Constructions of Resolvable Mendelsohn Designs

Zhang Xuebin

Teaching and Research Section of Mathematics
Nanjing Architectural and Civil Engineering Institute
Nanjing, 210009
People's Republic of China

Abstract. In this paper, we shall establish some construction methods for resolvable Mendelsohn designs, including constructions of the product type. As an application, we show that the necessary condition for the existence of a $(v,4,\lambda)$-RPMD, namely, $v \equiv 0$ or $1 \pmod 4$, is also sufficient for $\lambda > 1$ with the exception of pairs (v,λ) where $v = 4$ and λ odd. We also obtain a $(v,4,1)$-RPMD for $v = 57$ and 93.

1. Introduction and preliminaries.

The notion of a perfect cyclic design was introduced by N.S. Mendelsohn [13]. This concept was developed and studied in subsequent papers by various authors (see, for example, [1 - 6, 12 - 13, 15]). These designs were also called Mendelsohn designs by Hsu and Keedwell in [6]. The following are some definitions on Mendelsohn designs.

Definition 1.1: In the ordered k-tuple $(a_1, a_2, a_3, \dots, a_k)$, a_i and a_j are said to be *t-apart* if $j - i \equiv t \pmod k$. If $B = \{a_1, a_2, a_3, \dots, a_k\}$ then we say that $(a_1, a_2, a_3, \dots, a_k)$ is to be a *cyclically ordered k-subset of B*.

Definition 1.2: Let v, k and λ be positive integers. A (v,k,λ)-*Mendelsohn design* (briefly (v,k,λ)-MD) is a pair $(X, \mathbf{B})$ where X is a v-set (of points) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called blocks) such that every ordered pair of points of X are 1-apart in exactly λ of the blocks of $\mathbf{B}$. The (v,k,λ)-MD is called γ-*perfect* (briefly (v,k,λ)-r-PMD) if each ordered pair of points of X appears r-apart in exactly λ of the blocks of $\mathbf{B}$.

It is easy to show that the number of blocks in a (v,k,λ)-r-PMD is $\lambda v(v-1)/k$, and, hence, an obvious necessary condition for its existence is $\lambda v(v-1) \equiv 0 \pmod k$. We next define the notion of resolvability of a (v,k,λ)-r-PMD where $v \equiv 0$ or $1 \pmod k$.

Definition 1.3: If the blocks of a (v,k,λ)-r-PMD for which $v \equiv 1 \pmod k$ can be partitioned into λv sets each containing $(v-1)/k$ blocks which are pairwise disjoint (as sets), we say that the (v,k,λ)-r-PMD is *resolvable* (briefly (v,k,λ)-r-RPMD) and each set of $(v - 1,)/k$ pairwise disjoint blocks will be called *a parallel class*.

A resolvable PMD and parallel classes by Definition 1.3 are usually called an almost resolvable PMD and almost parallel classes. For convenience, we use Definition 1.3 in this paper.

Definition 1.4: If the blocks of a (v, k, λ)-r-PMD for which $v \equiv 0 \pmod{k}$ can be partitioned into $\lambda(v-1)$ sets each containing v/k blocks which are pairwise disjoint (as sets), we shall also say that the (v, k, λ)-r-PMD is *resolvable* (briefly (v, k, λ)-r-RPMD) and each set of v/k pairwise disjoint blocks will be called *a parallel class*.

Remark 1.5: Let $(X, \mathbf{A})$ be a (v, k, λ)-r-RPMD where $X = \{x_1, x_2, \dots, x_v\}$. By Definition 1.3 and 1.4, it is easy to see that for $v \equiv 1 \pmod{k}$ $\mathbf{A}$ can be partitioned into λv parallel classes $\mathbf{A}_{ij}$ such that $\mathbf{A}_{ij}$ is a partition of $X \setminus \{x_i\}$ where $1 \leq i \leq v$ and $1 \leq j \leq \lambda$, and for $v \equiv 0 \pmod{k}$ $\mathbf{A}$ can be partitioned into $\lambda(v-1)$ parallel classes $\mathbf{A}_j$ such that $\mathbf{A}_j$ is a partition of X where $1 \leq j \leq \lambda(v-1)$.

The following are the known results on (v, k, λ)-RPMDs and (v, k, λ)-PMDs, of which a survey can be found in [2, 4, 5].

Theorem 1.6. *A $(v, 3, 1)$-RPMD exists if and only if $v \equiv 0$ or $1 \pmod{3}$, $v \neq 6$.*

Theorem 1.7. *A $(v, 4, 1)$-RPMD exists for $v \equiv 1 \pmod{4}$, with the possible exception of $v = 57, 93$ (see Theorem 2.4 in [5]).*

Theorem 1.8. *The necessary condition for the existence of $(v, 4, \lambda)$-PMD, namely, $\lambda v(v-1) \equiv 0 \pmod{4}$, is also sufficient, except for $v = 4$ and λ odd, $v = 8$ and $\lambda = 1$, and possibly excepting $v = 12$ and $\lambda = 1$.*

Theorem 1.9. *Let p be an odd prime and $r \geq 1$, then there exists a $(p^r, p, 1)$-PMD.*

Theorem 1.10. *Let $v = p^r$ be any prime power and $k > 2$ be such that $k \mid (v-1)$, then there exists a $(v, k, 1)$-RPMD.*

Theorem 1.11. *A $(v, 4, 1)$-RPMD exists for all sufficiently large v with $k \geq 3$ and $v \equiv 1 \pmod{k}$.*

Theorem 1.12. *Let v, k and λ be positive integers. Suppose there exists a PBD $B(\{k_1, k_2, \dots, k_r\}, 1; v)$ and for each k_i there exists a (k_i, k, λ)-PMD. Then there exists a (v, k, λ)-PMD.*

Compared with (v, k, λ)-PMD the existence question for (v, k, λ)-RPMD seems much more open (see [12]). In fact, the existence of a (v, k, λ)-RPMD for which $v \equiv 0 \pmod{k}$ and k even has not been studied, for example, $k = 4$ and $v \equiv 0 \pmod{4}$.

The purpose of this paper is to establish some construction methods for (v, k, λ)-r-RPMDs. As an application, we show that the necessary condition for the existence of a (v, k, λ)-RPMD where $\lambda > 1$, namely, $v \equiv 0$ or $1 \pmod{4}$, is also sufficient with the exception of pairs (v, λ) where $v = 4$ and λ odd. We also obtain a $(v, 4, 1)$-RPMD for $v = 57$ and 93.

We mention the definition and the result on Whist tournaments for later use (see [10, 14]).

Definition 1.13: Let X be a set of v objects, called players.

(i) A Whist-table, donoted $[x_1, x_2; x_3, x_4]$ is a set of four players with the pairs $[x_1, x_2]$ and $[x_3, x_4]$ known as partners and the remaining four pairs known as opponents.

(ii) A Whist-round is a set of Whist-tables such that each player occurs at exactly one Whist-table.

(iii) A Whist-tournament denoted WH$[v]$, is a set of Whist-rounds such that any two players are partners at exactly one Whist-table and opponents at exactly two Whist-tables.

Apparently condition (ii) implies that $v = 4n$ and each Whist-round must contain n Whist-tables.

Theorem 1.14. *There exist $WH[v]$ for all $v \equiv 0 \pmod 4$ except possibly $v = 264$.*

We assume that the reader is familiar with the concept of a group divisible design (GDD), a transversal design (TD) and a resolvable transversal design (RTD) (see [8, 9]).

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. The following results are well known (see [7, 8, 9]).

Lemma 1.15. *The existence of a $TD[k, 1; n]$ is equivalent to $N(n) \geq k-2$.*

Lemma 1.16. *The existence of a $TD[k+1, 1; n]$ implies the existence of a resolvable $TD[k, 1; n]$ (briefly $RTD[k, 1; n]$)*

Lemma 1.17. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the factorization of n into powers of distinct primes p_i, then $N(n) \geq \min_{1 \leq i \leq k} \{p_i^{\alpha}\} - 1$.*

Lemma 1.18. *If $n \neq 2, 6$ then $N(n) \geq 2$; if $n \neq 2, 3, 6, 10$ then $N(n) \geq 3$.*

We need to define the following terminology which will be used later.

Definition 1.19: A subset of blocks in a (v, k, λ)-r-RPMD is called a *partial parallel class* if the subset consists of pairwise disjoint blocks.

Definition 1.20: Let $(X, \mathbf{A})$ be a (v, k, λ)-r-RPMD having parallel classes $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_r$. Let $(Y, \mathbf{B})$ be a (u, k, λ)-r-RPMD having parallel classes $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_t$. If $X \supset Y$ and $\mathbf{A}_i \supset \mathbf{B}_i$, $1 \leq i \leq t$, we say that the first design contains the second *as a subdesign*.

Definition 1.21: Let X be a v-set (of points), $Y (\subset X)$ be a n-set (of points), $\mathbf{B}$ be a collection of cyclically ordered subsets of X (called blocks) with block size k, and $v, n \equiv 0$ or $1 \pmod k$. *A (v, k, λ)-r-incomplete RPMD with a hole of size n* (briefly, $(v, k, \lambda; n)$-r-IRPMD) is a triple $(X, Y, \mathbf{B})$ where $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2$ such that (1) every ordered pair (x, y) of points of X with $\{x, y\} \not\subset Y$ appears r-apart in λ blocks of $\mathbf{B}$; (2) every ordered pair (x, y) of points of X with $\{x, y\} \subset Y$ appears in no block of $\mathbf{B}$; (3) $\mathbf{B}_1$ can be partitioned into some parallel classes of X and $\mathbf{B}_2$ can be partitioned into h parallel classes of $X \setminus Y$ where $h = \lambda(n-1)$ when $n \equiv 0 \pmod{k}$ and $h = \lambda n$ when $n \equiv 1 \pmod{k}$.

It is easy to see that if there is a (v, k, λ)-r-RPMD containing a (u, k, λ)-r-RPMD as a subdesign then there is a $(v, k, \lambda; u)$-r-IRPMD.

Definition 1.22: Let Y be a v-set, $B = \{a_1, a_2, \ldots, a_k\}$, a k-set and $A = (a_1, a_2, \ldots, a_k)$, a cyclically of ordered set of B. Let $(B \times Y, \mathbf{G}, \mathbf{C})$ be an RTD $[k, \lambda; v]$ and $C(A) = ([a_1, y_1], [a_2, y_2], \ldots, [a_k, y_k])$ for every $C = \{[a_1, y_1], [a_2, y_2], \ldots, [a_k, y_k]\} \in \mathbf{C}$. We say that the $(A \times Y, \mathbf{G}, \mathbf{C}(A))$ is a cyclically ordered RTD $[k, \lambda; v]$ where $\mathbf{C}(A) = \{C(A) \mid C \in \mathbf{C}\}$.

Definition 1.23: Let v, k and n_i $(1 \le i \le h)$ be positive integers. Let $M = \{n_i \mid 1 \le i \le h\}$ and $X = \cup_{1 \le i \le h} X_i$, where X_i $(1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$ and $v = \sum_{1 \le i \le h} n_i$. An r-perfect GDD $[k, \lambda, M; v]$ (briefly, r-PGMD $[k, \lambda, M; v]$) is a triple $(X, \mathbf{G}, \mathbf{B})$ where $\mathbf{G} = \{X_i | 1 \le i \le h\}$ and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X such that (1) every ordered pair (x, y) with two vertices x and y from different sets X_i and X_j appears r-apart in exactly λ of the blocks of $\mathbf{B}$; (2) every ordered pair (x, y) with $\{x, y\} \subset X_i$ $(1 \le i \le h)$ appears in no blocks of $\mathbf{B}$. If $n_1 = n_2 = \ldots = n_h = n$, we denote the r-PGMD $[k, \lambda, M; v]$ by r-PGMD $[k, \lambda, n; v]$.

Definition 1.24: Let $(X, \mathbf{G}, \mathbf{B})$ be an r-PGMD $[k, \lambda, M; v]$. If $\mathbf{B}$ can be partitioned into λv partial parallel classes: $\mathbf{P}_{ije}$, $1 \le i \le h$, $1 \le j \le n_i$, $1 \le e \le \lambda$ such that $\mathbf{P}_{ije}$ is a partition of $X \setminus X_i$, then the r-PGMD $[k, \lambda, M; v]$ is said to be an *r-$(k, \lambda, M; v)$-frame*. If $\mathbf{B}$ is the union of $\lambda(v-h)$ partial parallel classes: $\mathbf{P}_{ije}$, $1 \le i \le h$, $1 \le j \le n_i - 1$, $1 \le e \le \lambda$ and $\lambda(h-1)$ parallel classes, where $\mathbf{P}_{ije}$ is a partition of $X \setminus X_i$, then the r-PGMD $[k, \lambda, M; v]$ is said to be an *r-$(k, \lambda, M; v)$-semiframe*.

Definition 1.25: A perfect MD (briefly, PMD) is an MD that is r-perfect for $1 \le r \le k$. Similarly, we can define an RPMD, an IRPMD, a PGMD, a $(k, \lambda, M; v)$-frame and a $(k, \lambda, M; v)$-semiframe.

