Graceful and Harmonious Labelings of Prism Related Graphs

Joseph A. Gallian, John Prout, and Steven Winters

Department of Mathematics and Statistics
University of Minnesota, Duluth
Duluth, MN 55812

Abstract. We provide graceful and harmonious labelings for all vertex deleted and edge deleted prisms. We also show that with the sole exception of the cube all prisms are harmonious.

1. Introduction

A connected graph with v vertices and e edges is called *graceful* if it is possible to label the vertices x with distinct integers $f(x)$ from $0, 1, \cdots, e$ in such a way that, when each edge xy is labeled with $|f(x) - f(y)|$, the resulting edge labels are distinct. A graceful labeling is called an *α-labeling* if there is an interger k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph with an α-labeling is necessarily bipartite so that graphs that contain a cycle of odd length do not have α-labelings. The concepts of gracefulness and α-labelings were first introduced by Rosa in 1966 [13].

In 1980, as an additive analog of gracefulness, Graham and Sloane [10] defined a *harmonious* graph as a connected graph with v vertices and e edges if it is possible to label the vertices x with distinct elements from $0, 1, \cdots, e-1$ in such a way that, when each edge xy is labeled with $(f(x) + f(y))$ modulo e, the resulting edge labels are distinct.

In [9] Grace defines a *sequential* labeling of a graph with e edges and v vertices as one where the vertices x are assigned distinct integers $f(x)$ from $0, 1, \cdots, e-1$ in such a way that, when each edge xy is labled with $f(x) + f(y)$, the resulting edge labels are distinct consecutive integers. Every sequential labeling induces a harmonious labeling. It is an open question whether or not every harmonious graph admits a sequential labeling.

By a prism $D_n (n \geq 3)$ we mean the cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices. For convenience, we denote the vertices of D_n by $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$ where the a's are consecutive vertices of one n-cycle and the b's are consecutive vertices of the other n-cycle and a_i is connected to b_i. See Figure 1. We mention that D_n is the Cayley graph of the dihedral group of order $2n$ generated by a rotation and a reflection.

The first result on labeling prisms was by Bodendiek, Schumacher and Wegner [4] in 1977. They proved that the prism D_n has an α-labeling when $n \equiv 0 \bmod 4$. According to a survey by Bermond [1], T. Gangopadyay and S. P. Rao Hebbare did the case that n is even about the same time. In 1979 Frucht [5] stated, without proof, that he had proved that all prisms are graceful. In [6] Frucht and

Gallian showed that all prisms have graceful labelings, and when n is even, the labelings are α-labelings. In this paper we do the same for vertex deleted and edge deleted prisms. Ropp [12] has just proved the prisms with pendant points attached are graceful.

In their paper introducing harmonious graphs Graham and Sloane [10] give a harmonious labeling for $C_m \times P_n$, m odd, and mention that a computer check had revealed that $C_4 \times P_2$, the cube, is not harmonious. This motivated us to investigate $C_m \times P_2$ where m is even. Here we prove that the cube is the only prism that is not harmonious. We also prove that all vertex deleted and edge deleted prisms are harmonious, indeed sequential.

Jungreis and Reid [11] investigated the existence of graceful and harmonious graphs of the form $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$. They handled many but not all of these graphs. Interesting, they showed how α-labelings for certain graphs can be used to produce graceful labelings and sequential labelings for related graphs.

Graham and Sloane [10] gave an extensive survey of graceful and harmonious labelings up to 1980. Gallian [7] brought this up to date as of 1988 and included a list of open problems. A broad overview of graph labeling results and techniques as of 1982 was given by Grace in his Ph.D. thesis [8]. Applications of graph labelings are detailed in [2] and [3]. Despite the large number of papers (over 150) on the subject there are very few general results or methods on graph labeling. Indeed, nearly all papers focus on particular classes of graphs and employ ad hoc methods. The present paper is no exception.

