it has at least two vertices x, y with no incoming arcs. The lemma follows from letting D be the digraph obtained from the union of D_1, D_2 such that the isolated vertices in $I_{k(G_2)}$ are removed and all arcs going to them are instead going to x or y, or both, depending on $k(G_2) = 1$ or 2. The rest of the lemma is obvious. $\square$

By Lemma 2.4, Conjecture 2.1 only needs to be considered on connected graphs. Therefore, in the rest of this paper, any graph without special notice is always connected. Now we introduce the critical graphs which will play the central role in this paper. We call a graph *critical* if for all complete subgraphs C (not necessarily maximal) of G, there is a $w \in C$ such that $\theta(N(w) - C) = 2$. A *non-critical* graph is simply a graph not critical, i.e., there is a clique C such that either for all $w \in C$ we have $\theta(N(w) - C) = 1$ or for some $w \in C$ we have $\theta(N(w) - C) \geq 3$.

It follows from the definition of critical graphs that line graphs and graphs satisfying the conditions in Kim and Roberts' Theorem 2.2 are non-critical. We will show that Conjecture 2.1 is true for non-critical graphs.

Now we prove that collection of non-critical graphs satisfying $\theta(N(v)) \leq 2$ for all $v \in V(G)$ are closed under taking induced subgraphs. One fact used throughout the rest of this paper without further notice is that if $\theta(N_G(v)) \leq 2$ for all $v \in V(G)$, then $\theta(N_H(v)) \leq 2$ for all $v \in V(H)$ for all induced subgraphs H of G.

Theorem 2.5 *Let G be a graph such that $\theta(N(v)) \leq 2$ for all $v \in V(G)$. G has no critical induced subgraphs if and only if G is non-critical.*

Proof. To prove the "*if*" part, suppose on the contrary that there are connected non-critical graphs such that $\theta(N_G(v)) \leq 2$ for all $v \in V(G)$ and such that some of their induced subgraphs are critical. Let G be such a graph with smallest number of vertices. Then by the minimality of G, for any $w \in V(G)$, $G - w$ is critical. On the other hand, by G critical, there is a complete subgraph C of G such that for all $w \in C$, $\theta(N_G(w) - C) \leq 1$. Let $v_0 \in G - C$ (exists since G cannot be a complete graph). Then we have that for all $w \in C$, $\theta(N_{G-v_0}(w) - C) \leq 1$. This is contrary to $G - v_0$ critical. The *"only if"* part is trivial. So the theorem is proved. □

Theorem 2.6 *Suppose that G is connected and $\theta(N(v)) \leq 2$ $\forall v \in V(G)$. Then $k(G) = 1$ if and only if G has a simplicial vertex.*[2]

Proof. The *"only if"* follows from Lemma 2.3. To prove the *"if"* part, inductively suppose that the lemma is true for graphs of fewer vertices. (For graph of at most 2 vertices, the theorem is true.) Suppose that there is $v_0 \in V(G)$ such that $\theta(N_G(v_0)) = 1$. If the subgraph $G(N_G[v_0])$ is G, then G is a complete graph and $k(G) = 1$ since $k(K_m) = 1$ for any complete graph of size $m \geq 2$.

If $G(N_G[v_0])$ is not G, let $C = N_G[v_0]$ and let $G_1, ..., G_k$ be the connected components of $G - C$. Then $k \geq 1$. For each G_i, choose a vertex $v_i \in C$ such that $N(v_i) \cap V(G_j) \neq \emptyset$ if and only if $i = j$. Such v_i exits since G is connected. Let $\widehat{G}_i = G_i \cup \{v_i\}$. Since v_i is a simplicial vertex of $\widehat{G}_i$, it follows from induction that $k(\widehat{G}_i) = 1$. Since $|V(\widehat{G}_i)| \geq 2$, by Lemma 2.4, $k(\widehat{G}) = 1$ where $\widehat{G} = \cup_i \widehat{G}_i$.

Let $\widehat{D}$ be an acyclic digraph such that $C(\widehat{D}) = \widehat{G} \cup \{a\}$ where a is an isolated vertex. Let $\{v_{k+1}, ..., v_n, v_{n+1} = v_0\}$ be the remaining vertices in $C - \{v_1, ..., v_k\}$. Notice that for any v_i, $N(v_i) \cap v(\widehat{G})$ is a clique (possibly empty). Now it is easy to check that the following digraph D is such that $C(D) = G \cup \{a\}$:

$$
\begin{aligned}
V(D) =& V(G) \cup \{a\} \\
A(D) =& A(\widehat{D}) - \{(v,a) | (v,a) \in A(\widehat{D})\} \\
& \cup \{(v, v_{k+1}) | (v,a) \in A(\widehat{D})\} \\
& \cup_{i=k+1}^{n} \{(v, v_{i+1}) | (v, v_i) \in E(G) \textit{ and } v \notin N_G[v_0]\} \\
& \cup_{i=k+1}^{n} \{(v_i, v_{i+1})\} \\
& \cup_{i=0}^{n} \{(v_i, a)\}
\end{aligned}
$$

So $k(G) = 1$. □

Theorem 2.7 *Conjecture 2.1 is true for non-critical graphs.*

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[2]The author thanks the referee for suggesting this theorem.

