NASH-WILLIAMS CONDITIONS AND THE EXISTENCE OF k-FACTORS

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ABSTRACT. **Sufficient conditions depending on the minimum degree and the independence number of a simple graph for the existence of a k-factor are established.**

1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard unless indicated otherwise. A good reference for any undefined terms is [1].

The set of vertices of a graph G is denoted by V. If v is a vertex of G, then the neighbourhood $N_G(v)$ of v is the set of all vertices in V adjacent to v, and the degree $d_G(v)$ of v is $|N_G(v)|$. We use δ for the minimum degree, and α to denote the independence number. A spanning subgraph H of G is called k-factor, if $d_H(v) = k$ for all $v \in V$. If G and H are disjoint graphs, then the union is denoted by $G \cup H$ and the join by $G + H$.

Nash-Williams [4] proved the following sufficient condition for a Hamiltonian circuit, which is also sufficient for a 2-factor.

Theorem 1. *Let G be a 2-connected graph with n vertices. If G satisfies $\delta \geq \frac{1}{3}(n+2)$ and $\delta \geq \alpha$, then G is Hamiltonian.*

The aim of this paper is to prove sharp sufficient conditions for the existence of k-factors depending on the minimum degree and the independence number. Our main theorem is

Theorem 2. *Let $k \geq 2$ be an integer and G be a graph with n vertices. If k is odd, then suppose that n is even and G is connected. Let G satisfy*

$$n > 4k + 1 - 4\sqrt{k+2}\,, \tag{1}$$

$$\delta \geq \frac{k-1}{2k-1}(n+2) \quad \textit{and} \tag{2}$$

(3) $$\delta > \frac{1}{2k-2}\Big((k-2)n + 2\alpha - 2\Big).$$

Then G has a k-factor.

Inequality (3) of the theorem contains the minimum degree, the independence number and, if $k \geq 3$, the number of vertices. This condition can be replaced by a condition only depending on minimum degree and independence number.

Theorem 3. *Theorem 2 remains true, if (3) is replaced by*

(4) $$k\delta > 2(k-1)\alpha - 2(k-1)^2.$$

Recently, Tokushige [6] and Woodall [8] proved independently the following closely related theorem, where Woodall used instead of (1) the slightly weaker $n \geq 4k - 6$.

Theorem 4. *The conclusion of Theorem 2 is true, if (3) is replaced by: For every non-empty independent set $X \subseteq V$ the inequality*

(5) $$|N_G(X)| \geq \frac{1}{2k-1}\Big((k-1)n + |X| - 1\Big)$$

holds.

We can not use this result to prove our theorems, because Theorem 2 or Theorem 3 and Theorem 4 do not imply each other (see Section 3). But we will use a major part of their proofs (cf. Theorem 5 below).

2. Proofs

Theorem 2 implies Theorem 3. Let G be a graph satisfying the hypotheses of Theorem 3. Using (2) and (4) we get (3) as follows:

$$\begin{aligned}
\delta &= \frac{(k-2)(2k-1)}{2(k-1)^2}\delta + \frac{k\delta}{2(k-1)^2} \\
&> \frac{k-2}{2(k-1)}(n+2) + \frac{2(k-1)\alpha - 2(k-1)^2}{2(k-1)^2} \\
&= \frac{1}{2k-2}\Big((k-2)n + 2\alpha - 2\Big).
\end{aligned}$$

So G has a k-factor by Theorem 2. ∎

To prove Theorem 2 we need the following version of Theorem 4.

Theorem 5. *Let $k \geq 2$ be an integer and G be a graph with n vertices. If k is odd, then suppose that n is even and G is connected. Let G satisfy (1) and (2). Then G has a k-factor or there exist disjoint non-empty sets $A,B \subset V$ such that*

$$\omega \geq k|A| - k|B| + \sum_{v \in B} d_{G\setminus A}(v) + 2, \tag{6}$$

where ω denotes the number of components of $G \setminus (A \cup B)$ with at least three vertices, and

$$X := \{x \in B \mid d_{G\setminus A}(x) = 0\} \neq \emptyset. \tag{7}$$

In fact, this version was not stated in Tokushige's or Woodall's paper, but their proofs, which are based on Tutte's factor theorem [7], use (5) only to show that the situation described by (6) and (7) cannot occur.

Proof of Theorem 2. The proof is by contradiction. Let G be a graph satisfying the hypotheses of Theorem 2, which has no k-factor. By Theorem 5 there exist disjoint non-empty sets $A,B \subset V$ such that (6) and (7) hold. By (7) we get

$$|A| \geq \delta\,. \tag{8}$$

Let $Y := \{v \in B \mid d_{G\setminus A}(v) = 1\}$. Next we show

$$\alpha \geq \omega + |X| + \frac{1}{2}|Y|\,. \tag{9}$$

To see this, let $H_1,\ldots,H_{\omega_1}$ be the components of $G \setminus (A \cup B)$ with at least three vertices, which have a vertex without neighbour in Y. Let S_1 be a set containing one such vertex from every H_i, for $i = 1,\ldots,\omega_1$. The further components of $G \setminus (A \cup B)$ with at least three vertices are denoted by $F_1,\ldots,F_{\omega_2}$.

Furthermore let $Y_1 := \{v \in Y \mid N_{G\setminus A}(v) \subseteq B\}$ and $Y_2 := Y \setminus Y_1$. Then the graph induced by Y_1 in G has maximum degree at most 1. Let S_2 be a maximum independent set of this graph. Clearly, S_2 has cardinality at least $\frac{1}{2}|Y_1|$.