2. Graceful Labelings of Vertex Deleted Prisms

Since D_n is vertex symmetric, the graph obtained by deleting a vertex is independent of the vertex selected. We delete b_1. To specify our graceful labelings we will take advantage of that fact that they have a zigzag pattern. Thus we define the FTB (first-top-bottom) *vertex label sequence* for the vertex deleted prism D_n as the labels for the vertices $a_1, b_2, a_3, b_4, \cdots$ and the STB (second-top-bottom) *vertex label sequence* for the vertex deleted prism D_n as the labels for the vertices $a_2, b_3, a_4, b_5, \cdots$. Then we may define a vertex labeling of the vertex deleted prism D_n by specifying the labels for a_1 and a_2 and the **increments** from one member of the vertex label sequences to the next. For example, the FTB vertex label sequence for D_{12} (see Figure 2): $0,1,2,3,4,5,6,7,8,9,10,11$ can be specified by listing the value of a_1 and the sequence of increments: $0; 1,1,1,1,1,1,1,1,1,1,1$. Similarly, the STB vertex sequence for D_{12} : $32,30,28,24,22,20,18,16,14,12,33$ can be determined from the label of a_2 and the sequence of increments: $32; -2,-2,-4,-2,-2,-2,-2,-2,-2,21$. Because the increment sequences often contain long strings of the same integer it is convenient to use $k * t$ to denote a sequence of k $t's$. Analogously, $k * (s,t)$ denotes the string $s,t,s,t,\cdots,s,t$ of length $2k$. With this notation we can compactly convey graceful labelings of the vertex deleted prisms.

Theorem 1. *For all $n > 2$, the vertex deleted prism D_n is graceful. Moreover, when n is even the graph has an α-labeling.*

Proof: The proof breaks up into six cases depending upon the congruence class of n modulo 6. When $n < 6$ the graphs are done ad hoc. For $n \geq 6$, let $m = \lfloor n/6 \rfloor$.

Case 1. $n \equiv 0 \mod 6$.

FTB increments: $0; (n-1) * 1$.
STB increments: $3n-4; (2m-2) * -2, -4, (n-2m-2) * -2, 2n-3$.

See Figure 2 for D_{12}

Case 2. $n \equiv 1 \mod 6$.

FTB increments: $0; (n-3) * 1, 2, 2n-2$.
STB increments: $3n-4; (2m-1) * -2, -4, (n-2m-4) * -2, -1, -1$.

Case 3. $n \equiv 2 \mod 6$.

FTB increments: $0; (n-3) * 1, 2, 1$.
STB increments: $3n-4; (2m-1) * -2, -4, (n-2m-4) * -2, -2, -1,$
$2n-4$.

Case 4. $n \equiv 3 \mod 6$.

FTB increments: $0; 1, (n-2m-3) * 2, 4, (2m-1) * 2, n$.
STB increments: $3n-4; (n-2) * -1$.

Case 5. $n \equiv 4 \mod 6$.

FTB increments: $0; (n-4) * 1, 2, 2, 1$.
STB increments: $3n-4; 2m * -2, -4, (n-2m-6) * -2, -1, -1, 2n-5$.

Case 6. $n \equiv 5 \mod 6$.

FTB increments: $0; (n-3) * 1, 3, 2n-3$.
STB increments: $3n-4; (2m+1) * -2, -4, (n-2m-7) * -2, -3, 2, -1$.

∎

3. Graceful Labelings of Edge Deleted Prisms

There are two kinds of edge deleted prisms: Those where an edge from a cycle has been deleted and those where an edge joining the cycles has been deleted. We denote these by D_n^r and D_n^s, respectively (r for "rim" and s for "spoke").

As was the case for the vertex deleted prisms, our labelings have a zigzag pattern. Here we define the TB (top-bottom) *vertex label sequence* for D_n as the labels for the vertices $a_1, b_2, a_3, b_4, \cdots$ and the BT (bottom-top) *vertex label* *sequence* for D_n as the labels for the vertices $b_1, a_2, b_3, a_4, \cdots$. As before, we specify our graceful labelings by listing the initial vertex label of each sequence and the successive increments from one label to the next.

Theorem 2. *For all $n > 2$ the edge deleted prisms D_n^r and D_n^s are graceful. Moreover, when n is even the graphs D_n^r have α-labelings.*

Proof: We first consider the graphs D_n^r. When $n < 6$, the graphs are done ad hoc. For $n \geq 6$, the proof breaks into two cases when n is odd and three cases when n is even. We delete the edge joining a_1 to a_n.