Proof. Let G be a non-critical graph. Then G has no critical induced subgraphs by Theorem 2.5. By Lemmas 2.3 and Theorem 2.6, we only need to show that if $\theta(N_G(v)) = 2$ for all $v \in V(G)$, then $k(G) = 2$. We do induction on $|V(G)|$. When $|V(G)|$ is small, say $|V(G)| \leq 4$, it can be checked that the theorem is true. Inductively assume that the theorem is true for graphs of fewer vertices than G.

Now suppose $\theta(N_G(v)) = 2$ for all $v \in V(G)$. Since G is not critical, there is a clique C such that $\theta(N_G(v) - C) \leq 1$ $\forall v \in C$. Let $C_1 = C = \{v_0, v_1, ..., v_s\}$ and $C_2 = (N_G(v_0) - C) \cup \{v_0\}$. Then $N_{\widehat{G}}(v_0)$ in $\widehat{G} = G - \{v_1, ..., v_s\}$ is a non-empty clique since $\theta(N_G(v_0)) \neq 1$. By Theorem 2.6, $k(\widehat{G}) = 1$. Let $\widehat{D}$ be an acyclic digraph such that $\widehat{G} \cup \{a\} = C(\widehat{D})$. Then the following is an acyclic digraph $D = (V(D), A(D))$ such that $C(D) = G \cup \{a, b\}$ where a, b are two isolated vertices:

$$\begin{aligned}
V(D) =& V(G) \cup \{a, b\} \\
A(D) =& A(\widehat{D}) - \{(v, a) | (v, a) \in A(\widehat{D})\} \\
& \cup \{(v, v_1) | (v, a) \in A(\widehat{D})\} \\
& \bigcup_{i=2}^{s} \{(v, v_i) | v \in N_G(v_{i-1}) - V(C_1)\} \\
& \bigcup \{(v, a) | v \in N(v_s) - V(C_1)\} \\
& \bigcup_{i=2}^{s} \{(v_{i-1}, v_i)\} \\
& \bigcup_{i=0}^{s} \{(v_i, b)\} \\
& \bigcup \{(v_s, a)\}
\end{aligned}$$

Combining this result with Lemma 2.3, we see that $k(G) = 2$. The theorem is proved. □

The digraph construction in the proof of Theorem 2.7 is similar to the constructions Kim and Roberts used in [4], which in turn are generalizations of those used by Opsut in [7]. Now our main result of this section is ready.

Theorem 2.8 *Conjecture 2.1 is true if and only if* $k(G) = 2$ *for critical graphs.*

Proof. The *"only if"* part is trivial. The *"if"* part is a corollary of Theorems 2.6 and 2.7. □

Theorem 2.7 generalizes the results of Opsut [7] and Kim and Roberts [4] since it is easy to see that graphs satisfying conditions in their results are non-critical graphs and there are non-critical graphs which do not satisfy conditions in their results. The graph in Figure 2.1 is a such example.

3 K_4-free Critical Graphs

In this section, we characterize K_4-free critical graphs. This will enable us to prove that Conjecture 2.1 is true for K_4-free graphs.

Before beginning our discussion on K_4-free graphs, we mention that it is easy to show that Conjecture 2.1 is true for K_3-free graphs. For if G is a K_3-free graph such that $\theta(N(v)) \leq 2$ for all $v \in V(G)$, then each connected component of G is either a path or a cycle of length larger than 3. So G is non-critical. Then that Conjecture 2.1 is true for K_3-free graphs follows from Theorem 2.7. Now we characterize K_4-free critical graphs.

Let $G = (V(G), E(G))$ be a graph. For $S \subseteq V(G)$, let $G^c(S)$ denote the complement of the subgraph of G induced by S. A *claw* is a graph $G = (\{a, b, c, d\}, \{ab, ac, ad\})$. If $\theta(N(v)) \leq 2$, $G^c(N(v))$ induces a bipartite subgraph of G^c. The converse is also true. Thus we have

Lemma 3.1 *Let $G = (V(G), E(G))$ be a graph. $\theta(N(v)) \leq 2$ for all $v \in V(G)$ if and only if $G^c(N(v))$ is bipartite for all $v \in V$.*

Lemma 3.1 implies that if $\theta(N(v)) \leq 2$ for all $v \in V(G)$, then G is claw-free.

Lemma 3.2 *Let $G = (V(G), E(G))$ be a critical graph, $v \in V(G)$. Let C_1, C_2 be two cliques covering vertices of $N[v]$ and $v \in C_1 \cap C_2$. Then there is a vertex $w \in C_1$ such that $\exists\ x, y \in N(w) - C_1$, $x \not\sim y$ and $x \sim v$.*

Proof. Let $v \in V(G)$ be a vertex of G. Since G is critical, if C_1, C_2 are two cliques covering vertices of $N[v]$ and $v \in C_1 \cap C_2$ then there must be some $w \in C_1$ such that $\exists x, y \in N(w) - C_1$, $x \not\sim y$. If $x \not\sim v$, $y \not\sim v$, $\{x, y, v, w\}$ would induce a claw, which is impossible. So $\{x, y\} \cap N(v) \neq \emptyset$. □

Theorem 3.3 *Let G be a critical graph, and let δ_G be the minimum degree of a vertex of G. Then $\delta_G \geq 4$. In particular, K_4-free critical graphs are 4-regular.*

Proof. It is clear that G has no vertex of degree ≤ 1. Suppose that G has a vertex v of degree 2 with two distinct neighbors a, b. By Lemma 3.2 one of those, say a, must have two non-adjacent neighbors in $N(a) - \{v\}$, one of them adjacent to v. Thus one has to be b and $\theta(N(v)) = 1$, contradicting that G is critical.