Since every vertex of every F_i has a neighbour in Y_2 and since these neighbours are necessarily distinct, we have

$$|Y_2| \geq \sum_{i=1}^{\omega_2} |V(F_i)| \geq 3\omega_2 \geq 2\omega_2\,. \tag{10}$$

By our definitions the set $X \cup S_1 \cup S_2 \cup Y_2$ is an independent set of G. Using (10) this yields (9) as follows:

$$\begin{aligned} \alpha &\geq |X| + |S_1| + |S_2| + |Y_2| \\ &\geq |X| + \omega_1 + \frac{1}{2}|Y_1| + \frac{1}{2}|Y_2| + \omega_2 \\ &= |X| + \omega + \frac{1}{2}|Y| . \end{aligned}$$

Now, applying (9), (6) and (8), we get

$$\begin{aligned} \alpha &\geq \omega + |X| + \frac{1}{2}|Y| \\ &\geq k|A| - k|B| + \sum_{v \in B} d_{G \setminus A}(v) + 2 + |X| + \frac{1}{2}|Y| \\ &\geq k\delta - k|B| + \sum_{v \in B \setminus (X \cup Y)} d_{G \setminus A}(v) + 2 + |X| + \frac{3}{2}|Y| \\ &\geq k\delta - k|B| + 2|B \setminus (X \cup Y)| + 2 + |X| + \frac{3}{2}|Y| \\ &= k\delta - k|B| + 2|B| + 2 - \left(|X| + \frac{1}{2}|Y|\right) \\ &\geq k\delta - (k-2)|B| + 2 - \alpha \end{aligned}$$

and so

$$(11) \qquad (k-2)|B| \geq k\delta - 2\alpha + 2 .$$

If $k = 2$, then (11) is equivalent to $\alpha \geq \delta + 1$, contradicting (3). If $k \geq 3$, then we get by (8), (11) and (3)

$$\begin{aligned} 0 &\leq |V \setminus (A \cup B)| = n - |A| - |B| \\ &\leq n - \delta - \frac{1}{k-2}(k\delta - 2\alpha + 2) \\ &= n - \frac{1}{k-2}\Big((2k-2)\delta - 2\alpha + 2\Big) \quad < \quad 0 . \end{aligned}$$

This contradiction completes the proof of Theorem 2. ∎

3. Remarks

We first verify that Theorem 2 (or Theorem 3, respectively) and Theorem 4 do not imply each other.

For $k \geq 2$ fixed, we choose integers r and p, such that $r \geq 4k$ and p is even with

$$\frac{2(k-1)}{k}(r+2) \leq p < \frac{2k-1}{k}r + \frac{2k-3}{k} .$$

Such an integer p exists, since

$$\frac{2k-1}{k}r + \frac{2k-3}{k} - \frac{2(k-1)}{k}(r+2) = \frac{1}{k}(r-2k+1) \geq \frac{1}{k}(2k+1) > 2\,.$$

The reader may verify that the graph $K_p + (rK_1 \cup K_{r+2})$ has a k-factor by Theorem 2 or Theorem 3, but for $X = V(rK_1)$ condition (5) is not fulfilled.

Now we consider examples of the form $K_p + rK_{1,2}$. We choose $r \geq 4k$ and p such that $p + r$ is even and

$$\frac{1}{k}(3kr - 3r - 1) \leq p \leq \frac{1}{k}(3kr - 2r - 2k)\,.$$

This is possible, since the right-hand-side minus the left-hand-side is equal to $(r - 2k + 1)/k \geq 2$. Here the reader can verify that such a graph has a k-factor by Theorem 4, but neither condition (3) nor condition (4) is satisfied.

To see that the conditions of Theorem 2 are best possible, we consider graphs of the form

$$K_{r+2(pk-p-1)} + \left(rK_1 \cup (pk-1)K_2\right),$$

where $r \geq 0$ and $p \geq k$. Woodall [8] has shown that these graphs have no k-factor. If $r = 0$ or $r = 2k$, respectivily, these graphs show that (2) or (3), respectivily, is best possible. Examples showing that (1) is needed can be found in [6].

The occurence of a condition like (1) is somewhat surprising, and so it is a natural question, what happens, if (1) is not satisfied. In this case the situation changes dramatically. Clearly, it is possible to require stronger conditions depending on minimum degree and independence number, but, remarking the fact that

$$\delta > n + 2k - 2\sqrt{kn+2} \tag{12}$$

always guarantees a k-factor (see [2],[8]), a minimum degree condition has to be weaker than (12). However, the independence number of the graphs given by Katerinis and Woodall [3], which show that (12) is best possible, depends only on $\frac{n}{k}$. So, compared with Theorem 2, where $\alpha \leq \frac{n}{4}$ always suffices, a condition for the independence number has to be very restrictive.

Finally, we have not considered 1-factors, because it seems that this case does not fit into the pattern of Theorem 2. Using a different method the author obtained in [5] the following lower bound for the size of a maximum matching as a corollary.

Theorem 6. *Let G be a connected graph of even order n and let m be an odd integer. If $m\,\delta \geq n$ holds, then the number of unsaturated vertices of a maximum matching is at most* $\max\{m-3, m-4+\alpha-\delta\}$.

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