2. Construction methods.

In this section we shall etablish some construction methods for (v, k, λ)-r-RPMDs. We always adopt the notation of Remark 1.5, that is, the blocks of a (v, k, λ)-r-RPMD of $(X, \mathbf{A})$ where $X = \{x_1, x_2, \ldots, x_v\}$, can be partitioned into $\lambda(v-1)$ parallel classes: $\mathbf{A}_{ij}$, $1 \le i \le \lambda(v-1)$ for $v \equiv 0 \pmod{k}$ and λv parallel classes: $\mathbf{A}_{ij}$, $1 \le i \le v$, $1 \le j \le \lambda$ and $\cup_{A \in \mathbf{A}_{ij}} A = X \setminus \{x_i\}$ for $v \equiv 1 \pmod{k}$.

We first establish constructions of the product type.

Theorem 2.1. *Suppose there exist a (u,k,λ)-r-RPMD and a (v,k,μ)-r-RPMD, where $u,v\equiv 0$ or $1 \pmod k$. Then there exists a $(uv,k,\lambda\mu)$-r-RPMD.*

Proof: Let $X=\{x_1,x_2,\ldots,x_u\}$, $Y=\{y_1,y_2,\ldots,y_v\}$, $Z=X\times Y$, $(X,\mathbf{A})$ be a (u,k,λ)-r-RPMD and $(Y,\mathbf{B})$ be a (v,k,μ)-r-RPMD. Let $A=(x_1,x_2,\ldots,x_k)$, $B=(y_1,y_2,\ldots,y_k)$ and $B^i=(y_i,y_{i+1},\ldots,y_k,y_1,\ldots,y_{i-1})$ where $i=1,2,\ldots,k$. Denote $\langle A,B^i\rangle=([x_1,y_i],[x_2,y_{i+1}],\ldots,[x_{k-i+1},y_k],[x_{k-i+2},y_1],\ldots,[x_k,y_{i-1}])$ where $i=1,2,\ldots,k$. $\langle A,B\rangle=\{\langle A,B^i\rangle \mid 1\le i\le k\}$ $\{x_i\}\times B=([x_i,y_i],[x_i,y_j],\ldots,[x_i,y_j])$ $A\times\{y_j\}=([x_1,y_j],[x_2,y_j],\ldots,[x_i,y_1],[x_i,y_2],\ldots,[x_i,y_k])$ $A\times Y=\{A\times\{y_j\}\mid 1\le j\le v\}$, $\langle \mathbf{A},B\rangle=\bigcup_{A\in\mathbf{A}}\langle A,B\rangle$, $\langle\mathbf{A},\mathbf{B}\rangle=\bigcup_{B\in\mathbf{B}}\langle\mathbf{A},B\rangle$ and $\langle\mathbf{A}\times Y=\bigcup_{A\in\mathbf{A}}A\times Y$. Taking each of the blocks of $\mathbf{A}\times Y$ μ times, we have a collection of blocks denoted by $\mu\mathbf{A}\times Y$. Similarly, we have $\lambda X\times\mathbf{B}$. We are to prove that $(Z,\mathbf{D})$ is a $(uv,k,\lambda\mu)$-r-RPMD where $\mathbf{D}=\langle\mathbf{A},\mathbf{B}\rangle\cup\mu\mathbf{A}\times Y\cup\lambda X\times\mathbf{B}$.

We first prove that $(Z,\mathbf{D})$ is a $(uv,k,\lambda\mu)$-r-PMD. Let (z_1,z_2) be an of ordered pair of points of Z and $z_1=[x_i,y_j]$, $z_2=[x_\ell,y_n]$. We consider the following cases.

(a) If $x_i\ne x_\ell$ and $y_j\ne y_n$, then (z_1,z_2) appears r-apart in $\lambda\mu$ blocks of $\langle\mathbf{A},\mathbf{B}\rangle$.
(b) If $x_i\ne x_\ell$ and $y_j=y_n$, then (z_1,z_2) appears r-apart in $\lambda\mu$ blocks of $\mu\mathbf{A}\times Y$.
(c) If $x_i=x_\ell$ and $y_j\ne y_n$, then (z_1,z_2) appears r-apart in $\lambda\mu$ blocks of $\lambda X\times\mathbf{B}$.

In what follows we shall prove that $(Z,\mathbf{D})$ is resolvable.

Case 1: $u\equiv v\equiv 0\pmod k$. Since $\langle\mathbf{A},\mathbf{B}\rangle=\bigcup_{\substack{1\le i\le(u-1)\lambda\\1\le j\le(v-1)\mu}}\langle\mathbf{A}_i,\mathbf{B}_j\rangle$ $\mu\mathbf{A}\times Y=\bigcup_{1\le i\le(u-1)\lambda}\mu\mathbf{A}_i\times Y$, $\lambda X\times\mathbf{B}=\bigcup_{1\le j\le(v-1)\mu}\lambda X\times\mathbf{B}_j$ and each of $\langle\mathbf{A}_i,\mathbf{B}_j\rangle$, $\mathbf{A}_i\times Y$ and $X\times\mathbf{B}_j$ is a parallel class, so $\mathbf{D}$ can be partitioned into $(uv-1)\lambda\mu$ parallel classes.

Case 2: $u\equiv v\equiv 1\pmod k$. Since

$$\langle\mathbf{A},\mathbf{B}\rangle=\bigcup_{\substack{1\le i\le u,1\le j\le\lambda\\1\le\ell\le v,1\le n\le\mu}}\langle\mathbf{A}_{ij},\mathbf{B}_{\ell n}\rangle,\quad \mathbf{A}\times Y=\bigcup_{\substack{1\le i\le u,1\le j\le\lambda\\1\le\ell\le v}}\mathbf{A}_{ij}\times\{y_\ell\},$$

$$X\times\mathbf{B}=\bigcup_{\substack{1\le j\le u,1\le\ell\le v\\1\le n\le\mu}}\{x_i\}\times\mathbf{B}_{\ell n},\quad \mathbf{D}=\bigcup_{\substack{1\le i\le u,1\le j\le\lambda\\1\le\ell\le v,1\le n\le\mu}}\mathbf{D}_{ij\ell n}$$

where $\mathbf{D}_{ij\ell n}=\langle\mathbf{A}_{ij},\mathbf{B}_{\ell n}\rangle\cup\mathbf{A}_{ij}\times\{y_\ell\}\cup\{x_i\}\times\mathbf{B}_{\ell n}$ and $\bigcup_{D\in\mathbf{D}_{ij\ell n}}D=Z\backslash\{[x_i,y_\ell]\}$ (that is, $\mathbf{D}_{ij\ell n}$ is a parallel class), we have that $(Z,\mathbf{D})$ is resolvable.

Case 3: $u\equiv 0\pmod k$, $v\equiv 1\pmod k$. It is easy to see that $\langle\mathbf{A},\mathbf{B}\rangle\cup\mu\mathbf{A}\times Y=\bigcup_{\substack{1\le i\le(u-1)\lambda\\1\le i\le v,1\le n\le\mu}}(\langle\mathbf{A}_i,\mathbf{B}_{\ell n}>\cup\mathbf{A}_i\times\{y_\ell\})$, can be partitioned into $\lambda\mu(u-1)v$

parallel classes. We first show that for $1 \le i \le (u-1)\lambda$ and $1 \le n \le \mu$, $\mathbf{A}_i \times \{y_1\} \cup \langle \mathbf{A}_i, \mathbf{B}_{1n} \rangle \cup (\cup_{1 \le s \le v} X \times \mathbf{B}_{sn})$ can be partitioned into v parallel classes.

Let $F(B^m) = \cup_{D \in \langle \mathbf{A}_i, B^m \rangle} D$ for $B \in \mathbf{B}_{1n}$, $F(\mathbf{B}_{1n}) = \cup_{1 \le m \le k, B \in \mathbf{B}_{1n}} F(B^m)$ $\mathbf{M}(B^m) = \cup_{[x_j, y_s] \in F(B^m)} \{x_j\} \times \mathbf{B}_{sn}$ and $\mathbf{M}(\mathbf{B}_{1n}) = \cup_{1 \le m \le k, B \in \mathbf{B}_{1n}} \mathbf{M}(B^m)$. It is easy to see that $|F(B^m)| = u$, $|F(B^m) \cap \{x_j\} \times Y| = 1$, for $1 \le j \le u$ and $\cup_{D \in \{x_j\} \times \mathbf{B}_{sn}} D = \{x_j\} \times Y \setminus \{[x_j, y_s]\}$. Hence, $\langle \mathbf{A}_i, \mathbf{B}^m \rangle \cup \mathbf{M}(B^m)$ is a parallel class. Since $F(\mathbf{B}_{1n}) = \cup_{D \in \langle \mathbf{A}_i, \mathbf{B}_{1n} \rangle} D = X \times Y \setminus X \times \{y_1\}$, we have $\mathbf{M}(\mathbf{B}_{1n}) = \cup_{1 \le m \le k, B \in \mathbf{B}_{1n}} \mathbf{M}(B^m) = \cup_{[x_j, y_s] \in F(\mathbf{B}_{1n})} \{x_j\} \times \mathbf{B}_{sn} = \cup_{[x_j, y_s] \in X \times Y \setminus X \times \{y_1\}} \{x_j\} \times \mathbf{B}_{sn} = \cup_{2 \le s \le v} X \times \mathbf{B}_{sn}$. Therefore, $\mathbf{A}_i \times \{y_1\} \cup \langle \mathbf{A}_i, \mathbf{B}_{1n} \rangle \cup (\cup_{1 \le s \le v} X \times \mathbf{B}_{sn}) = [\cup_{1 \le m \le k, B \in \mathbf{B}_{1n}} \langle \mathbf{A}_i, \mathbf{B}^m \rangle \cup \mathbf{M}(B^m)] \cup (\mathbf{A}_i \times \{y_1\} \cup X \times \mathbf{B}_{1n})$ can be partitioned into v parallel classes.

Taking $\lambda\mu$ parallel classes in $\langle \mathbf{A}, \mathbf{B} \rangle \cup \mu\mathbf{A} \times Y$: $\mathbf{A}_i \times \{y_1\} \cup \langle \mathbf{A}_i, \mathbf{B}_{1n} \rangle$, $1 \le i \le \lambda$, $1 \le n \le \mu$ together with $\lambda X \times \mathbf{B}$, we can obtain $\lambda\mu v$ parallel classes in the same way as above. Hence, $\mathbf{D}$ can be partitioned into some parallel classes. The proof is now complete. ∎

Remark 2.2: It is easy to see that the $(uv, k, \mu\lambda)$-r-RPMD constructed in the proof of Theorem 2.1 contains a (u, k, λ)-r-RPMD and a (v, k, μ)-r-RPMD, respectively, as a subdesign.

Theorem 2.3. *Suppose there exist a (u, k, λ)-r-RPMD and a (v, k, λ)-r-RPMD, where $N(v) \ge k-1$ and $u, v \equiv 0$ or $1 \pmod k$. Then there exists a (uv, k, λ)-r-RPMD.*

Proof: Let $X = \{x_1, x_2, \ldots, x_u\}$, $Y = \{y_1, y_2, \ldots, y_v\}$, $Z = X \times Y$, $(X, \mathbf{A})$ be a (u, k, λ)-r-RPMD and $(Y, \mathbf{B})$ be a (v, k, λ)-r-RPMD. For every $A \in \mathbf{A}$, since $N(v) \ge k-1$ we can let $(A \times Y, \mathbf{G}, \mathbf{C}(A))$ be a cyclically ordered RTD $[k, 1; v]$ having parallel classes $\mathbf{C}(A)_1, \mathbf{C}(A)_2, \ldots, \mathbf{C}(A)_v$. Let $\mathbf{C}(\mathbf{A}) = \cup_{A \in \mathbf{A}} \mathbf{C}(A)$ and $\mathbf{D} = X \times \mathbf{B} \cup \mathbf{C}(\mathbf{A})$. We are to prove that $(Z, \mathbf{D})$ is a (uv, k, λ)-r-RPMD.

We first show that $(Z, \mathbf{D})$ is a (uv, k, λ)-r-PMD.

Let (z_1, z_2) be an ordered pair of points of Z. If $z_1, z_2 \in \{x_i\} \times Y$, then (z_1, z_2) appears r-apart in λ blocks of $\{x_i\} \times \mathbf{B}$; if $z_1 \in \{x_i\} \times Y$ and $z_2 \in \{x_j\} \times Y$ where $i \ne j$, then (x_i, x_j) appears r-apart in λ blocks of $\mathbf{A}$ and (z_1, z_2) appears r-apart in λ blocks of $\mathbf{C}(\mathbf{A})$.

In what follows, it is shown that $(Z, \mathbf{D})$ is resolvable.