Now, we prove that G has no vertex of degree 3. Suppose on the contrary that there is a vertex $v \in V(G)$ such that $d_v = 3$, with neighbors a, b, c. Since $\theta(N(v)) = 2$ for all $v \in V(G)$, without loss generality, suppose $a \sim b$ and $a \not\sim c$. Let $C_1 = \{v, c\}$, $C_2 = \{v, a, b\}$. C_1, C_2 cover vertices of $N[v]$ and $v \in C_1 \cap C_2$. By Lemma 3.2 c has two non-adjacent neighbors outside of C_1. One of them is adjacent to v. It can only be b. So $c \sim b$. Let the other neighbor of c be d. Then $d \not\sim b$ and $d \not\sim v$.

Since $d_v = 3$ and $\theta(N(v)) = 2$, all neighbors of c other than b must be in a clique with d. Let $C_1' = \{v, b, c\}$ and let C_2' contain c, d and all other neighbors of c. Then C_1', C_2' are two cliques covering vertices of $N[c]$ and $c \in C_1' \cap C_2'$. By Lemma 3.2 there are two non-adjacent vertices, say e, f, in $N(b) - C_1'$ such that one of them, say e, is adjacent to c. It cannot be d. Therefore e is another neighbor of c. If $a \neq f$, $\{b, e, f, v\}$ would induce a claw. So $a = f$. Thus, we conclude that there is $e \notin \{a, b, c, v\}$ so that $e \sim c$, $e \sim b$, $e \not\sim a$, $e \not\sim v$. Also, $e \neq d$. By symmetry of a and c, there is a vertex $g \notin \{a, b, c, v\}$, so that $g \sim a$, $g \sim b$, $g \not\sim c$, $g \not\sim v$. Moreover, $g \neq e$ because $e \sim c$. (Figure 3.1) It follows that $\{a, c, e, g, v\} \subseteq N(b)$ and a, c, g, e, v, a is a 5-cycle in $G^c(N(b))$. Thus $G^c(N(b))$ is not bipartite contrary to Lemma 3.1.

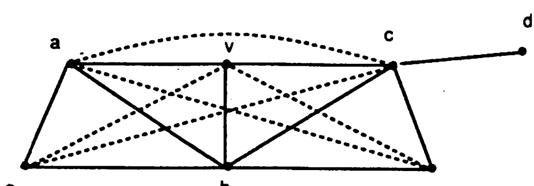

Figure 3.1

Hence if G is critical, $\delta_G \geq 4$. When G is K_m-free critical, by $\theta(N(v)) = 2$ for all $v \in V(G)$, we have that the maximum of degrees of vertices of G is at most $2m - 4$. Hence the theorem follows. □

Lemma 3.4 *Let $G = (V(G), E(G))$ be a K_4-free critical graph, $v \in V(G)$. If C_1, C_2 are two cliques covering vertices of $N[v]$, both containing v, then there are $x \in C_1$, $y \in C_2$ such that x and y are non-adjacent and $\theta(N(y) - C_2) = 2$.*

Proof. Since $\theta(N(v)) = 2$ for all $v \in V(G)$, we have that $\theta(N(v) - S) \leq 2$ for all $S \subseteq V(G)$. Let $v \in V$. By Theorem 3.3, $d_v = 4$. Let $N(v) = \{a, b, c, d\}$. Let $C_1 = \{v, a, b\}$ and $C_2 = \{v, c, d\}$ be two cliques covering vertices of $N[v]$. (neither C_1 nor C_2 could be K_4). Since $\theta(N(v)) = 2$, without loss of the generality, we may assume that $a \not\sim c$. If $\theta(N(c) - C_2) = 2$, then let $x = a$ and $y = c$, and we are done. If $\theta(N(c) - C_2) \leq 1$, by Lemma 3.2, d must have two non-adjacent neighbors in $G - C_2$ so that one of them is either a or b. If it is a, let $x = b$, $y = d$; if it is b, let $x = a$, $y = d$. The lemma follows since d could not be adjacent to both a and b (else G has a K_4). □

Let B_n, $n \geq 4$, be a graph of $2n$ vertices such that there is labeling v_1, , v_2, ... , v_{2n} of $V(B_n)$ so that the adjacency of B_n is given as follows (Figure 3.2):

$$
\begin{aligned}
&v_{2i-1} \sim v_{2i+1}, && i = 1, 2, ..., n-1;\\
&v_{2i} \sim v_{2i+2}, && i = 1, 2, ..., n-1;\\
&v_i \sim v_{i+1}, && i = 1, 2, ..., 2n-1;\\
&v_{2n-1} \sim v_1, \quad v_{2n} \sim v_1, && v_{2n} \sim v_2
\end{aligned}
$$

