# Degrees, Neighborhood Unions and Hamiltonian Properties①

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**ABSTRACT:** Let G be a 2–connected simple graph of order n ($\geqslant 3$) with connectivity k. One of our results is that if there exists an interge t such that for any distinct vertices u and v, d(u,v) = 2 implies $|N(u) \bigcup N(v)| \geqslant n-t$, and for any independent set S of cardinality k+1, $\max\{d(u)|u \in S\} \geqslant t$, then G is hamiltonian. This unifies many known results for hamiltonian graphs. We also obtain a similar result for hamilton–connected graphs.

This paper uses terms and notations of [1]. Throughout G denotes an undirected connected simple graph of order n($\geqslant 3$) with connectivity k and independence number $\alpha$. Let L be a subset of V(G), F a subgraph of G and v a vertex in G. Define $N_L(v) = \{u \in L | uv \in E(G)\}$, $N_L(F) = \bigcup_{v \in V(F)} N_L(v)$. Specially, if L = V(G), we simple write it as N(v) and N(F). If no ambiguity can arise we sometimes write F instead of V(F). Let $S \subseteq V$, define $\Delta(S) = \max\{d(u)|u \in S\}$.

The study of the theory of hamiltonian graphs has given rise to many results. Many of these results use edge density conditions to force the existence of a hamiltonian cycle. Recently, it has been determined that less stringent edge density requirements can be placed on a graph by considering the cardinality of neighborhood unions rather than degree sums. In this paper, we establish the relations among these results. The result of [2] is a special case of our result. Ore's[3]Theorem and Chvátal and Erdös's[4]Theorem are corollaries of our result.

**Theorem 1** *Let G be a 2–connected graph of order n($\geqslant 3$) and connectivity k. If there exists an interger t such that for any vertices u,v, d(u,v) = 2 implies $|N(u) \bigcup N(v)| \geqslant n-t$, and for any independent set S of cardinality k+1, $\Delta(S) \geqslant t$, then G is hamiltonian.*

**Proof:** Let $C = v_1 v_2 \cdots v_r v_1$ be a longest cycle in G. If G is not hamiltonian, let B be any component of $G \setminus V(C)$, $N_C(B) = \{v_{i_1}, v_{i_2}, \cdots,$

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$v_{i_m}\}$. G is k-connected implies $m \geqslant k$. Let $x_j \in N_B(v_{i_j})$. It is possible that $x_p = x_j$ for $p \neq j$. Put $N^- = \{v_{i_1-1}, v_{i_2-1}, \cdots, v_{i_m-1}\}$, $N^+ = \{v_{i_1+1}, v_{i_2+1}, \cdots, v_{i_m+1}\}$. The following Claim 1 is clear.

**Claim 1** *For any j $(1 \leqslant j \leqslant m)$, $N^- \cup \{x_j\}$ and $N^+ \cup \{x_j\}$ are independent.*

For any $j (1 \leqslant j \leqslant m)$, Claim 1 implies $v_{i_{j-1}+1} \notin N(v_{i_j+1})$. Thus, since C is longest and $v_{i_j} \in N(v_{i_j+1})$, there exists a vertex $v_{h_j}$, $i_{j-1}+1 \leqslant h_j \leqslant i_j-1$, such that $v_{h_j} \notin N(v_{i_j+1})$, yet $v_q \in N(v_{i_j+1})$ for all $h_j+1 \leqslant q \leqslant i_j$. Put $N = \{v_{h_1}, v_{h_2}, \cdots, v_{h_m}\}$.