Case 1: $u, v \equiv 0 \pmod k$. Since $(\mathbf{C}(\mathbf{A}_i))_\ell$ and $X \times \mathbf{B}_j$ are two parallel classes, $X \times \mathbf{B} = \cup_{1 \le j \le \lambda(v-1)} X \times \mathbf{B}_j$ and $\mathbf{C}(\mathbf{A}) = \cup_{1 \le i \le \lambda(u-1), 1 \le \ell \le v} \mathbf{C}(\mathbf{A}_i)_\ell$, so $(Z, \mathbf{D})$ is resolvable.

Case 2: $u, v \equiv 1 \pmod k$. $(Z, \mathbf{D})$ is resolvable since $(\mathbf{C}(\mathbf{A}_{is}))_j \cup \{x_i\} \times \mathbf{B}_{js})$ is a parallel class and $\mathbf{C}(\mathbf{A}) \cup X \times \mathbf{B} = \cup_{1 \le i \le u, 1 \le s \le \lambda, 1 \le j \le v} (\mathbf{C}(\mathbf{A}_{is}))_j \cup \{x_i\} \times \mathbf{B}_{js})$.

Case 3: $u \equiv 0 \pmod k$, $v \equiv 1 \pmod k$. Without loss of generality we assume $\mathbf{C}(\mathbf{A}_n)_1 = \mathbf{A}_n \times Y$ for $1 \le n \le \lambda$. Since both $(X \times \mathbf{B}_{\ell n} \cup \mathbf{A}_n \times \{y_\ell\})$

and $C(A_i)_j$ are parallel classes and $X \times B \cup (\cup_{1\leq n\leq\lambda} C(A_n)_1) = \cup_{1\leq \ell\leq v, 1\leq n\leq\lambda} (X \times B_{\ell n} \cup A_n \times \{y_\ell\})$, it is easy to see that D can be partitioned into $\lambda(uv-1)$ parallel classes.

Case 4: $u \equiv 1 \pmod k$, $v \equiv 0 \pmod k$. Without loss of generality we assume $C(A_{ij})_1 = A_{ij} \times Y$ for $2 \leq i \leq u$, $1 \leq j \leq \lambda$, and $C(A_{1j})_1 = \{\langle A_{1j}, N^t\rangle \mid 1 \leq t \leq v\}$ where $N^t = (y_t, y_{t+1}, \dots, y_{t+k-1})$ and $t+i$ is taken modulo v.

Let $E_m(A) = \{\langle A, N^t\rangle \mid t = m, k+m, 2k+m, \dots, (v/k-1)k+m\}$ for $1 \leq m \leq k$ and $A \in A_{1j}$, $F_m(A) = \cup_{D\in E_m(A)} D$ and $M_m(A) = \cup_{[X_i,y_s]\in F_m(A)} A_{ij} \times \{y_s\}$ then $E(A_{1j}) = \cup_{1\leq m\leq k, A\in A_{1j}} E_m(A) = C(A_{1j})_1$ $F(A_{1j}) = \cup_{D\in E(A_{1j})} D = \cup_{D\in C(A_{1j})_1} D = X \times Y \setminus \{x_1\} \times Y$. $M(A_{1j}) = \cup_{1\leq m\leq k, A\in A_{1j}} M_m(A) = \cup_{[x_i,y_s]\in F(A_{1j})} A_{1j} \times y_s = \cup_{[x_i,y_s]\in X\times Y\setminus\{x_1\}\times Y} A_{ij} \times y_s = \cup_{2\leq i\leq u} A_{ij} \times Y$. It is easy to see that $E_m(A)$ is a partial parallel class and $|F_m(A) \cap X \times \{y_\ell\}| = 1$ for $1 \leq \ell \leq v$. Hence, $E_m(A) \cup M_m(A)$ is a parallel class. Since $\cup_{1\leq i\leq u} C(A_{ij})_1 = (\cup_{2\leq i\leq u} A_{ij} \times Y) \cup C(A_{1j})_1 = M(A_{1j}) \cup E(A_{1j}) = \cup_{1\leq m\leq k, A\in A_{1j}} (E_m(A) \cup M_m(A))$, so $\cup_{1\leq i\leq u, 1\leq j\leq\lambda} C(A_{ij})_1$ can be partitioned into $\lambda(u-1)$ parallel classes.

Since $C(A_{ij})_\ell \cup \{x_i\} \times B_n$ is a parallel class, and $D \setminus (\cup_{1\leq i\leq u, 1\leq j\leq\lambda} C(A_{ij})_1) = \cup_{1\leq s\leq v-1, 1\leq j\leq\lambda, 1\leq i\leq u} (C(A_{ij})_{s+1} \cup \{x_i\} \times B_{(j-1)(v-1)+s})$, we obtain that D can be partitioned into $\lambda(uv-1)$ parallel classes. We have completed the proof. ∎

Remark 2.4: It is easy to see that the (uv, k, λ)-r-RPMD constructed in the proof of Theorem 2.3 contains a (u, k, λ)-r-RPMD and a (v, k, λ)-r-RPMD, respectively as a subdesign.

Corollary 2.5. *Let p be an odd prime and $r \geq 1$, then there exists a $(p^r, p, 1)$-RPMD.*

Proof: Since the existence of a $(p, p, 1)$-RPMD is equivalent to the existence of a $(p, p, 1)$-PMD, the conclusion follows from Theorem 1.9 and 2.1. ∎

We now establish some constructions by using IRPMDs and frames.

By Definition 1.21, we have

Theorem 2.6. *Suppose there is a $(v, k, \lambda; n)$-r-IRPMD and there is an (n, k, λ)-r-RPMD. Then there exists a (v, k, λ)-r-RPMD.*

In what follows, we alway adopt the notation of Definition 1.23 and 1.24.

Theorem 2.7. *Suppose that (X, G, B) is an r-$(k, \lambda, M; v)$-frame where $M = \{n_1, n_2, \dots, n_h\}$, and for each n_i, there exists an (n_i, k, λ)-r-RPMD, $n_i \equiv 1 \pmod k$. Then there exists a (v, k, λ)-r-RPMD.*

Proof: Let $(X_i, H(i))$ be an (n_i, k, λ)-r-RPMD having λn_i parallel classes on X_i; for $1 \leq i \leq h$. Since the union of a parallel class on X_i and a partition of $X \setminus X_i$ is a parallel class on X, by Definition 1.24 it is easy to see that $(X, B \cup H)$ is a (v, k, λ)-r-RPMD where $H = \cup_{1\leq i\leq h} H(i)$. ∎

Theorem 2.8. *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is an* r-$(k, \lambda, M; v)$*-semiframe where* $M = \{n_1, n_2, \ldots, n_h\}$*, and for each* n_i *there exists an* (n_i, k, λ)*-*r*-RPMD,* $n_i \equiv 0 \pmod{k}$*. Then there exists a* (v, k, λ)*-*r*-RPMD.*

Proof: The proof is similar to that of Theorem 2.7 ∎

Theorem 2.9. *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is an* r-$(k, \lambda, M; v)$*-frame where* $M = \{n_1, n_2, \ldots, n_h\}$*, and for each* n_i *there exists an* $(n_i + 1, k, \lambda)$*-*r*-RPMD,* $n_i + 1 \equiv 1 \pmod{k}$*. Then there exists a* $(v + 1, k, \lambda)$*-*r*-RPMD.*

Proof: Let $(X \cup \{\theta\}, \mathbf{H}(i))$ be an $(n_i + 1, k, \lambda)$-r-RPMD having $\lambda(n_i, +1)$ parallel classes $\mathbf{H}(i)_{j\ell}$, $H(i)_{\theta\ell}$, $1 \leq j \leq n_i$, $1 \leq \ell \leq \lambda$ where $\cup_{H \in \mathbf{H}(i)\theta\ell} H = X_i$ for $1 \leq i \leq h$. Since both $\mathbf{H}(i)_{j\ell} \cup \mathbf{P}_{ij\ell}$ and $\cup_{1 \leq i \leq h} \mathbf{H}(i)_{\theta\ell}$ are parallel classee on $X \cup \{\theta\}$, so $(X \cup \{\theta\}, \mathbf{B} \cup \mathbf{H})$ is a $(v + 1, k, \lambda)$-r-RPMD where $\mathbf{H} = \cup_{1 \leq i \leq h} \mathbf{H}(i)$. ∎

Theorem 2.10. *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is an* r-$(k, \lambda, M; v)$*-frame where* $M = \{n_1, n_2, \ldots, n_h\}$*, and for each* n_i *there exists an* $(n_i + 1, k, \lambda)$*-*r*-RPMD,* $n_i + 1 \equiv 0 \pmod{k}$*. Then there exists a* $(v + 1, k, \lambda)$*-*r*-RPMD.*

Proof: Let $(X_i \cup \{\theta\}, \mathbf{H}(i))$ be an $(n_i + 1, k, \lambda)$-r-RPMD having λn_i parallel classes $\mathbf{H}(i)_{(j-1)\lambda+\ell}$, $1 \leq j \leq n_i$, $1 \leq \ell \leq \lambda$. Since $\mathbf{H}(i)_{(j-1)\lambda+\ell} \cup \mathbf{P}_{ij\ell}$ is a parallel class on $X \cup \{\theta\}$, $(X \cup \{\theta\}, \mathbf{B} \cup \mathbf{H})$ is a $(v + 1, k, \lambda)$-r-RPMD where $\mathbf{H} = \cup_{1 \leq i \leq h} \mathbf{H}(i)$. ∎

Theorem 2.11. *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is an* r-$(k, \lambda, M; v)$*-frame where* $M = \{n_i \mid 1 \leq i \leq h\}$ *and* $n_i \equiv 0 \pmod{k}$ *for* $1 \leq i \leq h$*, satisfying (1) there exists an* $(n_i + w, k, \lambda; w)$*-*r*-IRPMD for* $1 \leq i \leq h - 1$*; (2) there exists an* $(n_h + w, k, \lambda)$*-*r*-RPMD, where* $w \equiv 0 \pmod{k}$*. Then there exists a* $(v + w, k, \lambda)$*-*r*-RPMD.*

Proof: Let $(X_i \cup W, W, \mathbf{H}(i))$ be an $(n_i + w, k, \lambda; w)$-r-IRPMD having λn_i parallel classes on $X_i \cup W$ and $\lambda(w - 1)$ parallel classes on X_i for $1 \leq i \leq h - 1$ and $(X_h \cup W, \mathbf{H}(h))$ be an $(n_h + w, k, \lambda)$-r-RPMD having $\lambda(n_h + w - 1)$ parallel classes on $X_h \cup W$. We are to prove that $(X \cup W, \mathbf{B} \cup \mathbf{H})$ is a $(v + w, k, \lambda)$-r-RPMD where $\mathbf{H} = \cup_{1 \leq i \leq h} \mathbf{H}(i)$. Since the union of parallel class on $X_i \cup W$ and a partition of $X \setminus X_i$ is a parallel class on $X \cup W$, so we can get $\sum_{1 \leq i \leq h} \lambda n_i$ parallel class on $X \cup W$ by Definition 1.24 and the remaining blocks can be partitioned into $\lambda(w - 1)$ parallel classes on $X \cup W$. This completes the proof. ∎

Corollary 2.12. *Suppose* $(X, \mathbf{G}, \mathbf{B})$ *is an* r-$(k, \lambda, M; v)$*-frame and for each* $n_i \in M$ *there exists an* $(n_i + w, k, \lambda)$*-*r*-RPMD having a* (w, k, λ)*-*r*-RPMD as a subdesign where* $n_i + w \equiv w \equiv 0 \pmod{k}$*. Then there exists a* $(v + w, k, \lambda)$*-*r*-RPMD.*

Lemma 2.13. *Suppose that there exist an r-$(k, \lambda, M; u)$-frame and an RTD$[k, \mu; v]$, then there exists an r-$(k, \lambda\mu, N; uv)$-frame where $N = \{ug \mid g \in M\}$.*

Proof: Let Y be a v-set, $(X, \mathbf{G}, \mathbf{B})$ be an r-$(k, \lambda, M; u)$-frame and $(B \times Y, \mathbf{G}_B, \mathbf{C}(B))$ be a cyclically ordered RTD$[k, \mu; v]$ for every $B \in \mathbf{B}$. It is not difficult to see that $\mathbf{C}(\mathbf{P}_{ije})$ is the union of μv partitions of $X \times Y \setminus X_i \times Y$, and then we obtain that $(X \times Y, \mathbf{G} \times Y, \mathbf{C}(\mathbf{B}))$ is an r-$(k, \mu\lambda, N; uv)$-frame where $\mathbf{G} \times Y = \{X_i \times Y \mid 1 \le i \le h\}$. ∎

Theorem 2.14. *Suppose that there exist a $(u, k, \lambda,)$-RPMD and a $(v+1, k, \lambda)$-r-RPMD, where $u \equiv 1 \pmod{k}$ and $v + 1 \equiv 0$ or $1 \pmod{k}$. If there is an RTD$[k, \mu; v]$, then there exists a $(uv + 1, k, \lambda\mu)$-r-RPMD.*

Proof: Since a (u, k, λ)-r-RPMD is also an r-$(k, \lambda, 1; u)$-frame and from Lemma 2.13 we have a $(k, \lambda\mu, v; uv)$-frame. Therefore, by Theorem 2.9 and 2.10, we obtain a $(uv + 1, k, \lambda\mu)$-r-RPMD. ∎

Remark 2.15: It is easy to see that the $(uv+1, k, \lambda\mu)$-r-RPMD constructed in the proof of Theorem 2.14 contains a (u, k, λ)-r-RPMD and a $(v+1, k, \lambda)$-r-RPMD, respectively, as a subdesign.