Case 1. $n \equiv 1 \bmod 4$. Let $m = \lfloor n/4 \rfloor$.

TB increments: $0; m * (1,2), 3, (n-2m-3)/2 * (1,2), 1$.
BT increments: $3n-1; m * (-2,-1), -1, (n-2m-3)/2 * (-2,-1), -2$.

See Figure 3 for D_9^r

Case 2. $n \equiv 3 \bmod 4$. Let $m = \lfloor n/4 \rfloor$.

TB increments: $0, m * (1,2), 1, 3, m * (2,1)$.
BT increments: $3n-1; (m+1) * (-2,-1), -1, (n-2(m+1)-3)/2 * (-2,-1), -2$.

Case 3. $n \equiv 0 \bmod 6$. Let $m = n/6$.

TB increments: $0; (n-2m-1) * 2, 3, (2m-1) * 2$.
BT increments: $3n-1; (n-1) * -1$.

Case 4. $n \equiv 2 \bmod 6$. Let $m = \lfloor n/6 \rfloor$.

TB increments: $0; (n-2m-2) * 2, 3, (2m-1) * 2, 1$.
BT increments: $3n-1; (n-2) * -1, -2$.

Case 5. $n \equiv 4 \bmod 6$. Let $m = \lfloor n/6 \rfloor$.

TB increments: $0; 2(m+1) * 1, 2, (n-2m-5) * 1, 2$.
BT increments: $3n-1; (2m+1) * -2, -3, -1, (n-2m-5) * -2, -1$.

We now consider D_n^s. The cases $n \leq 12$ are done ad hoc. For $n > 12$, we consider five cases. We delete the edge joining a_n and b_n.

Case 1. $n \equiv 0 \bmod 4$. Let $m = n/4$.

TB increments: $0; m * (2,1), 3, (m-3) * (1,2), 1, 4, -3, 7$.
BT increments: $3n-1; (m-1) * (-1,-2), -1, -3, (m-3) * (-1,-2)$,
$-1, -4, 3, -7, 2$.

See Figure 4 for D_{16}^s.

Case 2. $n \equiv 2 \bmod 4$. Let $m = \lfloor n/4 \rfloor$.

TB increments: $0; m * (2,1), 3, (m-2) * (1,2), 1, 4, -3, 7$.

BT increments: $3n-1; m*(-1,-2),-1,-3,(m-3)*(-1,-2),$ $-1,-4,3,-7,2.$

Case 3. $n \equiv 1 \bmod 6$. Let $m = \lfloor n/6 \rfloor$.

TB increments: $0; (n-6)*1,6,11,-6,9,2n-15.$
BT increments: $3n-2; 2m*-2,-4,(n-2m-7)*-2,-4,2,$ $-4,1,-2.$

Case 4. $n \equiv 3 \bmod 6$. Let $m = \lfloor n/6 \rfloor$.

TB increments: $0; (n-6)*1,23,-6,-9,7,2n-10.$
BT increments: $3n-2; 2m*-2,-4,(n-2m-5)*-2,1,-4,-2.$

Case 5. $n \equiv 5 \bmod 6$. Let $m = \lfloor n/6 \rfloor$.

TB increments: $0; (n-4)*1,4,6,2n-7.$
BT increments: $3n-2; 2(m+1)*-2,-4,(n-2m-9)*$ $-2,-4,3,-5,3,-2.$ ∎

As mentioned in the introduction, graphs which contain an odd cycle cannot have an α-labeling, However, our labelings above for D_n^r, n odd, motivate us to introduce the following generalization of an α-labeling. We say a graceful labeling of a graph is a *weakly α-labeling* if there is an integer k so that for each edge xy either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$. Thus our labelings for D_n^r are weakly α-labelings when $n > 3$ while our labeling for D_n^s, n odd, are not.

The integer k is called the value that *characterizes* the labeling. (The same terminology applies the α-labelings). Notice that while the value that characterizes an α-labeling is the lesser of the two labels whose difference is 1, the value that characterizes a weakly α-labeling can be either of the two labels whose difference is 1. The next theorem shows that although graphs that contain odd cycles can admit weakly α-labelings there is a severe restriction on the cycles and the labeling.