Let M_n, $n \geq 4$, be a graph of $2n - 1$ vertices such that there is labeling v_1, v_2, ..., v_{2n-1} of $V(M_n)$ so that the adjacency of M_n is given as follows (Figure 3.3):

$$
\begin{aligned}
&v_{2i-1} \sim v_{2i+1}, && i = 1, 2, ..., n-1;\\
&v_{2i} \sim v_{2i+2}, && i = 1, 2, ..., n-2;\\
&v_i \sim v_{i+1}, && i = 1, 2, ..., 2n-2;\\
&v_{2n-1} \sim v_1, \quad v_{2n-1} \sim v_2, && v_{2n-2} \sim v_1.
\end{aligned}
$$

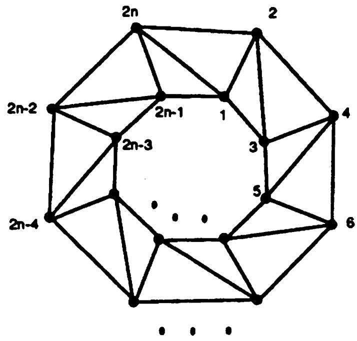

Figure 3.2

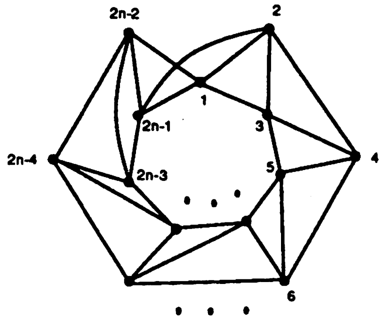

Figure 3.3

Let C be the collection of all B_n's and M_n's, $n \geq 4$. It is easy to check that all graphs in C are K_4-free and critical. We are going to show that these are the only connected K_4-free critical graphs.

Let graph D_n, $n \geq 3$, be a graph of $2n-1$ vertices such that there is a labeling of its vertices with adjacency given as the following (Figure 3.4):

$$v_{2i-1} \sim v_{2i+1}, \qquad i = 1, 2, ..., n-1;$$
$$v_{2i} \sim v_{2(i+1)}, \qquad i = 1, 2, ..., n-2;$$
$$v_i \sim v_{i+1}, \qquad i = 1, 2, ..., 2n-2.$$

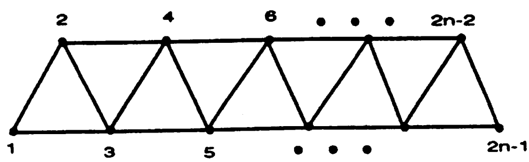

Figure 3.4

Lemma 3.5 *If G is a K_4-free critical, then every closed neighborhood $N[x]$, $x \in V(G)$ of G induces a subgraph D_3.*

Proof. By Theorem 3.3 $d_v = 4$ for all $v \in V(G)$. Thus $|V(G)| = n \geq 5$. So by $\theta(N(v)) = 2$, for all $v \in V(G)$, $N[v]$ is covered by two triangles. Suppose that $v_0 \in V(G)$. We show that $N[v_0]$ induces a D_3. Let $C_1 = \{v_1, v_2, v_0\}$ and $C_2 = \{v_0, v_3, v_4\}$ be two triangles covering vertices of $N[v_0]$. By Lemma 3.4 there are, say $v_1 \in C_1$, $v_4 \in C_2$, such that $v_1 \not\sim v_4$ and $N(v_4) - C_2$ has two non-adjacent vertices. By $\theta(N(v_4)) = 2$, one of them is adjacent to v_0 (otherwise there would be a claw in G). Since G is 4-regular and $v_1 \not\sim v_4$, it must be v_2. Let the other one be v_5. Then $v_5 \not\sim v_2$. Since G is K_4-free and $\theta(N(v_4)) = 2$, we have $v_2 \not\sim v_3$, $v_5 \not\sim v_0$ and $v_5 \sim v_3$.

Now we show that $v_3 \not\sim v_1$. Suppose on the contrary that $v_3 \sim v_1$. Let the other neighbor of v_2 other than v_1, v_0, v_4 be x. Since $v_5 \not\sim v_2$, $x \neq v_5$ and x, v_2, v_1, v_4 cannot induce a claw, x is adjacent to either v_1 or v_4. It follows from $N(v_4) = \{v_0, v_2, v_3, v_5\}$ that $x \sim v_1$. $C'_1 = \{x, v_1, v_2\}$ and $C'_2 = \{v_1, v_0, v_3\}$ are two cliques covering $N[v_1]$ such that $v_1 \in C'_1 \cap C'_2$. But for all $w \in C'_2$, $N(w) - C'_2$ is a clique. Therefore, $v_3 \not\sim v_1$. It follows that $N[v_0] = \{v_0, v_1, v_2, v_3, v_4\}$ induces a D_3 shown in Figure 3.5. □

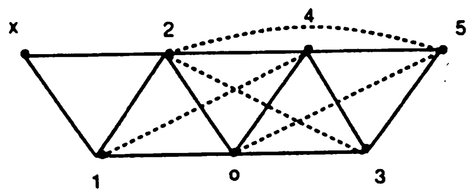

Figure 3.5

Theorem 3.6 *A graph G is K_4-free critical if and only if $G \in \mathcal{C}$.*

Proof. Let $G = (V(G), E(G))$ be a K_4-free critical graph. By Lemma 3.5 every closed neighborhood of G induces a subgraph D_3. Let m be the maximum such that G contains an induced subgraph H which is isomorphic to D_m. Without loss of the generality, suppose that the vertices in subgraph H of G are labeled in the same way as in the definition of adjacency of D_m and other vertices of G are unlabeled. We now prove by induction on m that $G = M_{m+1}$ or $G = B_{m+1}$.