Theorem 2.16. *Suppose*

(1) *There exists a (u, k, λ)-r-RPMD where $u \equiv 1 \pmod{k}$;*
(2) *there exists a $(v+w, k, \lambda; w)$-r-IRMD and a $(v+w, k, \lambda)$-r-RPMD where $v + w \equiv w \equiv 0 \pmod{k}$;*
(3) *there is an RTD $[k, \mu; v]$.*

Then there exists a $(uv + w, k, \lambda\mu)$-r-RPMD.

Proof: The proof that there exists an r-$(k, \lambda\mu, v; uv)$-frame is similar to that of Theorem 2.14. From Theorem 2.11 the conclusion follows. ∎

Theorem 2.17. *Suppose that (1) $(X, \mathbf{G}, \mathbf{B})$ is an r-$(k, \lambda, u; hu)$-semiframe where $u \equiv 0 \pmod{k}$; (2) there exists a $(uv, k, \lambda\mu)$-r-RPMD; (3) there is an RTD $[k, \mu; v]$. Then there exists a $(huv, k, \mu\lambda)$-r-RPMD.*

Proof: We adapt the notation of Theorem 2.3 and Definition 1.24. Let Y be a v-set, $(X_i \times Y, \mathbf{H}(i))$ be a $(uv, k, \lambda\mu)$-r-RPMD having $\lambda\mu(uv - 1)$ parallel classes: $\mathbf{H}(i)_j$, $1 \le j \le \lambda\mu\,(uv - 1)$ for $1 \le i \le h$. It is easy to see that $(\cup_{1 \le j \le \mu\lambda(u-1)v}\, \mathbf{H}(i)_j) \cup \mathbf{C}(\mathbf{P}_i)$ can be partitioned into $\mu\lambda(u - 1)v$ parallel classes where $\mathbf{P}_i = \cup_{1 \le j \le u-1, 1 \le e \le \lambda}\, \mathbf{P}_{ije}$ for $1 \le i \le h$, and $\cup_{1 \le i \le h}\, \mathbf{H}(i)_j$ is a parallel class for $\mu\lambda(u - 1)v + 1 \le j \le \mu\lambda(uv - 1)$ and $\mathbf{C}(\mathbf{B} \setminus \mathbf{P})$ can be partitioned into $\mu\lambda(h - 1)v$ parallel classes where $\mathbf{P} = \cup_{1 \le i \le h}\, \mathbf{P}_i$. Hence, it is clear that $(X \times Y, \mathbf{H} \cup \mathbf{C}(\mathbf{B}))$ is an $(huv, k, \mu\lambda)$-r-RPMD where $\mathbf{H} = \cup_{1 \le i \le h}\, \mathbf{H}(i)$. ∎

The idea of the following theorem can be found [2, 3, 10].

Theorem 2.18. *If there exists a GDD*$[K,1,M;v]$ *satisfying for each* $h \in K$ *there exists an* (h,k,λ)*-*r*-RPMD,* $h \equiv 1 \pmod{k}$. *Then there exists an* r*-*$(k,\lambda,M;v)$*-frame.*

Proof: Let $(X,\mathbf{G},\mathbf{B})$ be a GDD$[K,1,M;v]$, $\mathbf{B}(x) = \{B \mid x \in B \in \mathbf{B}\}$ for every $x \in X$, and for every $B \in \mathbf{B}$, provided $|B| = h$, $(B,\mathbf{A}(B))$ be an (h,k,λ)-r-RPMD having λh parallel classes: $\mathbf{A}(B)_{xj}, x \in B, 1 \le j \le \lambda$ where $\mathbf{A}(B)_{xj}$ is a partition of $B \setminus \{x\}$. Provided $x \in X_i$, then $\cup_{B\in\mathbf{B}(x)} \mathbf{A}(B)_{xj}$ is a partition of $X \setminus X_i$, for $1 \le j \le \lambda$, and it is easy to see that $(X,\mathbf{G},\mathbf{A})$ is an r-$(k,\lambda,M;v)$-frame where $\mathbf{A} = \cup_{B\in\mathbf{B}} \mathbf{A}(B)$. ∎

A special case of a GDD $[K,1,M;v]$ is $M = \{1\}$, and we can regard it as PBD $B(K,1;v)$. Hence, it follows from Theorem 2.18 that

Corollary 2.19. *Let* v, k *and* λ *be positive integers. Suppose there exists a PBD* $B(\{k_1,k_2,\dots,k_r\},1;v)$ *where* $k_i \equiv 1 \pmod{k}$ *for* $1 \le i \le r$ *and for each* k_i *there exists* $a(k_i,k,\lambda)\,r$*-RPMD. Then there exists a* (v,k,λ)*-*r*-RPMD and there exists a* (v,k_i,λ)*-*r*-RPMD, as a subdesign, for* $1 \le i \le r$.

We shall adapt the following notation:

$$\text{dev}\,\mathbf{B} = \{B+g: \quad B \in \mathbf{B} \text{ and } g \in G\}$$

where $\mathbf{B}$ is the collection of base blocks of the design and G is a given group.

Theorem 2.20. *Suppose that*

(1) *there exists a* $u+1,k,\lambda)$*-*r*-RPMD, a* (u,k,λ)*-*r*-RPMD and a* (w,k,λ)*-*r*-RPMD where* $u+1 \equiv w \equiv 1 \pmod{k}$;
(2) *there exists a* (p,k,λ)*-*r*-RPMD of* $(Y, dev\,\mathbf{B})$ *where* $Y = Z_p$, $p \equiv 1 \pmod{k}$, *an odd prime and* $u+1, w \le p$;
(3) $\mathbf{B}$ *can be partitioned into* λ *partitions of* $Z_p \backslash \{0\}$ $\mathbf{B}_1,\mathbf{B}_2,\dots,\mathbf{B}_\lambda$ *satisfying that there is a partial parallel class* $\mathbf{B}_i' \subset \mathbf{B}_i$ *containing* $(w-1)/k$ *blocks for* $1 \le i \le \lambda$ *such that* $\cup_{B\in\mathbf{B}_i'} B = \cup_{B\in\mathbf{B}_j'} B$ *for* $1 \le i,j \le \lambda$.

Then there exists a $(pu+w,k,\lambda)$*-*r*-RPMD.*

Proof: Let

$$\begin{aligned}
Y &= \{0,1,2,\dots,u\} \times Z_p, W = \{0\} \cup (\cup_{B\in\mathbf{B}_1'} B),\\
Q &= Z_p \setminus W, X = \{1,2,\dots,u\} \times Z_p \cup \{0\} \times W,\\
G_i &= \{[i,0],[i,1],\dots,[i,p-1]\}, \mathbf{G} = \{G_i | 0 \le i \le u\}\\
A_m(i) &= \{[0,i],[1,i+m],[2,i+2m],\dots,[u,i+um]\}\\
A_m'(i) &= \{[1,i+m],[2,i+2m],\dots,[u,i+um]\}\\
\mathbf{A}_m(W) &= \{A_m(i) | i \in W\}, \mathbf{A}_m'(Q) = \{A_m'(i) | i \in Q\}
\end{aligned}$$

Since p is an odd prime and $p \geq u+1$, it is readily checked that $\{A_m(i) \mid i, m \in Z_p\}$ is a collection of blocks of RTD $[u+1,1;p]$. Delete $(p-w)$ points from G_o to get a GDD$[\{u+1,u\},1,\{p,w\};pu+w]$ of which $\cup_{m\in Z_p}(\mathbf{A}_m(W) \cup \mathbf{A}'_m(Q))$ is a collection of blocks.

Let $(A_m(\ell), \mathbf{D}(A_m(\ell)))$ be a $(u+1,k,\lambda)$-r-RPMD having $\lambda(u+1)$ parallel classes: $\mathbf{D}(A_m(\ell))_{ij}$, $0 \leq i \leq u$, $1 \leq j \leq \lambda$ where $A_m(\ell) \in \mathbf{A}_m(W)$ and $\mathbf{D}(A_m(\ell))_{ij}$ is a partition of $A_m(\ell) \setminus [i, \ell + im]$. Let $(A', \mathbf{D}(A'))$ be a (u,k,λ)-r-RPMD having $\lambda(u-1)$ parallel classes: $\mathbf{D}(A')_j$, $1 \leq j \leq \lambda(u-1)$ where $A' \in \mathbf{A}'_m(Q)$. Let $(\{0\} \times W, \mathbf{F})$ be a (w,k,λ)-r-RPMD having λw parallel classes: $\mathbf{F}_{ij}$, $i \in W$, $1 \leq j \leq \lambda$, $(\{i\} \times Z_p, \{i\} \times \text{dev } \mathbf{B})$ be a (p,k,λ)-r-RPMD for $i = 1,2,\ldots,u$, and $\mathbf{D} = [\cup_{m \in Z_p} (\mathbf{D}(\mathbf{A}_m(W)) \cup \mathbf{D}(\mathbf{A}'_m(Q)))] \cup \mathbf{F} \cup (\cup_{1 \leq i \leq u} \{i\} \times \text{dev } \mathbf{B})$.

By Theorem 1.12 $(X, \mathbf{D})$ is a $(pu+w,k,\lambda)$-r-PMD. We are to prove that it is resolvable. We briefly denote $\mathbf{A}_m(W)$ and $\mathbf{A}'_m(Q)$ by $\mathbf{A}_m$ and $\mathbf{A}'_m$, respectively.

Let

$$\begin{aligned}
\mathbf{D}_{ijm} &= \mathbf{D}(\mathbf{A}_m)_{ij} \cup \{\{i\} \times (B + mi) \mid B \in \mathbf{B}'_j\} \\
&\quad \cup \mathbf{D}(\mathbf{A}'_m)_{(i-2)\lambda + j} \text{for } 2 \leq i \leq u, 1 \leq j \leq \lambda, m \in Z_p; \\
\mathbf{D}_{1jm} &= \mathbf{D}(\mathbf{A}_m)_{1j} \cup \{\{1\} \times (B + m) \mid B \in \mathbf{B}'_j\} \\
&\quad \cup \{\{i\} \times (B + mi) \mid B \in \mathbf{B}_j \setminus \mathbf{B}'_j, 1 \leq i \leq u\} \text{for } 1 \leq j \leq \lambda, m \in Z_p; \\
\mathbf{D}_{ij} &= \bigcup_{m \in Z_p} \mathbf{D}(A_m(i))_{oj} \cup \mathbf{F}_{ij} \text{for } 1 \leq j \leq \lambda, i \in W.
\end{aligned}$$

It is readily checked that both each $\mathbf{D}_{ijm}$ and each $\mathbf{D}_{ij}$ are parallel classes on X and $\mathbf{D}$ is the union of them. This completes the proof. ∎

Theorem 2.21. *Suppose that*

(1) *there exists a $(u+1,k,\lambda)$-r-RPMD, a (u,k,λ)-r-RPMD and a $(w+1,k,\lambda)$-r-RPMD where $u \equiv w+1 \equiv 0 \pmod{k}$;*

(2) *there exists a $(p+1,k,\lambda)$-r-RPMD of $(Y, \text{dev}\,\mathbf{B})$ where $Y = Z_p \cup \{\infty\}$, $p+1 \equiv 0 \pmod{k}$, an odd prime and $u+1, w \leq p$;*

(3) *$\mathbf{B}$ can be partitioned λ partitions of Y $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_\lambda$ satisfying that there is a partial parallel class $\mathbf{B}'_i \subset \mathbf{B}_i$ containing $(w+1)/k$ blocks for $1 \leq i \leq \lambda$ such that $\cup_{B \in \mathbf{B}'_i} B = \cup_{B \in \mathbf{B}'_j} B$ for $1 \leq i, j \leq \lambda$ and $\infty \in \cup_{B \in \mathbf{B}'_1} B$.*

Then there exists a $(pu+w+1,k,\lambda)$-r-RPMD.

Proof: We adopt the notation of Theorem 2.20. Let $W = (\cup_{B \in \mathbf{B}'_1} B) \setminus \{\infty\}$, $X = \{1,2,\ldots,u\} \times Z_p \cup \{0\} \times W \cup \{\infty\}$, $(\{0\} \times W \cup \{\infty\}, \mathbf{F})$ be a $(w+1,k,\lambda)$-r-RPMD having λw parallel classes: $\mathbf{F}_j$, $1 \leq j \leq \lambda w$, for convenience, we denote the λw parallel classes by $\mathbf{F}_{ij}$, $1 \leq j \leq \lambda$, $i \in W$. Let $(\{i\} \times Z_p \cup \{\infty\}, \{i\} \times \text{dev } \mathbf{B})$ be a $(p+1,k,\lambda)$-r-RPMD for $1 \leq i \leq u$ where we set $[i,\infty] = \infty$. Let $\mathbf{D}_{ijm}$ and $\mathbf{D}_{ij}$ be as in Theorem 2.20. Similarly, we have that $(X, \mathbf{D})$ is a $(pu+w+1,k,\lambda)$-r-RPMD.