Theorem 3. *If a graph has a weakly α-labeling characterized by k then the vertex labeled k must be on every odd cycle.*

Proof: Suppose there is a odd cycle C that does not contain the vertex labeled k and vertex x is labeled with $f(x)$. Then $V = \{x \in C | f(x) < k\}$ and $W = \{x \in C | f(x) > k\}$ are partite sets for C. But odd cycles are not bipartite. ∎

4. Harmonious Labelings of Prisms

As was the case for our graceful labelings, to specify our harmonious labelings for D_n we will take advantage of the fact that they have a zigzag pattern (See Figure 5). Thus we define the TB (top-bottom) *vertex label sequence* for D_n as the labels for the vertices $a_1, b_2, a_3, b_4, \cdots$ and the BT (bottom-top) *vertex label sequence* for D_n as the labels for the vertices $b_1, a_2, b_3, a_4, \cdots$. Then we may define a vertex labeling of D_n by specifying the labels of a_1 and b_1 together with the sequence of increments from one member of the vertex sequence to the next as we did for graceful labelings.

Theorem 4. *All prisms except the cube are harmonious.*

Proof: By the results of Graham and Sloane [10] we need only consider D_n where n is even and $n \geq 6$. We first handle special the case $n = 6$. The remaining possibilities are done in three cases. In every case we label a_1 with 0 and let $m = \lfloor n/6 \rfloor$.

Case 1. $n = 6$. Label b_1 with 8.

TB increments: $3 * 1, 3, 3$.
BT increments: $2, 2, 5, -2, 1$.

Case 2. $n \equiv 0 \bmod 6, n > 6$. Label b_1 with $n + 1$.

TB increments: $(4m-1) * 1, 2, 1, 2, (2m-4) * 1, 2$.
BT increments: $2m*2, 3, (2m-3)*2, 3, 1, 2, 1, (2m-4)*2, 1$.

Case 3. $n \equiv 2 \bmod 6$. Label b_1 with $n - 1$.

TB increments: $(n-2) * 1, 2$.
BT increments: $2m * 2, 3, (2m-1) * 2, 3, (2m-1) * 2, 1$.

Case 4. $n \equiv 4 \bmod 6$. Label b_1 with n.

TB increments: $(n-3) * 1, 2, 2$.
BT increments: $(2m+1)*2, 3, (2m-1)*2, 3, (2m-1)*2, 1, 1$.

∎

5. Harmonious Labelings of Edge Deleted Prisms

As as the case for D_n, our labelings for D_n^r and D_n^s have a zigzag pattern and we define the TB and BT vertex label sequences in the same way. We will always label a_1 with 0.

Theorem 5. *All edge deleted prisms are harmonious.*

Proof: We first do D_n^r. In every case we delete the edge joining b_1 to b_n and let $m = \lfloor (n-2/4 \rfloor$. The case $n = 4$ must be done separately.

Case 1. $n = 4$. Label b_1 with 7.

TB increments: $5, -4, 1$.
BT increments: $1, 1, -5$.

Case 2. $n \equiv 0 \bmod 4, n > 4$. Label b_1 with $3n/2 - 2$.

TB increments: $m * (1, 2), 1, 3, 1, m * (1, 2)$.
BT increments: $(m+1) * (2, 1), 2, m * (2, 1)$.

See Figure 6 for D_{12}^r.

Case 3. $n \equiv 2 \bmod 4, n > 4$. Label b_1 with $3n/2 - 2$.

TB increments: $(m+1) * (1, 2), 2, (m-1) * (1, 2)$.
BT increments: $m * (2, 1), 3, 1, 1, (m-1) * (2, 1)$.

Case 4. n is odd. Label b_1 with $3(n-1)/2+1$.

TB increments: $(n-1)/2) * (1,2)$.
BT increments: $(n-1)/2 * (2,1)$.

We now do D_n^s. In every case we delete the edge joining a_n to b_n and let $m = \lfloor n/4 \rfloor$.

Case 1. $n \equiv 0 \mod 4$. Label b_1 with $3n/2-2$.