**Claim 2** *Let p,q be two integers satisfing $1 \leqslant p < q \leqslant m$. Then for any $v_{j_1} \in \{v_{h_p}, v_{h_p+1}, \cdots, v_{i_p-1}\}$, $v_{j_2} \in \{v_{h_q}, v_{h_q+1}, \cdots, v_{i_q-1}\}$, $v_{j_1} v_{j_2} \notin E(G)$.*

If there exist such vertices $v_{j_1}$, $v_{j_2}$ with $v_{j_1} v_{j_2} \in E(G)$, the cycle

$$v_{j_1} v_{j_2} v_{j_2-1} \cdots v_{i_p+1} v_{j_1+1} v_{j_1+2} \cdots v_{i_p} x_p \cdots x_q v_{i_q} v_{i_q-1} \cdots v_{j_2+1} v_{i_q+1} v_{i_q+2} \cdots v_{j_1}$$

is longer than C. A contradiction.

**Claim 3** *$\Delta(V(B)) < t$.*

In fact, if $\Delta(V(B)) \geqslant t$, then for any $u, v \in \{v_{h_1}, v_{i_2-1}, v_{i_3-1}, \cdots, v_{i_m-1}\}$ or $N^+$, $|N(u) \cup N(v)| \leqslant n-m-|B| \leqslant n-t-1$. This and the condition of the theorem imply $N(u) \cap N(v) = \phi$. So, put $D = N(v_{h_1}) \cup N(v_{i_1+1})$, $D \cap (N_C(B) \setminus \{v_{i_1}\}) = \phi$. This implies that $|D| \leqslant n-|B|-(|N_C(B)|-1)-2 \leqslant n-t-2$. However, since $d(v_{h_1}, v_{i_1+1}) = 2$, the condition of Theorem 1 implies $|D| \geqslant n-t$. This leads to a contradiction.

Put $S = \{v_{h_1}, v_{h_2}, \cdots, v_{h_k}, x_1\}$. By Claim 2, S is an independent set of cardinality k+1. Then the condition of Theorem 1 implies $\Delta(S) \geqslant t$. By Claim 3, without loss of generality, suppose $d(v_{h_1}) \geqslant t$. Consider $v_j \in F = N(v_{i_2-1}) \cup N(x_2)$. By Claim 2, $v_j \notin \{v_{h_1}, v_{h_1+1}, \cdots, v_{i_1-1}\}$. We prove the following Claim 4.

**Claim 4** *(1) If $i_1 \leqslant j \leqslant i_2-2$, then $v_{h_1} v_{j+1} \notin E(G)$;*

*(2) If $i_2 \leqslant j \leqslant h_1-1$, then $v_{h_1} v_{j-1} \notin E(G)$.*

Proof: If there exists j ($i_1 \leqslant j \leqslant i_2-2$) with $v_{h_1} v_{j+1} \in E(G)$, then by the definition of $N_C(B)$, $v_j \in N(v_{i_2-1})$, the cycle

$$v_{h_1} v_{j+1} v_{j+2} \cdots v_{i_2-1} v_j v_{j-1} \cdots v_{i_1+1} v_{h_1+1} v_{h_1+2} \cdots v_{i_1} x_1 \cdots x_2 v_{i_2} v_{i_2+1} \cdots v_{h_1}$$

is longer than C. If there exists j ($i_2 \leqslant j \leqslant h_1-1$), $v_{h_1} v_{j-1} \in E(G)$, then by the definition of $N_C(B)$ and Clanim 2, $v_j \in N(v_{i_2-1})$, the cycle

$$v_{h_1} v_{j-1} v_{j-2} \cdots v_{i_2} x_2 \cdots x_1 v_{i_1} v_{i_1-1} \cdots v_{h_1+1} v_{i_1+1} v_{i_1+2} \cdots v_{i_2-1} v_j v_{j+1} \cdots v_{h_1}$$

is longer than C. These are contradiction.