In what follows, we give weight $(w+1)$ to one point of an r-$(k, \lambda, v; uv)$-frame, we wish to construct an r-$(k, \lambda, \{v, v+w\}; uv+w)$-frame or a $(uv+w, k, \lambda)$-r-RPMD. For this we need a special RTD $[k, 1; v]$ satisfying that there are v pairwise disjoint blocks which come from v parallel classes.

Since v blocks containing a certain point of an RTD $[k, 1; v]$ come from v parallel classes, we can get v pairwise disjoint blocks with size $k-1$ by deleting the point; that is,

Lemma 2.22. *If there is an RTD $[k+1, 1; v]$ then there is an RTD $[k, 1; v]$ with v pairwise disjoint blocks which come from v parallel classes.*

Theorem 2.23. *Suppose that*

(1) *there exists a (u, k, λ)-r-RPMD where $u \equiv 1 \pmod{k}$;*
(2) *there is an RTD $[k+1, 1; v]$;*
(3) *there exists a $(u+w, k, \lambda; w+1)$-r-IRPMD where $w = 0 \pmod{k}$.*

Then there exists an r-$(k, \lambda, \{v, v+w\}; uv+w)$-frame.

Proof: Let Y be a v-set, $(X, \mathbf{B})$ be a (u, k, λ)-r-RPMD having λu parallel classes: $\mathbf{B}_{ij}$, $1 \leq i \leq u$, $1 \leq j \leq \lambda$ such that each $\mathbf{B}_{ij}$ is a parition of $X \setminus \{x_i\}$.

Let $B = (a_1, a_2, \ldots, a_k) \in \mathbf{B}$, from Lemma 2.22 we can let $(B \times Y, \mathbf{G}_B, \mathbf{C}(B))$ be a cyclically ordered RTD $[k, 1; v]$ having v parallel classes $\mathbf{C}(B)_j$, $1 \leq j \leq v$ such that $E(B)_j = ([a_1, y_j]\ [a_2, y_j], \ldots, [a_k, y_j]) \in \mathbf{C}(B)_j$ if $x_1 \notin B$ and $E(B)_j = ([x_1, y_1], [a_2, y_j], \ldots, [a_k, y_j]) \in \mathbf{C}(B)_j$ if $x_1 \in B$, provided $B = (x_1, a_2, a_3, \ldots, a_k)$.

Let W be a w-set and $W \cap X = \phi$, $L = (\{x_1\} \cup W) \times \{y_1\}$, $M_t = L \cup (X \setminus \{x_1\}) \times \{y_t\}$ and $(M_t, L, \mathbf{A}^t)$ be a $(u+w, k, \lambda; w+1)$-r-IRPMD for $1 \leq t \leq v$, which has $\lambda(u-1)$ parallel classes $\mathbf{A}^t_{ij}$, $2 \leq i \leq u$, $1 \leq j \leq \lambda$ such that each $\mathbf{A}^t_{ij}$ is a partition of $M_t \setminus [x_i, y_t]$ and $\lambda(w+1)$ partitions of $(X \setminus \{x_1\}) \times \{y_t\}$: $\mathbf{D}^t_s$, $1 \leq s \leq \lambda(w+1)$. Define

$$\mathbf{P}_{1jt} = (\mathbf{C}(\mathbf{B}_{1j})_t \setminus \mathbf{E}(\mathbf{B}_{1j})_t) \cup \mathbf{D}^t_j$$

$$\mathbf{P}_{1j\ell} = \bigcup_{1 \leq t \leq v} \mathbf{D}^t_{(\ell-v)\lambda+j}$$

$$\mathbf{P}_{ijt} = (\mathbf{C}(\mathbf{B}_{1j})_t \setminus \mathbf{E}(\mathbf{B}_{ij})_t) \cup \mathbf{A}^t_{ij}$$

where $1 \leq j \leq \lambda$, $1 \leq t \leq v$, $v+1 \leq \ell \leq v+w$, $2 \leq i \leq u$ and $\mathbf{E}(\mathbf{B}_{ij})_t = \{E(B)_t \mid B \in \mathbf{B}_{ij}\}$.

Let

$$Z = X \times Y \cup W \times \{y_1\}$$
$$G_1 = \{x_1\} \times Y \cup W \times \{y_1\}$$
$$G_i = \{x_i\} \times Y \text{ for } 2 \leq i \leq u$$
$$\mathbf{G} = \{G_i \mid 1 \leq i \leq u\}$$
$$\mathbf{P} = \left(\cup_{1 \leq i \leq u, 1 \leq j \leq \lambda, 1 \leq t \leq v} \mathbf{P}_{ijt}\right) \cup \left(\cup_{1 \leq j \leq \lambda, v+1 \leq e \leq v+w} \mathbf{P}_{ije}\right)$$

Since each $\mathbf{P}_{ijt}$ is a partition of $Z \setminus G_i$, so $(Z, \mathbf{G}, \mathbf{P})$ is an r-$(k, \lambda, \{v, v+w\}; uv+w)$-frame. ∎

Theorem 2.24. *Suppose that*

(1) *there exists a* (u, k, λ)-r-*RPMD where* $u \equiv 1 \pmod{k}$;
(2) *there is RTD* $[k+1, 1; v]$;
(3) *there exists a* $(u+w, k, \lambda; w+1)$-r-*IRPMD where* $w+u \equiv 0 \pmod{k}$;
(4) *there exists a* (v, k, λ)-r-*RPMD and a* $(w+v, k, \lambda)$-r-*RPMD where* $v \equiv 1 \pmod{k}$ *and* $w+v \equiv 0 \pmod{k}$.

Then there exists a $(uv+w, k, \lambda)$-r-*RPMD.*

Proof: We adopt the notation of Theorem 2.23. Let $M_t, L, \mathbf{A}^t)$ be a $(u+w, k, \lambda; w+1)$-r-IRPMD for $1 \le t \le v$ having $(u-1)\lambda$ parallel classes $\mathbf{A}^t_{(i-1)\lambda+j}$ $1 \le j \le \lambda$, $2 \le i \le u$ and λw parallel classes of $(X \setminus \{x_i\}) \times \{y_t\}$: $\mathbf{D}^t_s$, $1 \le s \le \lambda w$.
Let

$$\mathbf{H}_{1jt} = (\mathbf{C}(\mathbf{B}_{1j})_t \setminus E(\mathbf{B}_{1j})_t) \cup \mathbf{D}^t_j$$

$$\mathbf{H}_{1j\ell} = \bigcup_{1 \le t \le v} \mathbf{D}^t_{(\ell-v)\lambda+j}$$

$$\mathbf{H}_{ijt} = (\mathbf{C}(\mathbf{B}_{ij})_t \setminus E(\mathbf{B}_{ij})_t) \cup \mathbf{A}^t_{(i-1)\lambda+j}$$

where $1 \le j \le \lambda$, $1 \le t \le v$, $v+1 \le \ell \le v+w-1$, $2 \le i \le u$.
Let $(G_1, \mathbf{D}')$ is a $(v+w, k, \lambda)$-r-RPMD having $(w+v-1)\lambda$ parallel classes $\mathbf{D}'_{(e-1)\lambda+j}$, $1 \le e \le v+w-1$, $1 \le j \le \lambda$ and $(G_i, \mathbf{D}^i)$ is a (v, k, λ)-r-RPMD having $v\lambda$ parallel classes $\mathbf{D}^i_{tj}$, $1 \le t \le v$, $1 \le j \le \lambda$ for $2 \le i \le u$.
Let

$$\mathbf{F}_{1je} = \mathbf{H}_{1je} \cup \mathbf{D}'_{(e-1)\lambda+j}$$

$$\mathbf{F}_{ijt} = \mathbf{H}_{ijt} \cup \mathbf{D}^i_{tj}$$

$$\mathbf{F} = \left(\bigcup_{\substack{1 \le j \le \lambda \\ 1 \le e \le v+w-1}} \mathbf{F}_{1je} \right) \cup \left(\bigcup_{\substack{2 \le j \le u \\ 1 \le j \le \lambda \\ 1 \le t \le v}} \mathbf{F}_{ijt} \right)$$

Since each $\mathbf{F}_{ijt}$ is a parallel class of Z, so $(Z, \mathbf{F})$ is a $(uv+w, k, \lambda)$-r-RPMD.
∎

3. The construction of $(v, 4, \lambda)$-RPMD, $\lambda > 1$.

In this section we shall show that the necessary condition for the existence of a $(v, 4, \lambda)$-RPMD for $\lambda > 1$, namely, $v \equiv 0$ or $1 \pmod{4}$, is also sufficient except for $v = 4$ and λ odd. We, also obtain a $(v, 4, 1)$-RPMD for $v = 57$ and 93.

In fact, we need only establish the reult for the cases $\lambda = 2$ and $\lambda = 3$, since if there exists a (v, k, λ_1)-r-RPMD and a (v, k, λ_2)-r-RPMD then there exists a $(v, k, \lambda_1 + \lambda_2)$-$r$-RPMD.

From Theorem 1.7, we have

Theorem 3.1. *A $(v, 4, \lambda)$-RPMD exists for every positive integer $v \equiv 1 \pmod 4$ with the possible exception of $v = 57$ and 93.*

Theorem 3.2. *A $(v, 4, 2)$-RPMD exists for any positive integer $v \equiv 0 \pmod 4$.*

Proof: Suppose that $(X, \mathbf{A})$ is a Whist-tournament, denoted Wh$[v]$ where $\mathbf{A}$ is a collection of Whist tables. Let $B(A) = (x_1, x_3, x_2, x_4)$ and $B'(A) = (x_4, x_2, x_3, x_1)$ for every $A = [x_1, x_2; x_3, x_4] \in \mathbf{A}$ and $\mathbf{B} = \{B(A), B'(A) \mid A \in \mathbf{A}\}$. By Definition 1.13, it is easy to see that every ordered pair of points of X appear t-apart in two blocks of $\mathbf{B}$ for $t = 1, 2$, and then $(X, \mathbf{B})$ is a $(v, 4, 2)$-RPMD. Therefore, there exists a $(v, 4, 2)$-RPMD for all $v \equiv 0 \pmod 4$ and $v \neq 264$ by Theorem 1.14.

For $v = 264$, since there is a $(33, 4, 1)$-RPMD from Theorem 3.1, and there is an $(8, 4, 2)$-RPMD, so we have a $(264, 4, 2)$-RPMD by Theorem 2.1. ∎

Theorem 3.3. *There exists a $(v, 4, 1)$-RPMD for $v = 57$ and 93.*

Proof: In each of the following two cases for $v = 57$ and 93, we let $\mathbf{G} = Z_{v-n}$, $X = Z_{v-n}$ and $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$ for $n = 13, 21$. From Theorem 3.1 we can let $(Y, \mathbf{A})$ be an $(n, 4, 1)$-RPMD with parallel classes $\mathbf{A}_i$, $1 \leq i \leq n$. We then present a collection of base blocks $\mathbf{B}$ and n parallel classes of blockks based on X, namely $\mathbf{D}_i$, $1 \leq i \leq n$, as defined. Since $\mathbf{B}$ is a parallel class of $X \cup Y$, it is easily checked that $(X \cup Y, \text{dev } \mathbf{B} \cup \mathbf{D} \cup \mathbf{A})$ is a $(v, 4, 1)$-RPMD where $\mathbf{D} = \cup_{1 \leq i \leq n} \mathbf{D}_i$, and $\mathbf{A}_i \cup \mathbf{D}_i$ is a parallel class of the design.