If $m = 3$, then $H = N[v_3]$. v_5 must have two neighbors which are unlabeled. Label them v_6, v_7 respectively. $v_6 \not\sim v_3, v_7 \not\sim v_3$ since

$N(v_3) = \{v_1, v_2, v_4, v_5\}$. Applying Lemma 3.5 to $N[v_5]$, we have that $v_6 \sim v_7$ and one of them, say v_6, is adjacent to v_4.

If $v_6 \sim v_1$, applying Lemma 3.5 to $N[v_6]$ we have $v_7 \sim v_1$. Applying Lemma 3.5 to $N[v_1]$ we have (since $d_{v_6} = 4$) $v_2 \sim v_7$. Then we have M_4 as an induced subgraph of G. Since G is 4-regular and G is connected, $G = M_4$.

Now suppose that $v_6 \not\sim v_1$. Applying Lemma 3.5 to $N[v_4]$, we have $v_2 \not\sim v_6$. If $v_2 \sim v_7$, applying Lemma 3.5 to $N[v_2]$, it follows (since $d_{v_4} = 4$) that $v_7 \sim v_1$. By $d_{v_6} = 4$, v_6 has a neighbor other than $v_1, v_2, v_3, v_4, v_5, v_7$ but not adjacent to any of v_4, v_5, v_7 since $d_{v_4} = d_{v_5} = d_{v_7} = 4$. Thus $N[v_6]$ has a claw, which is impossible. So $v_2 \not\sim v_7$. Then it must be $v_7 \sim v_1$ otherwise G would have D_4 as an induced subgraph, contrary to the choice of m. v_6 must have a neighbor which is not labeled. Label it v_8. Applying Lemma 3.5 to $N[v_6]$, it follows that $v_8 \sim v_7$. Applying Lemma 3.5 to $N[v_7]$ we have that $v_8 \sim v_1$. Finally, applying Lemma 3.5 to $N[v_1]$ we have that $v_2 \sim v_8$. Now, all labeled vertices induce B_4 as subgraph of G. Since G is 4-regular and connected, $G = B_4$.

Now let $m > 3$ and suppose that G has D_m as an induced subgraph, but not D_{m+1}. Suppose by using induction that if $m - 1 \geq 3$ and G has D_{m-1} as an induced subgraph but not D_m, then $G = M_m$ or $G = B_m$. Now we replace vertices v_3 and v_5 by a new vertex v_{35}, replace vertices v_2 and v_4 by a new vertex v_{24} and remove all edges incident with v_2, v_3, v_4, v_5. Let $N(v_{35}) = \{v_1, v_{24}, v_6, v_7\}$. Let $N(v_{24}) = (N(v_2) \cup N(v_4) \cup \{v_{35}\}) - \{v_2, v_3, v_4, v_5\}$. It is easy to see that the newly obtained graph G' is still K_4-free critical and the largest k such that D_k is an induced subgraph of G' is $k = m - 1$ (since the adjacency of the neighborhood of any vertex other than v_{24}, v_{35} is unchanged and the neighborhoods of v_{24}, v_{35} are similar to those of v_2, v_3). By induction, $G' = B_m$ or $G' = M_m$. So it follows that $G = B_{m+1}$ or $G = M_{m+1}$. □

Now we are going to show that $k(G) = 2$ for all $G \in C$. The following theorem was first given by Lundgren and Maybee in [6] with a small error, and was corrected by Kim [3].

Theorem 3.7 *(Lundgren and Maybee [6], Kim [3]) If G is a graph with n vertices and $m \leq n$, then $k(G) \leq m$ if and only if G has an* *edge clique covering $\{C_1, C_2, ..., C_{n+m-2}\}$ and a labeling $v_1, v_2, ..., v_n$ of $V(G)$ so that if $v_i \in C_j$, then $i \geq j - m + 1$, where $n = |V(G)|$.*

Lemma 3.8 *$k(B_n) = 2$ and $k(M_n) = 2$, for $n \geq 4$.*

Proof. By Lemma 2.3, we only need to show that $k(G) \leq 2$ for all $G \in \mathcal{C}$. Let B_n, $n \geq 4$, be re-labeled as shown in Figure 3.6. An edge clique covering is given as follows:

$$\begin{aligned} C_1 &= \{1, n, 2n\}; & \\ C_i &= \{i-1, i, n+i-1\}, & i = 2, ..., n; \\ C_{n+i} &= \{i-1, i\}, & i = n+2, ..., 2n; \\ C_{n+1} &= \{n+1, 2n\}; & \end{aligned}$$