We define the function f: F→V(G) by:

$$f(u) = \begin{cases} u & for\ u \notin V(C) \\ v_{j+1} & for\ i_1 \leqslant j \leqslant i_2-3\ and\ u=v_j \\ v_{h_1} & for\ j=i_2-2\ and\ u=v_j \\ v_{j-1} & for\ i_2 \leqslant j \leqslant h_1-1\ and\ u=v_j \end{cases}$$

From the previous arguments and Claim 4, for any u∈F, we have $uv_{h_1} \notin E$. By the condition of Theorem 1, $d(v_{i_2-1}, x_2)=2$ means $|F| \geqslant n-t$. Therefore, Note that $x_2 \notin f(F)$ and $x_2 v_{h_1} \notin E(G)$, we obtain $d(v_{h_1}) \leqslant n-(|F|)-1 = t-1$. This is a contradiction. This implies Theorem 1 holds. □

**Corollary 1**[4] *Let G be a simple graph of order n⩾ 3,connectivity k and independence number α. If α⩽ k, then G is hamiltonian.*

Proof: When $\alpha \leqslant k$ this implies that these do not exist any independent set of cardinality k+1. Thus we need only to show that there exists an integer t, such that for any vertices u,v, d(u,v)=2 implies $|N(u) \bigcup N(v)| \geqslant n-t$. This is clear (for example, taking t=n−1). □

**Corollary 2** *Let G be a 2−connected simple graph of order n(⩾ 3). If for every pair of nonadjacent vertices u and v, $|N(u) \bigcup N(v)| \geqslant (2n-2)/3$, then G is hamiltonian.*

Proof: Take $t=\left[\frac{n+2}{3}\right]$, where $[x]$ denotes the largest integer to be less than or equal to x. Since $|N(u) \bigcup N(v)|$is integer, $|N(u) \bigcup N(v)| \geqslant \frac{2n-2}{3}$ implies $|N(u) \bigcup N(v)| \geqslant n-\left[\frac{n+2}{3}\right]$. Let S be any independent set of cardinality k+1. If there exist vertices u,v∈S such that $|N(u) \bigcap N(v)| \geqslant 2$, then $\Delta(S) \geqslant \frac{1}{2}(\frac{2n-2}{3} - 2)+2 = \frac{2n+2}{6} \geqslant t$. Theorem 1 implies the corollary holds. Otherwise, for any u, $v \in S$, $|N(u) \bigcap N(v)| \leqslant 1$. Let u,v,w be any three vertices in S, $p = |N(u) \bigcap N(v)| + |N(u) \bigcap N(w)| + |N(w) \bigcap N(v)|$. Then $p \leqslant 3$ and

$$\frac{1}{2}(3 \cdot \frac{2n-2}{3} + 1) \leqslant d(u)+d(v)+d(w) \leqslant n-(k+1)+p,$$

that is, $p \geqslant 2k$. $k \geqslant 2$ implies $p \geqslant 4$, and this leads to a contradiction. □

Corollary 2 improves Theorem 2 in [2].

**Corollary 3** *Let G be a 2–connected simple graph of order $n(\geqslant 3)$. If for every pair of distinct vertices u and v, $d(u,v)=2$ implies $|N(u) \bigcup N(v)| \geqslant n-\delta$, then G is hamiltonian.*

The hypothesis of Corollary 3 is weaker than the hypothesis of Theorem 2 in [5].

**Corollary 4** *Let G be a 2–connected graph of order $n(\geqslant 3)$. If for any distinct vertices u,v, $d(u,v)=2$ implies $|N(u) \bigcup N(v)| \geqslant \frac{n}{2}$, and for any independent set S of cardinality k+1, $\Delta(S) \geqslant \frac{n}{2}$, then G is hamiltonian.*

Corollary 4 is more general than Ore's Theorem.

Now we discuss hamilton–connected property of graphs.