The case $v = 57$ and $n = 13$

Let

$$\begin{aligned}\mathbf{B} = \{&(-6, 2, 6, -2), (\infty_1, -13, 4, -11), (\infty_2, 12, -21, 18)\\ &(\infty_3, -8, 19, 0), (\infty_4, 14, -7, 5), (\infty_5, 9, -14, -1)\\ &(\infty_6, -16, 15, -5), (\infty_7, -20, -4, 11), (\infty_8, 20, 8, -10)\\ &(\infty_9, 7, 17, -9), (\infty_{10}, -17, -12, 10), (\infty_{11}, -15, 13, 3)\\ &(\infty_{12}, 22, 21, -3), (\infty_{13}, -19, -18, 1)\}.\end{aligned}$$

For $0 \leq i \leq 3$, define:

$$\begin{aligned}\mathbf{D}_{i+1} &= \{(0, 2, 5, 3) + 4j + i: \quad 0 \leq j \leq 10\}\\ \mathbf{D}_{i+5} &= \{(0, 6, 15, 9) + 4j + i: \quad 0 \leq j \leq 10\}\\ \mathbf{D}_{i+9} &= \{(0, 14, 7, -7) + 4j + i: \quad 0 \leq j \leq 10\}\\ \mathbf{D}_{13} &= \{(33, 22, 11, 0) + j: \quad 0 \leq j \leq 10\}\end{aligned}$$

The case $v = 93$ and $n = 21$

Let

$$\begin{aligned}\mathbf{B} = \{&(2,6,-2,-6),(10,22,-10,-22),(\infty_1,13,-15,14)\\&(\infty_2,32,3,-24),(\infty_3,12,27,-18),(\infty_4,19,4,-31)\\&(\infty_5,34,17,-20),(\infty_6,9,25,-30),(\infty_7,0,-16,23)\\&(\infty_8,18,-13,20),(\infty_9,-1,30,5),(\infty_{10},7,-29\ -4)\\&(\infty_{11},-9,11,-33),(\infty_{12},-5,-25,21),(\infty_{13},-3,35,-11)\\&(\infty_{14},29,-19,15),(\infty_{15},-8,-32,33),(\infty_{16},8,-34,-27)\\&(\infty_{17},-35,16,-14),(\infty_{18},31,-28,-7),(\infty_{19},26,-23,36)\\&(\infty_{20},-12,-21,28),(\infty_{21},-26,-17,1)\}\end{aligned}$$

For $0 \leq i \leq 3$, define:

$$\begin{aligned}\mathbf{D}_{i+1} &= \{(0,2,7,5) + 4j + i:\quad 0 \leq j \leq 17\}\\ \mathbf{D}_{i+5} &= \{(0,6,17,11) + 4j + i:\quad 0 \leq j \leq 17\}\\ \mathbf{D}_{i+9} &= \{(0,10,29,19) + 4j + i:\quad 0 \leq j \leq 17\}\\ \mathbf{D}_{i+13} &= \{(0,14,13,-1) + 4j + i:\quad 0 \leq j \leq 17\}\\ \mathbf{D}_{i+17} &= \{(0,22,19,-3) + 4j + i:\quad 0 \leq j \leq 17\}\\ \mathbf{D}_{21} &= \{(54,36,18,0) + j:\quad 0 \leq j \leq 17\}\end{aligned}$$

∎

Lemma 3.4. *There exists a* $(4,\lambda,4;4p)$*-frame and a* $(4,\lambda,4;4p)$*-semiframe for* $\lambda = 2,3$ *and* $p = 7,11,19,23$.

Proof: In each of the following four cases for p, we let $G = Z_p$, we then present three collections of base blocks $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, and it is readily checked that:

(a) Let

$$\mathbf{D}_i = \left(\bigcup_{1\leq t\leq 3} \langle E_t+i,\ \text{dev}\ \mathbf{B}\rangle\right)\cup\left(\bigcup_{4\leq t\leq 6}\langle E_t+i,\ \text{dev}\ \mathbf{A}\rangle\right)$$
$$\cup\left(\bigcup_{7\leq t\leq 8}\langle E_t+i,\ \text{dev}\ \mathbf{C}\rangle\right)\ \text{for}\ 0\leq i\leq 2,$$
$$\mathbf{D} = \bigcup_{0\leq i\leq 2}\mathbf{D}_i \text{ and } \mathbf{D}_\circ' = \{(d,c,b,a)\,|\,(a,b,c,d)\in\mathbf{D}_\circ\}$$

then $(X,\mathbf{G},\mathbf{D}_\circ\cup\mathbf{D}_\circ')$ is a PGMD $[4,2,4;4p]$ and $(X,\mathbf{G},\mathbf{D})$ is a PGMD $[4,3,4;4p]$.

(b) Each of

$$\langle E_1 + i,\ \mathbf{B} + g\rangle \cup \langle E_3 + i,\ \mathbf{B} + g\rangle,$$
$$\langle E_2 + i,\ \mathbf{B} + g\rangle \cup \langle E_4 + i,\ \mathbf{A} + g\rangle,$$
$$\langle E_5 + i,\ \mathbf{A} + g\rangle \cup \langle E_6 + i,\ \mathbf{A} + g\rangle, and$$
$$\langle E_7 + i,\ \mathbf{C} + g\rangle \cup \langle E_8 + i,\ \mathbf{C} + g\rangle,$$

is a partition $X \setminus X_g$.

(c) $\langle E_t + i, \text{dev } A\rangle$ is a parallel class of X for $0 \leq i \leq 2, 5 \leq t \leq 6$ and $A \in \mathbf{A}$.

Therefore, $(X, \mathbf{G}, \mathbf{D}_\circ \cup \mathbf{D}_\circ')$ is both a $(4,2,4;4P)$-frame and a $(4,2,4;4p)$-semiframe, and $(X, \mathbf{G}, \mathbf{D})$ is both a $(4, 3, 4; 4p)$-frame and a $(4, 3, 4; 4p)$-semiframe. For $p = 7$

$$\mathbf{A} = \{w^{2i}(w^3, w^\circ, w^2, w^5) \mid i = 0,1,2\}$$
$$\mathbf{B} = \{w^{2i}(w^\circ, w^3, w^5, w^2) \mid i = 0,1,2\}$$
$$\mathbf{C} = \{w^{2i}(w^\circ, w^2, w^3, w^5) \mid i = 0,1,2\}$$

where $w = 3$.

For $p = 11$

$$\mathbf{A} = \{w^{2i}(w^\circ, w^5, w^3, w^8) \mid i = 0,1,2,3,4\}$$
$$\mathbf{B} = \{w^{2i}(w^5, w^\circ, w^8, w^3) \mid i = 0,1,2,3,4\}$$
$$\mathbf{C} = \{w^{2i}(w^5, w^3, w^\circ, w^8) \mid i = 0,1,2,3,4\}$$

where $w = 2$.

For $p = 19$

$$\mathbf{A} = \{w^{2i}(w^\circ, w^9, w^{13}, w^4) \mid i = 0,1,2,\ldots,8\}$$
$$\mathbf{B} = \{w^{2i}(w^9, w^\circ, w^4, w^{13}) \mid i = 0,1,2,\ldots,8\}$$
$$\mathbf{C} = \{w^{2i}(w^9, w^{13}, w^\circ, w^4) \mid i = 0,1,2,\ldots,8\}$$

where $w = 2$.

For $p = 23$

$$\mathbf{A} = \{w^{2i}(w^\circ, w^{11}, w^5, w^{16}) \mid i = 0,1,2,\ldots,10\}$$
$$\mathbf{B} = \{w^{2i}(w^{11}, w^\circ, w^{16}, w^5) \mid i = 0,1,2,\ldots,10\}$$
$$\mathbf{C} = \{w^{2i}(w^{11}, w^5, w^\circ, w^{16}) \mid i = 0,1,2,\ldots,10\}$$

where $w = 5$.

∎

From Theorem 3.1 and 3.3, we have

Theorem 3.5. *A $(v,4,1)$-RPMD exists for every positive integer $v = 1 \pmod 4$.*

Summarizing the results given above and Theorem 3.1, we have

Theorem 3.6. *The necessary condition for the existence of a $(v,4,\lambda)$-RPMD for λ even, namely, $v \equiv 0$ or 1 $\pmod 4$ is also sufficient.*

Theorem 3.7. *A $(v,4,3)$-RPMD exists for $v = 12, 16, 24$.*

Proof: In each of the following three cases for v, we let $G = Z_{v-1}$ and $X = Z_{v-1} \cup \{\infty\}$. We then present a collection of base blocks $\mathbf{B}$, and it is readily checked that $(X, \text{dev } \mathbf{B})$ is a $(v,4,3)$-RPMD.

For $v = 12$

$$\begin{aligned}\mathbf{B} = \{&(\infty,0,1,3),(-1,5,-3,4),(2,-2,-4,-5);\\ &(\infty,0,2,-5),(-3,5,-1,-2),(4,-4,1,3);\\ &(\infty,0,-3,2),(3,-2,-4,-1),(5,-5,4,1)\ \}\end{aligned}$$

For $v = 16$

$$\begin{aligned}\mathbf{B} = \{&(0,1,7,10),(\infty,6,8,12),(3,11,14,13),(9,5,2,4)\\ &-(0,7,10,1),(\infty,8,12,6),(3,14,13,11),-(9,2,4,5)\\ &(0,10,1,7),(\infty,12,6,8),(3,13,11,14),(9,4,5,2)\}\end{aligned}$$

where $-(a,b,c,d)$ denotes (d,c,b,a).

For $v = 24$

$$\begin{aligned}\mathbf{B} = \{&(\infty,0,7,10),-(1,8,12,22),-(2,5,6,11)\\ &(\infty,10,0,7),-(1,22,8,12),(2,11,5,6)\\ &(\infty,7,10,0),(1,12,22,8),(2,6,11,5)\\ &(3,9,14,18,),(4,16,17,19),-(13,15,20,21)\\ &(3,18,9,14),(4,19,16,17),-(13,21,15,20)\\ &(3,14,18,9),-(4,17,19,16),-(13,20,21,15)\ \}\end{aligned}$$

where $-(a,b,c,d)$ denotes (d,c,b,a).

Theorem 3.8. *A $(v,4,3)$-RPMD exists for $v = 8, 20, 32, 44$ and 68.*

Proof: In each of the following five cases for v, we let w be a primitive root of $GF(v-1)$ and $X = GF(v-1) \cup \{\infty\}$. We take the additive group of $GF(v-1)$, and denote (ba_1, ba_2, ba_3, ba_4) by $b(a_1, a_2, a_3, a_4)$ where $a_i, b, ba_i \in GF(v-1)$ for $1 \le i \le 4$. We then present a collection of base blocks $\mathbf{B}$ and it is readily checked that $(X, \text{dev } (\mathbf{B} \cup b\mathbf{B} \cup b^2\mathbf{B}))$ is a $(v,4,3)$-RPMD where $b = w^{(v-2)/3}$ and $b\mathbf{B} = \{bB \mid B \in \mathbf{B}\}$.

For $v = 8, w = 3$

$$\mathbf{B} = \{(\infty, 1, w^2, w^4), (0, w, w^3, w^5)\}.$$

For $v = 20, w = 2$

$$\mathbf{B} = \{(\infty, 1, w^6, w^{12}), (0, w, w^7, w^{13})(w^{14}, w^8, w^2, w^5) (w^{17}, w^3, w^9, w^{15}), (w^{11}, w^4, w^{10}, w^{16}\}.$$

For $v = 32, w = 3$

$$\mathbf{B} = \{(\infty, 0, 2, 10), (-12, -10, 12, -2), (5, 14, 1, 8) (9, -8, -14, -9), (-6, -1, -5, 6), (13, -11, 15, 7) (3, -13, -15, -3), (4, -7, -4, 11)\}.$$

For $v = 44, w = 3$

$$\mathbf{B} = \{(1, 7, -6, -1), (-3, -18, 21, 3), (20, -20, 11, 9) (-16, 17, 10, 16), (-5, 13, -8, 5), (-19, 19, -4, -15) (2, 14, -12, -2), (-9, 6, -11, -7), (-17, 4, -10, 15) (-21, -14, 18, 12), (\infty, 8, 0, -13)\}.$$

For $v = 68, w = 2$

$$B = \{(-2, 7, 2, -9), (1, -29, -1, 30), (-18, 4, 14, -4) (\infty, 0, 15, -30), (-7, 32, 9, -22), (-14, -3, 18, 23) (-8, -6, -28, -21), (-16, -12, 11, 25), (29, -17, 33, -26) (-31, 8, 28, 31), (-5, -11, 16, 5), (-10, -32, 22, 10) (-20, 3, -23, 20), (-27, 21, 6, 27), (13, -25, 12, -13) (-24, 24, 17, 26), (19, -33, -15, -19)\}$$

∎

Lemma 3.9. *There exists a* $(v, 4, 3; n)$*-IRPMD for* $(v, n) = (16, 4), (20, 4), (24, 4), (36, 4), (52, 8)$.

Proof: In each of the following five cases, we let $G = Z_{v-n}, Y = \{\infty_1, \infty_2, \ldots, \infty_n\}, X = Y \cup \{0, 1, 2, \ldots, v-n-1\}$. We then present three collections of base blocks: $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$, and h parallel classes of $X \setminus Y$: $\mathbf{D}_i$, $1 \leq i \leq h$ where $h = 3(n-1)$ if $v \equiv n \equiv 0 \pmod 4$ or $h = 3n$ if $v \equiv n \equiv 1 \pmod 4$. It is readily checked that $\mathbf{B}_i$ is a parallel class of X for $i = 1, 2, 3$ and $(X, Y, (\text{dev}\,\mathbf{B}) \cup \mathbf{D})$ is a $(v, 4, 3; n)$-IRPMD where $\mathbf{D} = \cup_{1 \leq i \leq h} \mathbf{D}_i$ and $\mathbf{B} = \cup_{1 \leq i \leq 3} \mathbf{B}_i$.