Let M_n, $n \geq 4$, be re-labeled as shown in Figure 3.7. An edge clique covering of M_n is given as follows:

$$\begin{aligned} C_1 &= \{1, n, 2n-1\}; & \\ C_i &= \{i-1, n+i-1, i\}, & i = 2, ..., n; \\ C_i &= \{i-1, i\}, & i = n+1, ..., 2n-1. \end{aligned}$$

Then by Theorem 3.7, $k(M_n) \leq 2$, $k(B_n) \leq 2$. □

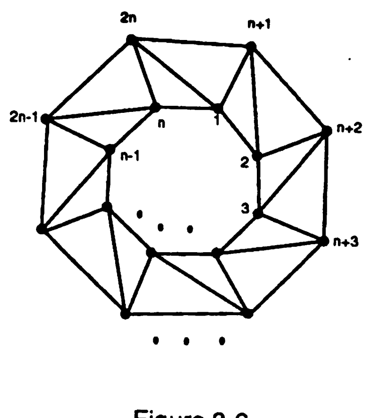

Figure 3.6

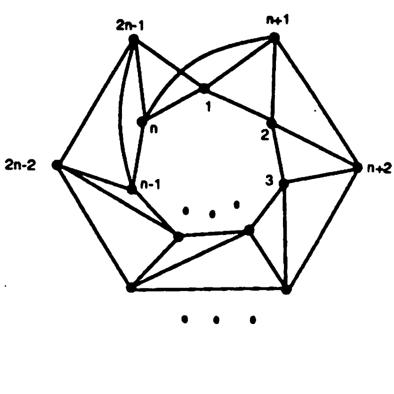

Figure 3.7

Theorem 3.9 *Conjecture 2.1 is true for K_4-free graphs.*

Proof. This follows from Theorem 3.6, Lemma 3.8, Theorems 2.6 and 2.7. □

Though characterizing critical graphs can be a possible approach to settle Conjecture 2.1, we expect that it is difficult. We shall see in the next section that there are critical graphs with arbitrarily large induced complete subgraphs and their structures are not as "regular" as those K_4-free critical graphs which we just characterized. Another approach to characterize critical graphs is proposed in next section.

4 Lift Critical Graphs

In this section, we study a graph operation as an approach to characterize critical graphs and to tackle Conjecture 2.1. We extend our results in section 3 to a larger class of critical graphs.

It is easy to see that any proper induced subgraph of a K_4-free critical graph is non-critical. So K_4-free critical graphs are minimal critical graphs under the process of taking induced subgraphs.

Let $G = (V(G), E(G))$ be a graph. If $S \subseteq V(G)$, a *lift* of G is the graph obtained by adding a new vertex w and a set of new edges $E' = \{(x, w) | x \in S\}$ to G. Denote the new graph as $G_S = (V(G_S), E(G_S))$. S is called a *lift set* of G.

Lemma 4.1 *Any critical graph can be obtained from some minimal critical graph by applying a series of lifts.*

Proof. Let G be a critical graph. Suppose that H is a minimal critical induced subgraph of G. If $G = H$ then we are done. Otherwise for any induced subgraph F of G such that H is an induced subgraph of F, F is critical by Theorem 2.5. So the lemma follows. □

We call a lift G_S a *critical lift* of a critical graph G if G_S is critical.

Theorem 4.2 *Conjecture 2.1 is true if and only if the following hold:*
(i) $k(G) = 2$ *for all minimal critical graphs; and*
(ii) Critical lifts do not increase competition number of a critical graph.

Proof. The "*only if*" part is trivial. The "*if*" part follows from Theorem 2.8 and Lemma 4.1. □

We have seen that all K_4-free critical graphs are minimal critical graphs. But K_4-free critical graphs are not the only minimal critical graphs known to date. Incidently, all critical graphs we have encountered can be obtained from the minimal critical graphs known by a series of special lifts which we will describe later and Conjecture 2.1 is true for them. While the problem of characterizing all minimal critical graphs remains open, in the following, we study some lift operations.

Lemma 4.3 *Let G be a critical graph. A lift G_S of G is critical if and only if $\theta(N_{G_S}(v)) = 2$ for all $x \in V(G_S)$.*

Proof. The "*only if*" part is trivial. The "*if*" part is also easy since if G_S is not critical, neither can G be by Theorem 2.5. □

An immediate observation is that a critical lift set must be the union of two distinct cliques. Among lift operations, we present three critical lifts defined by lift sets in the forms of $S = N[v]$, $S \subset N[v]$ and $S \supset N[v]$ for some $v \in S$.

A critical lift called *multiplication* is defined by a lift set as follows: let v be a non-isolated vertex of G and let the lift set $S = N_G[v]$. Then the corresponding lift of G is $G_S = (V(G) \cup \{w\}, E(G) \cup \{(w, x) | x \in N_G[v]\})$.

Lemma 4.4 (*a*) *Multiplications (on non-isolated vertices) do not increase competition number.*

(*b*) *G is critical if and only if any graph obtained by applying a series of multiplications (on non-isolated vertices) G is critical.*

Proof. We only need to show that the theorem is true after applying one multiplication on a critical graph.