**Theorem 2** *Let G be a 3–connected graph of order $n(\geqslant 3)$ and connectivity k. If there exists an interger t such that for any vertices u,v, $d(u,v)=2$ implies $|N(u) \bigcup N(v)| > n-t$, and for any independent set S of cardinality k, $\Delta(S) > t$ or there exist two distinct vertices $u,v \in S$ with $d(u)=t$, $d(v)=t$, then G is hamilton–connected.*

Proof: Suppose that G is not hamilton–connected. Then there exists some pair of vertices u and v such that no hamiltonian u–v path exists in G. Consider a longest u–v path $P = v_1 v_2 \cdots v_r$ in G, where $u = v_1$, $v = v_r$. Let B be any component of $G \setminus V(P)$, $N_P(B) = \{v_{i_1}, v_{i_2}, \cdots, v_{i_m}\}$. With the assumed connectivity, we have $m \geqslant k \geqslant 3$. Let $x_j \in N_B(v_{i_j})$. It is possible that $x_p = x_j$ for $p \neq i$. Put $N^+ = \{v_{i_1-1}, v_{i_2-1}, \cdots, v_{i_m-1}\}$, $N^- = \{v_{i_1+1}, v_{i_2+1}, \cdots, v_{i_m+1}\}$.

**Claim 5** *For any $j$ ($1 \leqslant j \leqslant m$), $N^+ \cup \{x_j\}$ and $N^- \cup \{x_j\}$ are independent.*

Similar to the proof of Theorem 1, we define $h_j$ for $j(2 \leqslant j \leqslant m)$, and if $i_m = r$, then $h_m = i_m - 1$. Put $N = \{v_{h_2}, v_{h_3}, \cdots, v_{h_m}\}$. We have:

**Claim 6** *Let $p$, $q$ be two integers satisfing $2 \leqslant p < q \leqslant m$. Then for any $v_{j_1} \in \{v_{h_p}, v_{h_p+1}, \cdots, v_{i_p-1}\}$, $v_{j_2} \in \{v_{h_q}, v_{h_q+1}, \cdots, v_{i_q-1}\}$, $v_{j_1} v_{j_2} \notin E(G)$.*

**Claim 7** $\Delta(V(B)) < t$.

Put $S = \{v_{h_2}, v_{h_3}, \cdots, v_{h_k}, x_2\}$. By Claim 6, S is an independent set of cardinality k. Then the condition of Theorem 2 implies $\Delta(S) \geqslant t$. By Claim 7, there exists $v_{h_j} \in N \cap S$ with $d(\bar{v}_{h_j}) \geqslant t$.

(1) $j < m$. Let $v_s \in N(v_{i_{j+1}-1}) \cup N(x_{j+1})$. By Claim 6, $v_s \notin \{v_{h_j}, v_{h_j+1}, \cdots, v_{i_j-1}\}$. Similar to Claim 4, we have:

**Claim 8** *(1) If $s < h_j$, or $s \geqslant i_{j+1}$, then $v_{s-1} \notin N(v_{h_j})$;*
*(2) If $i_j \leqslant s \leqslant i_{j+1} - 1$, then $v_{s+1} \notin N(v_{h_j})$.*

Hence, we obtain at least $|N(v_{i_{j+1}-1}) \cup N(x_{j+1})|$ vertices which are nonadjacent to $v_{h_j}$, by defining a function. This implies $d(v_{h_j}) < t$. A contradiction.

(2). $j = m$. If $i_1 > 1$, then $S' = \{v_{i_{j+1}-1}, v_{h_2}, v_{h_3}, \cdots, v_{h_{k-1}}, x_2\}$ is also an independent set of cardinality k. By the condition of theorem 2 we have $\Delta(S') \geqslant t$. By Claim 7 and the argument of (1), we can suppose $d(v_{i_1-1}) \geqslant t$. Consider $F = N(v_{i_2-1}) \cup N(x_2)$, we obtain a contradiction by similar argument of (1). Thus, $i_1 = 1$. By the symmetry, we can suppose $i_m = r$. Further, we can suppose $d(v_{h_m}) > t$ by the argument of (1) and the condition of Theorem 2. Let $v_j \in F$, if $i_2 < j < i_m - 1$, then $v_{j-1} \notin N(v_{h_m})$; if $1 < j < i_2 - 1$, then $v_{j+1} \notin N(v_{h_m})$. Define a function $f: F \to V(G)$ by:

$$f(u) = \begin{cases} u & \text{for } u \notin V(P) \\ v_{j-1} & \text{for } j \geqslant i_2 \text{ and } u = v_j \\ v_{j+1} & \text{for } 2 \leqslant j < i_2 - 2 \text{ and } u = v_j \\ x_2 & \text{for } j = i_2 - 2 \text{ and } u = v_j \end{cases}$$