For $v = 16, n = 4$

$$\mathbf{B}_1 = \{(\infty_1,2,7,0),(\infty_2,8,3,1),(\infty_i,4,11,6),(\infty_4,5,10,9)\}$$
$$\mathbf{B}_2 = \{(\infty_1,10,6,2),(\infty_2,3,4,5),(\infty_3,1,11,7),(\infty_4,8,0,9)\}$$
$$\mathbf{B}_3 = \{(\infty_1,4,10,0),(\infty_2,8,7,5),(\infty_3,9,1,2),(\infty_4,11,3,6)\}$$
$$\mathbf{D}_i = \{(1,3,2,4)+i+4j \mid 0 \le j \le 2\}$$
$$\mathbf{D}_{4+i} = \{(0,6,3,9)+i+4j \mid 0 \le j \le 2\} \text{ for } 1 \le i \le 4$$
$$\mathbf{D}_9 = \{(0,3,6,9)+j \mid 0 \le j \le 2\}$$

For $v = 20, n = 4$

$$\mathbf{B}_1 = \{(0,1,6,9),(\infty_1,7,3,14),(\infty_2,2,8,4)$$
$$(\infty_3,5,13,10),(\infty_4,12,11,15)\}$$
$$\mathbf{B}_2 = \{(0,9,1,6),(\infty_1,3,14,7),(\infty_2,4,2,8)$$
$$(\infty_3,11,5,12),(\infty_4,13,15,10)\}$$
$$\mathbf{B}_3 = \{(4,10,13,5),(\infty_1,7,14,3),(\infty_2,11,12,9)$$
$$(\infty_3,8,1,2),(\infty_4,6,0,15)\}$$
$$\mathbf{D}_i = \{(2,0,4,6)+2i+j \mid j = 0,1,8,9\}$$
$$\mathbf{D}_{4+i} = \{(0,2,5,3)+4j+i \mid j = 0,1,2,3\} \text{ for } 1 \le i \le 4$$
$$\mathbf{D}_9 = \{(0,4,8,12)+j \mid j = 0,1,2,3\}$$

For $v = 24, n = 4$

$$\mathbf{B}_1 = \{(0,11,4,3),(5,8,7,14),(\infty_1,13,17,1)$$
$$(\infty_2,12,10,2),(\infty_3,19,16,6)(\infty_4,18,15,9)\}$$
$$\mathbf{B}_2 = \{(0,12,3,7),(9,15,10,8),(\infty_1,4,13,1)$$
$$(\infty_2,18,17,2),(\infty_3,6,16,14),(\infty_4,11,19,5)\}$$
$$\mathbf{B}_3 = \{(0,9,1,4),(3,5,10,16),(\infty_1,8,11,7)$$
$$(\infty_2,19,6,14),(\infty_3,18,12,2),(\infty_4,17,13,15)\}$$
$$\mathbf{D}_i = \{(0,5,6,7)+i+4j \mid 0 \le j \le 4\}$$
$$\mathbf{D}_{4+i} = \{(0,9,11,6)+i+4j \mid 0 \le j \le 4\} \text{ for } 1 \le i \le 4$$
$$\mathbf{D}_9 = \{(15,10,5,0)+j \mid 0 \le j \le 4\}$$

For $v = 36, n = 4$

$$\mathbf{B}_1 = \{(0,1,7,3),(9,-15,4,-8),(5,15,-4,-9)$$
$$(-1,-7,10,13),(-13,-10,-11,-14)(\infty_1,-3,-5,11)$$
$$(\infty_2,2,12,16),(\infty_3,-2,-12,6),(\infty_4,8,14,-6)\}$$
$$\mathbf{B}_2 = \{(0,9,3,-11),(-5,8,12,-13),(15,13,5,-12)$$
$$(-3,11,-2,7),(-6,-1,10,-10),(\infty_1,-9,16,4)$$
$$(\infty_2,-14,2,-15),(\infty_3,-4,1,-8),(\infty_4,14,-7,6)\}$$
$$\mathbf{B}_3 = \{(0,-1,-5,9),(-15,-8,4,-7),(13,15,7,-4)$$
$$(-9,1,-6,-11),(5,11,2,-10),(\infty_1,12,-13,-14)$$
$$(3,6,8,\infty_2),(-3,-2,14,\infty_3),(-12,16,10,\infty_4)\}$$
$$\mathbf{D}_i = \{(0,-7,-9,10)+i+4j \mid 0 \le j \le 7\}$$
$$\mathbf{D}_{4+i} = \{(0,-3,2,-9)+i+4j \mid 0 \le j \le 7\} \text{ for } 1 \le i \le 4$$
$$\mathbf{D}_9 = \{(0,8,16,24)+j \mid 0 \le j \le 7\}$$

For $v = 52, n = 8$

$$\mathbf{B}_1 = \{(14,2,1,19),(-6,18,-5,9),(-14,-10,-1,-7)$$
$$(-19,6,-9,-2),(5,7,10,-18),(\infty_1,-8,20,-13)$$
$$(\infty_2,11,-4,8),(\infty_3,0,-21,22),(\infty_4,-17,-11,3)$$
$$(\infty_5,21,15,17),(\infty_6,-3,-15,13),(\infty_7,12,-12,4)$$
$$(\infty_8,16,-20,-16),\}$$
$$\mathbf{B}_2 = \{(-3,4,-7,6),(-19,17,-8,10),(21,16,5,2)$$
$$(1,13,12,18),(-15,-14,9,20),(\infty_1,0,22,15)$$
$$(\infty_2,-20,-4,7),(\infty_3,-1,8,-12),(\infty_4,-5,-21,-11)$$
$$(\infty_5,-9,11,3),(\infty_6,19,-13,-17),(\infty_7,-18,-16,-2)$$
$$(\infty_8,14,-6,-10)\}$$
$$\mathbf{B}_3 = \{(2,7,-16,1),(9,19,-12,18),(-7,-20,-5,-14)$$
$$(6,-4,-1,-19),(-9,8,5,10),(\infty_1,22,0,3)$$
$$(\infty_2,-6,11,-2),(\infty_3,12,20,-11),(\infty_4,4,-18,14)$$
$$(\infty_5,16,-8,-10),(\infty_6,-21,15,-3),(\infty_7,17,21,-15)$$
$$(\infty_8,13,-13,-17)\}$$

$D_i = \{(0,-5,-19,2) + i + 4j \mid 0 \le j \le 10\}$
$D_{4+i} = \{(0,-2,3,13) + i + 4j \mid 0 \le j \le 10\}$
$D_{8+i} = \{(0,-10,17,-1) + i + 4j \mid 0 \le j \le 10\}$
$D_{12+i} = \{(0,-10,9,3) + i + 4j \mid 0 \le j \le 10\}$
$D_{16+i} = \{(0,15,1,-6) + i + 4j \mid 0 \le j \le 10\}$ for $1 \le i \le 4$
$D_{21} = \{(0,33,22,11) + j \mid 0 \le j \le 10\}$

Theorem 3.10. *There exists a* $(52,4,3)$ *-RPMD.*

Proof: Since there exists a $(52,4,3;8)$-IRPMD from Lemma 3.9 and there exists an $(8,4,3)$-RPMD from Theorem 3.8. So we have that there exists a $(52,4,3)$-RPMD by Theorem 2.6 ∎

Theorem 3.11. *There exists a* $(4t,4,3)$ *-RPMD for* $t = 21,41,46,53,57$.

Proof: Apply Theorem 2.16 with $(u,v,w) = (5,16,4), (5,32,4,), (9,20,4), (13,16,4), (5,44,8)$ to obtain a $(4t,4,3)$-RPMD for $t = 21,41,46,53,57$. Here the required $(v+w,4,3;w)$-IRPMD and $(v+w,4,3)$-RPMD come from Lemma 3.9 and Theorem 3.7, 3.8, 3.10 and 3.20. ∎

Theorem 3.12. *There exists a* $(4t,4,3)$ *-RPMD for* $t = 59,61,67,71$ *and 73.*

Proof: Apply Theorem 2.21 with $(p,u,w) = (23,12,7)$ and $(23,12,15)$ to obtain a $(4t,4,3)$-RPMD for $t = 71,73$. Here the required $(24,4,3)$-RPMD comes from Theorem 3.7. Apply Theorem 2.21 with $(p,u,w) = (19,12,7)$, $(19,12,15)$ and $(31,8,19)$ to obtain a $(4t,4,3)$-RPMD for $t = 59,61,67$. Here the required $(p+1,4,3)$-RPMD for $p = 19,31$ comes from Theorem 3.8. That is, let $\mathbf{B}_j = b^{j-1}\,\mathbf{B}$, $\mathbf{B}'_j = b^{j-1}\mathbf{B}'$ for $1 \le j \le 3$, where $\mathbf{B}$ and b are as in Theorem 3.8. Take $\mathbf{B}' = \{(\infty,1,w^6,w^{12}), (0,w,w^7,w^{13})\}$, $\mathbf{B}' = \mathbf{B} \setminus \{(0,w,w^7,w^{13})\}$ and $\mathbf{B}' = \{(\infty,0,2,10), (-12,-10,12,-2), (5,14,1,8), (9,-8,-14,-9), (-6,-1,-5,6)\}$, respectively. ∎

Theorem 3.13. *There exists a* $(4t,4,3)$ *-RPMD for* $t = 28,35,56,76$, *and 77.*

Proof: Since there exists a $(4v,4,3)$-RPMD for $v = 4,5,8$ and 11, apply Theorem 2.17 with $u = 4$, $h = 7$, $\lambda = 3$ and $\mu = 1$ to obtain a $(28v,4,3)$-RPMD for $v = 4,5,8$ and 11. Here the required $(4,3,4;28)$-semiframe come from Lemma 3.4. Similarly, we can obtain a $(4\cdot 19\cdot 4,4,3)$-RPMD by applying Theorem 2.17 with $u = 4$, $h = 19$, $v = 4$. ∎

Theorem 3.14. *There exists a* $(v,4,3)$ *-RPMD for* $v = 88$, *124, 152.*

Proof: Apply Theorem 2.24 with $u = 17$, $v = 5$ and $w = 3$ to obtain an $(88,4,3)$-RPMD. Here the required $(20,4,3;4)$-IRPMD comes from Lemma 3.4.

Apply Theorem 2.23 with $(u,v,w) = (17,7,4)$ and $(21,7,4)$ to obtain a $(4,3,\{7,11\};\ 123)$ and a $(4,3,\{7,11\};\ 151)$. Here the required $(21,4,3;\ 5)$-IRPMD and $(25,4,3;\ 5)$-IRPMD come from Corollary 2.19 and the fact that there exist a $(21,5,1)$-BIBD and a $(25,5,1)$-BIBD. Hence, it follows from Theorem 2.10 that there exists a $(v,4,3)$-RPMD for $v = 124$ and 152. ∎

Lemma 3.15. *There is an RTD* $[4,3;3]$.

Proof: Let $X_i = \{[i,1],[i,2],[i,3]\}$ for $1 \le i \le 4$, $X = \cup_{1\le i\le 4} X_i$

$$E = (1,2,3,4)$$
$$\mathbf{A}_1 = \{(1,1,1,1),(2,2,2,2),(3,3,3,3),(1,2,3,1),(2,3,1,2)\ (3,1,2,3),(1,3,2,1),(2,1,3,2),(3,2,1,3)$$
$$\mathbf{A}_2 = \{(a,b,c,d+1) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_3 = \{(a,b,c,d+2) \mid (a,b,c,d) \in \mathbf{A}_1\}$$

where $d+1, d+2$ is taken modulo 3.