(*a*) Let G' be the graph obtained from G by multiplying a non-isolated vertex $v \in V(G)$ by w. Let $k(G) = m$ and $|V(G)| = n$. By Theorem 3.7, $G \cup I_m$ has an edge clique covering C_1, C_2, ... , C_{n+m-2} and a labeling $v_1, v_2, ..., v_n$ of $V(G)$ such that if $v_i \in C_j$, then $i > j - m + 1$. Now label w as v_{n+m+1}. For $i = 1, ..., n + m - 2$, let $C_i' = C_i$ if $v \notin C_i$, and $C_i' = C_i \cup \{w\}$ if $v \in C_i$. Let $C_{n+m+1} = \emptyset$. Now $C_1', \ldots C_{n+m+1}'$ is an edge clique covering of G' and if $v_i \in C_j$, then $i > j$. So $k(G') \leq k(G)$ by Theorem 3.7.

(*b*) Since multiplication does not change adjacency between vertices of $V(G)$, it follows that $\theta(N_{G'}(x)) \geq \theta(N_G(x))$ for all $x \in V(G)$. On the other hand, let $C_1, \ldots, C_k$ cover $N_G(x)$, $x \in V(G)$. Now let $C_i' = C_i \cup \{w\}$ if $v \in C_i$ and let $C_i' = C_i$ if $v \notin C_i$, $i = 1, \ldots, k$. Then $C_1', \ldots, C_k'$ cover $N_{G'}(x)$ and $\theta(N_G(x)) \geq \theta(N_{G'}(x))$. Thus we have that $\theta(N_G(x)) = \theta(N_{G'}(x))$ for all $x \in V(G)$. It is also clear that $\theta(N_{G'}(w)) = \theta(N_G(v))$. Therefore $\theta(N_G(x)) = 2$ for all $x \in V(G)$ if and only if $\theta(N_{G'}(y)) = 2$ for all $y \in V(G')$. By Lemma 4.3, G' is critical if G is.

Conversely, suppose G' is critical. We know that $\theta(N_G(v)) = 2$ for all $v \in V(G)$ since $\theta(N_{G'}(y)) = 2$ for all $y \in V(G')$. Let C be a clique of G. C is also a clique in G'. First suppose that $v \notin C$. Since G' is critical, there is $u \in C$ $(u \neq w)$ in G' such that u has two non-adjacent vertices $x, y \in N_{G'}(u) - C$. If $w \notin \{x, y\}$ then we are done since x, $y \in V(G)$ and $x \not\sim y$ in G. If $w \in \{x, y\}$, say $w = x$, then $u \neq v$ since $u \sim y$ but $w \not\sim y$. It follows that $v \sim u$ and $v \not\sim y$. Then $\{y, v\} \in G - C$ are two non-adjacent neighbors of u in G. Now suppose that $v \in C$. Let $C' = C \cup \{w\}$. Similarly, there is $u \in C'$ such some $x \not\sim y$ are neighbors of u in G', but $x, y \notin C'$. Therefore $x.y$ are neighbors of u in G which are not in C. So G is critical. The proof is finished. □

Given a graph $G = (V(G), E(G))$, vertices of G $x, y \in V(G)$ are called *equivalent* if $N[x] = N[y]$. Graph G is called *reduced* if it has no pair of distinct equivalent vertices. If x, y are two distinct equivalent vertices of G, the operation from G to $G - x$ is called *contracting* equivalent vertices. Repeating contraction, we obtain a graph without equivalent vertices, which we called the *reduced graph* of G. If the reduced graph of G does not contain H as an induced subgraph, we say that G is H-free reducible. Now, we can state Theorem 3.9 in a broader content.

Theorem 4.5 *Conjecture 2.1 is true for K_4-free reducible graphs.*

Proof. By Theorem 2.6 and 2.7, we only need to show that if G is critical, then $k(G) = 2$. Let G be a K_4-free reducible critical graph.

Let H be the reduced graph of G. Then by Lemma 4.4, H is critical and $k(G) = k(H)$. By Theorem 2.8 $k(H) = 2$. So it follows that $k(G) = 2$, i.e., Conjecture 2.1 is true for G. □

It is clear that the size of maximum cliques among K_4-free reducible graphs can be arbitrarily large. We list two other types of critical lift sets as follows. Let G be a critical graph. Then G has no isolated vertices. For a vertex $v \in V(G)$ there are always two cliques C_1, C_2 covering vertices of $N(v)$ ($v \notin C_1 \cup C_2$). Let C be a clique of G such that $C \cap N(v) = \emptyset$. If for all $x \in C$, $C_1 \subseteq N(x)$ and $N(x) - (C_1 \cup C)$ is a clique of G, then we call the critical lift defined by lift set $S = N[v] \cup C$ an *extended multiplication* of v.