We obtain that there exist at least $|F|-1$ vertices which are nonadjacent to $v_{h_m}$ . this implies that $d(v_{h_m -1}) \leqslant n-(|F|-1) < n+1-(n-t) = t+1$, that is $d(v_{h_m}) \leqslant t$. This is contrary to $d(v_{h_m}) > t$ and the proof of Theorem 2 is completed. □

**Corollary 5** *Let G be a simple graph of order $n \geqslant 3$, connectivity k and independence number $\alpha$. If $\alpha \leqslant k-1$, then G is hamilton–connected.*

**Corollary 6** *Let G be a 3–connected simple graph of order $n(\geqslant 3)$. If for every pair of nonadjacent vertices u and v, $|N(u) \bigcup N(v)| > (2n-2)/3$, then G is hamilton–connected.*

Proof: Take $t = \left[\frac{n+1}{3}\right]$, where $[x]$ denotes the largest integer to be less than or equal to x. Since $|N(u) \bigcup N(v)|$ is integer, $|N(u) \bigcup N(v)| > (2n-2)/3$ implies $|N(u) \bigcup N(v)| \geqslant n - \left[\frac{n+1}{3}\right]$. Let S be any independent set of cardinality k. If there exist vertices $u, v \in S$ such that $|N(u) \bigcap N(v)| \geqslant 2$, then $\Delta(S) \geqslant \frac{1}{2}(\frac{2n-1}{3} - 2) + 2 = \frac{2n+5}{6} > t$. Theorem 2 implies the corollary holds. Otherwise, for any $u, v \in S$, $|N(u) \bigcap N(v)| \leqslant 1$. Let u, v, w be any three vertices in S, $r = |N(u) \bigcap N(v)| + |N(u) \bigcap N(w)| + |N(w) \bigcap N(v)|$. Then $r \leqslant 3$ and

$$\frac{1}{2}(3 \cdot \frac{2n-1}{3} + r) \leqslant d(u) + d(v) + d(w) \leqslant n - k + r,$$

that is, $r \geqslant 2k-1$. $k \geqslant 3$ implies $r \geqslant 5$, and this leads to a contradiction. □

Corollary 6 improves Theorem 3 in [2].

**Corollary 7** *Let G be a 3–connected simple graph of order $n(\geqslant 3)$. If for every pair of distinct vertices u and v, $d(u,v) = 2$ implies $|N(u) \bigcup N(v)| > n-\delta$, then G is hamilton–connected.*

**Corollary 8** *Let G be a 3–connected graph of order $n(\geqslant 3)$. If for any distinct vertices u,v, $d(u,v) = 2$ implies $|N(u) \bigcup N(v)| > \frac{n+1}{2}$, and for any independent set S of cardinality k, $\Delta(S) \geqslant \frac{n+1}{2}$, then G is hamilton–connected.*

**Corollary 9** *Let G be a connected graph. If for any nonadjacent vertices u,v, $d(u)+d(v) \geqslant n+1$, then G is hamilton–connected.*

## References


[1] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications. Macmillan Co.,London, 1976.

[2] R.J.Faudree, R.J.Gould, M.S.Jacobson and R.H.Schelp, Neighborhood unions and Hamiltonian properties in graphs. J. Combin. Theory B 47(1989), 1–9.

[3] O.Ore, A note on hamiltonian circuits. Amer. Math. Monthly 67(1960),55.

[4] V. Chvátal and P. Erdös, A note on hamiltonian circuits. Discrete Math. 2(1972), 111–113.