Let $\mathbf{B}_i = \langle E, \mathbf{A}_i \rangle$, $\mathbf{B} = \cup_{1\le i\le 3}\mathbf{B}_i$. It is readily checked that $(X,\mathbf{G},\mathbf{B})$ is a TD $[4,3;3]$ and $\mathbf{B}_i$ can be partitioned into 3 parallel classes of X for $1 \le i \le 3$. So we have that $(X,\mathbf{G},\mathbf{B})$ is an RTD $[4,3;3]$. ∎

Lemma 3.16. *There exists a PGMD* $[4,1,4;16]$ *where the blocks can be partitioned into 12 parallel classes.*

Proof: Let $X_i = \{[1,i],[2,i],[3,i],[4,i]\}$ for $1 \le i \le 4$ and $X = \cup_{1\le i\le 4} X_i$. Let

$$E_1 = (2,2,4,4), E_2 = (1,1,3,3), E_3 = (1,3,2,4), E_4 = (3,1,4,2)$$
$$E_5 = (1,1,4,4), E_6 = (2,2,3,3), E_7 = (1,2,1,2), E_8 = (3,4,3,4)$$
$$\mathbf{A}_1 = \{(3,1,4,2),(2,4,1,3),(2,1,3,4),(4,3,1,2),(4,1,2,3),(3,2,1,4)\}$$
$$\mathbf{A}_2 = \{(b,a,d,c) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_3 = \{(b,c,a,d) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_4 = \{(d,a,c,b) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_5 = \{(a,b,d,c) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_6 = \{(b,a,c,d) \mid (a,b,c,d) \in \mathbf{A}_1\}$$
$$\mathbf{A}_7 = \mathbf{A}_8 = \mathbf{A}_3$$

Let $\mathbf{B}_i = \langle E_i, \mathbf{A}_i \rangle$ for $1 \le i \le 8$. It is readily checked that $(X,\mathbf{G},\mathbf{B})$ is a PGMD $[4,1,4;16]$ where $\mathbf{B} = \cup_{1\le i\le 8} \mathbf{B}_i$ and $\mathbf{B}_i \cup \mathbf{B}_{i+1}$ can be partitioned into 3 parallel classes of X for $i = 1,3,5,7$. ∎

Theorem 3.17. *There exists a* $(48,4,3)$*-RPMD.*

Proof: By Lemma 3.15 and 3.16 we see that we can let $(X, \mathbf{G}, \mathbf{B})$ be a PGMD $[4,3,12;48]$ where $\mathbf{B}$ can be partitioned into some parallel classes of X. Since there exists a $(12,4,3)$-RPMD, we can let $(G_i, \mathbf{A}_i)$ be a $(12,4,3)$-RPMD for $1 \le i \le 4$. Therefore $(X, \mathbf{B} \cup \mathbf{A})$ is a $(48,4,3)$-RPMD where $\mathbf{A} = \cup_{1 \le i \le 4} \mathbf{A}_i$.
∎

Theorem 3.18. *There exists a* $(v,4,3)$*-RPMD of* $v = 28$ *and 132.*

Proof: In each of the following two cases for $v = 28$ and 132 we let $G = Z_{v-n}$, $X = Z_{v-n}$ and $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$ for $n = 5$ and 33. From Theorem 3.1 we can let $(Y, \mathbf{A})$ be an $(n,4,1)$-RPMD with parallel classes $\mathbf{A}_i$, $1 \le i \le n$. We then present a collection of base blocks $\mathbf{B}$ and n partial parallel classes of blocks based on X, namely, $\mathbf{D}_i$, $1 \le i \le n$, as defined. From Theorem 3.2 we can let $(X \cup Y, \mathbf{E})$ be a $(v,4,2)$-RPMD having $2(v-1)$ parallel classes $\mathbf{E}_j$, $1 \le j \le 2(v-1)$, and without loss of generality we can let $\mathbf{E}_1$ be partitioned into n partial parllel classes $\mathbf{E}_{1j}$, $1 \le j \le n$, such that $\mathbf{E}_{1j} \cup (\mathbf{A}_j \cup \mathbf{D}_j)$ is a parallel class for $1 \le j \le n$. Since it is easy to see that $(X \cup Y, \mathbf{D} \cup \mathbf{A} \cup \text{dev } \mathbf{B})$ is a $(v,4,1)$-PMD where $\mathbf{D} = \cup_{1 \le i \le n} \mathbf{D}_i$ and $\mathbf{B}$ is a parallel class of $X \cup Y$, so it is not difficult to see that $(X \cup Y, \mathbf{E} \cup \mathbf{D} \cup \mathbf{A} \cup \text{dev } \mathbf{B})$ is a $(v,4,3)$-RPMD.

The case $v = 28$ and $n = 5$

$$\mathbf{B} = \{ (-4,4,9,-10), (-11,7,-7,0), (\infty_1,-2,11,-1)$$
$$(\infty_2,-5,5,-3), (\infty_3,-9,10,8), (\infty_4,2,1,-6), (\infty_5,-8,6,3) \}.$$

For $1 \le j \le 4$ define:

$$\mathbf{D}_j = \{(0,1,3,6) + 4(i+5j-5) : i = 0,1,2,3,4\}$$
$$\mathbf{D}_5 = \{(0,1,3,6) + 4i : i = -1,-2,-3\}$$

The case $v = 132$ and $n = 33$

$$\mathbf{B} = \{(\infty_1,33,-19,32), (\infty_2,18,49,-5), (\infty_3,24,-47,-15)$$
$$(\infty_4,-39,19,-36); (\infty_5,-44,-10,30), (\infty_6,16,-12,-43)$$
$$(\infty_7,10,-28,5), (\infty_8,-21,31,6), (\infty_9,-34,-16,8)$$
$$(\infty_{10},-11,27,-18), (\infty_{11},37,3,-33), (\infty_{12},-49,-29,-6)$$
$$(\infty_{13},17,-40,26), (\infty_{14},44,40,-24), (\infty_{15},-37,22,7)$$
$$(\infty_{16},-25,25,-14), (\infty_{17},-32,-3,34), (\infty_{18},14,29,-31)$$
$$(\infty_{19},-48,9,-35), (\infty_{20},-30,-46,35), (\infty_{21},2,21,48)$$
$$(\infty_{22},-2,41,-17), (\infty_{23},47,1,12), (\infty_{24},39,15,-8)$$
$$(\infty_{25},-26,-45,-9), (\infty_{26},43,0,-20), (\infty_{27},-13,13,38)$$
$$(\infty_{28},-42,28,-23), (\infty_{29},-41,23,-4), (\infty_{30},42,46,-7)$$
$$(\infty_{31},20,36,-1), (\infty_{32},-27,-38,11), (\infty_{33},4,-22,45)$$

For $1 \le j \le 33$ define:

$$\mathbf{D}_j = \{F + j + i : F \in \mathbf{F},\ \ i = 0, 33, 66\}$$

where

$$\mathbf{F} = \{(1,13,-1,-13),(2,8,-2,-8),(3,6,-3,-6)\ (4,9,-4,-9),(5,12,-5,-12),(7,15,-7,-15)\ (10,11,-10,-11),(14,16,-14,-16)\}$$

Summarizing the results given above we have

Theorem 3.19. *(1) There exists a* $(v,4,3)$*-RPMD for* $v \equiv 1 \pmod 4$*; (2) There exists a* $(4t,4,3)$*-RPMD for* $t \in \{2,3,4,5,6,7,8,11,12,13,17,21,22,28,31,33,35,38,41,46,53,56,57,59,61,67,71,73,76,77\}$.

Theorem 3.20. *There exists a* $(4t,4,3)$*-RPMD for* $1 \le t \le 80$ *and* $t \ne 1$.

Proof: From Theorem 3.19, it is clear that there exists a $(v,4,3)$-RPMD for v shown in Table A by applying Theorem 2.3 and 2.14. This completes the proof. ∎

Table A

$4 \cdot 9 = 5 \cdot 7 + 1$	$4 \cdot 10 = 8 \cdot 5$	$4 \cdot 14 = 5 \cdot 11 + 1$	$4 \cdot 15 = 12 \cdot 5$
$4 \cdot 16 = 8 \cdot 8$	$4 \cdot 18 = 8 \cdot 9$	$4 \cdot 19 = 5 \cdot 15 + 1$	$4 \cdot 20 = 16 \cdot 5$
$4 \cdot 23 = 13 \cdot 7 + 1$	$4 \cdot 24 = 12 \cdot 8$	$4 \cdot 25 = 20 \cdot 5$	$4 \cdot 26 = 8 \cdot 13$
$4 \cdot 27 = 12 \cdot 9$	$4 \cdot 29 = 5 \cdot 23 + 1$	$4 \cdot 30 = 17 \cdot 7 + 1$	$4 \cdot 32 = 8 \cdot 16$
$4 \cdot 34 = 8 \cdot 17$	$4 \cdot 36 = 16 \cdot 9$	$4 \cdot 37 = 21 \cdot 7 + 1$	$4 \cdot 39 = 12 \cdot 13$
$4 \cdot 40 = 8 \cdot 20$	$4 \cdot 42 = 8 \cdot 21$	$4 \cdot 43 = 9 \cdot 19 + 1$	$4 \cdot 44 = 5 \cdot 35 + 1$
$4 \cdot 45 = 36 \cdot 5$	$4 \cdot 47 = 17 \cdot 11 + 1$	$4 \cdot 48 = 8 \cdot 24$	$4 \cdot 49 = 5 \cdot 39 + 1$
$4 \cdot 50 = 8 \cdot 25$	$4 \cdot 51 = 12 \cdot 17$	$4 \cdot 52 = 16 \cdot 13$	$4 \cdot 54 = 24 \cdot 9$
$4 \cdot 55 = 44 \cdot 5$	$4 \cdot 58 = 8 \cdot 29$	$4 \cdot 60 = 12 \cdot 20$	$4 \cdot 62 = 13 \cdot 19 + 1$
$4 \cdot 63 = 12 \cdot 21$	$4 \cdot 64 = 17 \cdot 15 + 1$	$4 \cdot 65 = 20 \cdot 13$	$4 \cdot 66 = 8 \cdot 33$
$4 \cdot 68 = 16 \cdot 17$	$4 \cdot 69 = 25 \cdot 11 + 1$	$4 \cdot 70 = 56 \cdot 5$	$4 \cdot 72 = 8 \cdot 36$
$4 \cdot 74 = 8 \cdot 37$	$4 \cdot 75 = 13 \cdot 23 + 1$	$4 \cdot 78 = 24 \cdot 13$	$4 \cdot 79 = 5 \cdot 63 + 1$
$4 \cdot 80 = 64 \cdot 5$			

The following result is contained in [10, Lemma 4.1].

Lemma 3.21. *If* $N(n) \ge 15$*, then there exists a* $GDD[K,1,M;v]$ *of type* $n^{15}(n+4m_1)^1\ (n+4m_2)^1$ *where* $K = \{5,17\}$, $0 \le m_1, m_2 \le n$ *and* $v = 17n + 4m_1 + 4m_2$.

Theorem 3.22. *There exists a* $(4t,4,3)$*-RPMD for* $t \ge 81$.

Proof: We define RPMD $[4,3] = \{v \mid$ there exists a $(v,4,3)$-RPMD $\}$.

Taking $n = 19$, from Lemma 3.21 we have a GDD $[K, 1, M; v]$ of type 19^{15} $(19 + 4m_1)^1$ $(19 + 4m_2)^1$ where $K = \{5, 17\}$, $0 \le m_1, m_2 \le 19$ and $v = 17 \cdot 19 + 4(m_1 + m_2)$, then it is clear that if $19 + 4m_1 + 1$, $19 + 4m_2 + 1 \in$ RPMD $[4, 3]$ there exists a $(v + 1, 4, 3)$-RPMD by Theorem 2.18 and 2.10. Since $19 + 4m_1 + 1$, $19 + 4m_2 + 1 \in$ RPMD $[4, 3]$ for $0 \le m_1, m_2 \le 19$ and $m_1, m_2 \ne 2, 28$ from Theorem 3.20, and $\{s \mid s = m_1 + m_2, 0 \le m_1, m_2 \le 19$ and $m_1, m_2 \ne 2, 28\} = \{s \mid 0 \le s \le 38\}$. Therefore, there exists a $(4t, 4, 3)$-RPMD for $81 \le t \le 119$.

Taking $n = 23$, similarly, we have $23 + 4m_1 + 1$, $23 + 4m_2 + 1 \in$ RPMD $[4, 3]$ for $0 \le m_1, m_2 \le 23$ and $m_1, m_2 \ne 1, 27$, and $\{s \mid s = m_1 + m_2$ $0 \le m_1, m_2 \le 23$ and $m_1, m_2 \ne 1, 27\} \supset \{s \mid 2 \le s \le 46\}$. Therefore, there exists a $(4t, 4, 3)$-RPMD for $98 + 2 \le t \le 144$.

Similarly, taking $n = 31, 43, 59, 79, 103, 127, 179, 199, 271, 311, 383, 503, 719, 1019, 1427, 1831$, we can obtain $4t \in$ RPMD $[4, 3]$ for $132 \le t \le 11444$.

Similarly taking $n = 3^s$, $3^{s-3} \cdot 31$, $3^{s-4} \cdot 113$, $3^{s-3} \cdot 47$, $3^{s-3} \cdot 59$, $3^{s-3} \cdot 79$, $3^{s-3} \cdot 87$, $3^{s-3} \cdot 127$, $3^{s-1} \cdot 19$, $3^{s-1} \cdot 23$, 3^{s+2}, for $s = 7, 9, 11, \ldots$ we can obtain $4t \in$ RPMD $[4, 3]$ for $t \ge 11444$. ∎

Lemma 3.23. *There does not exist a* $(4, 4, \lambda)$ *-MD for any odd* λ *(see [4, Lemma 4.3]).*

Summarizing the results given above we readily obtain the following results.

Theorem 3.24. *The necessary condition for the existence of a* $(v, 4, \lambda)$ *-RPMD for* $\lambda = 3$, *namely,* $v \equiv 0$ *or* $1 \pmod 4$ *is also sufficient, except for* $v = 4$. *There exists a* $(v, 4, 1)$ *-RPMD for every positive integer* $v = 1 \pmod 4$.

Combining Theorem 3.6, 3.24, and 3.18, the main result of this paper can be summarized in the following theorem.

Theorem 3.25. *The necessary condition for the existence of a* $(v, 4, \lambda)$ *-RPMD for* $\lambda > 1$, *namely,* $v \equiv 0$ *of* $1 \pmod 4$, *is also sufficient with the exception of pairs* (v, λ) *where* $v = 4$ *and* λ *odd.*

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