Proposition 4.6 *The extended multiplication on a critical graph defines a critical lift.*

Proof. Let the extended multiplication of G be given by $G_S = (V(G) \cup \{w\}, E(G) \cup \{(w, x)|x \in S\})$. It is clear that $\theta(N_{G_S}(v)) \geq 2$ for all $v \in V(G_S)$. We only need to check that neighbors of any $x \in V(G_S)$ can be covered by exactly two cliques. Since $C_1 \subseteq N(x)$ for all $x \in C$ we have that $C_1 \cup C$ is a clique of G. So it is a clique of G_S. Hence, $N_{G_S}(w)$ can be covered by two cliques $C_2 \cup \{v\}$ and $C_1 \cup C$. $N_{G_S}(v)$ can be covered by C_2 and $C_1 \cup \{w\}$. If $x \in C$, then $N_{G_S}(x)$ can be covered by two cliques $(C - x) \cup C_1 \cup \{w\}$ and $N_G(x) - (C_1 \cup C)$. If $x \in N_G(v)$, $N_G(x)$ can be covered, say, by C_1' and C_2' where $v \in C_2'$ in G, then $N_{G_S}(x)$ can be covered by C_1' and $C_2' \cup \{w\}$ in G_S. If $x \notin S$, the neighbors of x in G_S have the same adjacency as in G, so the lemma follows. □

The graph in Figure 4.1 is a critical graph obtained by applying one extended multiplication on B_6. It can be checked that after deleting any vertex other than the newly added one w the graph is no longer critical. So it cannot be obtained by applying multiplications on any other smaller critical graph. Nevertheless, its competition number is 2.

Another class of critical lifts analogous to multiplication arises if instead of requiring $S = N_G[v]$, we have that $S \subset N_G[v]$, $v \in S$. We call it a *partial multiplication* of v. The graph in Figure 4.2 is a critical graph obtained by applying one partial multiplication on B_5. Similarly, it can be checked that after deleting any vertex other than the newly added one w the graph is no longer critical. Hence G cannot be obtained by applying multiplications or extended multiplications on any other smaller critical graph. It is easy to check that its competition number is also 2.

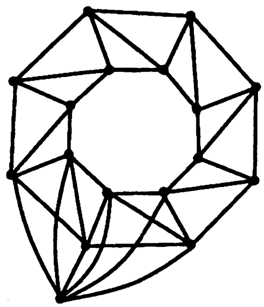

Figure 4.1

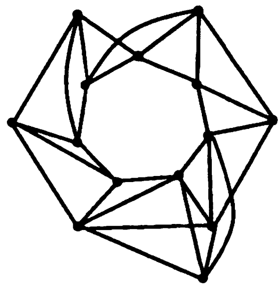

Figure 4.2

Our interests on the critical lifts multiplication, extended multiplication and partial multiplication given by $S = N[v]$, $S \supset N[v]$ and $S \subset N[v]$ for some $v \in S$ are that these critical lifts are the only critical lifts which we know to date. It can be proved that these three types of critical lifts are the only critical lifts on K_4-free critical graphs. We might ask if multiplications, extended multiplications and partial multiplications are the only possible critical lifts. Though we have not been able to answer this question completely, to end this section, we prove that $N(v)$ can never be a critical lift set for any $v \in V(G)$.

Proposition 4.7 *Let G be a connected graph such that $\theta(N(v)) = 2$ for all $v \in V(G)$. If there are $x \not\sim y \in G$ such that $N(x) = N(y)$ then G contains no critical induced subgraph and Conjecture 2.1 is true for G.*

Proof. Let $A = N(x) = N(y)$. Let $w \in A$. If there is $z \in N(w)$ such that $z \notin A$, then $\{x, y, w, z\}$ induces a claw, which is impossible. Hence $N(w) \subseteq A$ for all $w \in A$. So $V(G) = A \cup \{x, y\}$. To show that G has no induced critical subgraph, we only need to consider the induced subgraphs H of G such that $\theta(N_H(v)) = 2$ for all $v \in V(H)$. Let H be such an induced subgraph of G. Since $\theta(N_G(x)) = 2$, let C_1, C_2 be two cliques covering A with $x \notin C_1$. Then for all $w \in (C_1 \cup \{y\}) \cap H$ we have that $N_H(w) - (C_1 \cup \{y\}) \subseteq H \cap (C_2 \cup \{x\})$ which is a clique. So $\theta(N_H(w) - (C_1 \cup \{y\}) \cap H) \leq 1$. Hence H is not critical. □

Corollary 4.8 *If G is a critical graph then for any $v \in V(G)$, $N(v)$ cannot be a critical lift set.*

Proof. If $N(v)$ is a critical lift set for some $v \in V(G)$ then in the lift $G_{N(v)}$ of G obtained by adding new vertex w and new edges $\{wy | y \in N(v)\}$, we have that $N(v) = N(w)$. By Proposition 4.7, $G_{N(v)}$ is not critical. Since G is an induced subgraph of $G_{N(v)}$, G cannot be critical, contrary to our assumption. □

5 Closing Remark

Some problems related to Conjecture 2.1 remain unanswered. By Theorem 2.7, Opsut's Conjecture 2.1 now is true for all graphs which are disjoint unions of connected non-critical graphs and the settling of Conjecture 2.1 is reduced to connected critical graphs. Though we characterized all K_4-free critical graphs and proved that Conjecture 2.1 is true for those graphs and some other critical graphs, critical graphs have not been completely characterized. Also, the problems of characterizing all minimal critical graphs (under the process of taking induced subgraphs) and characterizing all critical lift sets remain unsolved. If Conjecture 2.1 is false, either there is some minimal critical graph G with $k(G) > 2$ or some critical lift increases competition number of some critical graph. Incidently, it is noticed that all critical lift sets known are in the forms of multiplication, extended multiplication and partial multiplication. Also, those critical lifts do not increase competition numbers in all encountered cases (otherwise we would have had a counterexample already).

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