TB increments: $(2m-2) * (1,2), 1, 4, -3$.
BT increments: $(m-1) * (2,1), 4, 1, (m-1) * (2,1), -2$.

See Figure 7 for D_{12}^s

Case 2. $n \equiv 2 \mod 4$. Label b_1 with $3n/2-2$.

TB increments: $2m * (1,2), 2$.
BT increments: $m * (2,1), 4, (m-1) * (1,2), 1, 1$.

Case 3. n is odd. Label b_1 with $3(n-1)/2+1$.

TB increments: $(n-1)/2 * (1,2)$.
BT increments: $(n-3)/2 * (2,1), 3, -2$.

∎

6. Harmonious Labelings of Vertex Deleted Prisms

Since D_n is vertex symmetric, the graph obtained by deleting a vertex is independent of the vertex selected. We delete b_n. (See Figure 8 for D_8.)

As before, we may define a vertex labeling of the vertex deleted prism D_n by specifying the labels of a_1 and b_1 and the increment sequences. In each case we label a_1 with 0.

Theorem 6. *All vertex deleted prisms are harmonious.*

Proof: We consider these cases. Let $m = \lfloor n/4 \rfloor$.

Case 1. $n \equiv 0 \mod 4$. Label b_1 with $3n/2-3$.

TB increments: $(2m-2) * (1,2), 1, 3$.
BT increments: $(m-1) * (2,1), 3, (m-1) * (1,2), 1, -2$.

See Figure 8 for D_8.

Case 2. $n \equiv 2 \mod 4$. Label b_1 with $3n/2-4$.

TB increments: $2m * (1,2)$.
BT increments: $m * (2,1), 3, (m-1) * (1,2), 1, 1$.

Case 3. n is odd. Label b_1 with $3(n-1)/2$.

TB increments: $(n-3)/2 * (1,2), 1, 1$.
BT increments: $(n-3)/2 * (2,1), 2$.

∎

We are indebted to Roberto Frucht for his generous comments. The labelings above for D_n^r, n odd, and $n \equiv 4 \mod 6$ are due to him. Our original labelings for

D_n^r, n odd, were done in three cases and our original labeling for $n \equiv 4 \mod 6$ was not an α-labeling.

The first and second authors were supported by the NSF (Grant Number DMS 8407498). the third author was supported by a University of Minnesota, Duluth Undergraduate Honors Research Stipend.

References

1. J. C. Bermond, *Graceful graphs, radio antennae and French windmills*, in "Graph Theory and Combinatorics", Pitman, London, 1979, pp. 13–37.
2. G. S. Bloom and S. W. Golomb, *Applications of numbered undirected graphs*, Proc. IEEE **65** (1977), 562–570.
3. G. S. Bloom and S. W. Golomb, *On Graceful digraphs and a problem in network addressing*, Congressus Numerantium **35** (1982), 91–103.
4. R. Bodendiek, H. Schumacher and H. Wegner, *Über graziöse graphen*, Math.-Phys. Semesterberichte **24** (1977), 103–126.
5. R. Frucht, *Graceful numberings of wheels and related graphs*, Ann. N. Y. Acad. Sci. **319** (1979), 219–229.
6. R. Frucht and J. Gallian, *Labeling prisms*, Ars Combinatoria **26** (1988), 69–82.
7. J. A. Gallian, *A survey - recent results, conjectures and open problems inlabeling graphs*, J. Graph Theory (to appear).
8. T. Grace, "Graceful, harmonious, and sequential graphs", Ph.D. thesis, Univ. Illinois at Chicago Circle, 1982.
9. T. Grace, *On sequential labeling of graphs*, J. Graph Theory 7 (1983), 195–201.
10. R. Graham and N. Sloane, *On additive bases and harmonious graphs*, SIAM J. Algebraic and Discrete Methods **4** (1980), 382–404.
11. D. Jungreis and M. Reid, *Labeling grids*, preprint.
12. D. Ropp, *Graceful labelings of prisms with pendant points*, preprint.
13. A. Rosa, *On certain valuations of the vertices of a graph*, in "Theory of Graphs: Proceedings of International Symposium, Rome, 1966", Gordon and Breach, New York, 1967, pp. 349–355.