A REMARK ON BENZ PLANES OF ORDER 9

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Abstract. Using the explicit determination of all ovals in the 3 non-desarguesian projective planes of order 9 given in [4] or [8], we prove that there are no other Benz planes of order 9 than the three miquelian planes and the Minkowski plane over the Dickson near-field of type $\{3,2\}$.

1. Introduction.

A Benz plane, whose algebraic properties have been studied in [2], comprises three types of circle planes: Möbius planes (or inversive planes); Laguerre planes; and Minkowski planes.

The miquelian Möbius plane, Laguerre plane, and Minkowski plane of order q (q being a prime power) is obtained as the geometry of nontrivial plane sections of an elliptic quadric, an elliptic cone, or a ruled quadric, respectively, in the 3-dimensional projective space over $GF(q)$.

In general, a Benz plane $\mathcal{B} = (P, \mathcal{K}, \mathcal{E})$ consists of a set P of points, a set $\mathcal{K}$ of circles (considered as subsets of P) and a set $\mathcal{E}$ of equivalence relations on P (parallelism) such that the following axioms hold (two points $p, q \in P$ are called parallel if and only if they are in relation pRq for some $R \in \mathcal{E}$; otherwise they are called nonparallel):

(B1) Any three pairwise nonparallel points can be joined uniquely by a circle passing through these points.

(B2) To every circle K and any two nonparallel points p, q where $p \in K$ and $q \notin K$ there is precisely one circle L which is tangential to K at p (that is, $K \cap L = \{p\}$) and passes through q.

(B3) For any parallelism every parallel class intersects any circle in a unique point.

(B4) Any two parallel classes to different parallelisms intersect in a unique point.

(B5) There are four pairwise nonparallel noncircular points.

In the case of a Laguerre plane $\mathcal{E}$ consists of precisely one equivalence relation; so axiom (B4) does not apply for Laguerre planes. The parallel class of $p \in P$ will be denoted by $|p|$.

In the case of a Minkowski plane $\mathcal{E}$ consists of precisely two equivalence relations called $(+)-$ and $(-)$-parallelism; the corresponding parallel classes of $p \in P$ will be denoted by $|p|_+$ and $|p|_-$, respectively.

In order to fit Möbius planes into this general setting we take $\mathcal{E}$ the empty set and we have to read nonparallel points as distinct points. In particular, the axioms (B3) and (B4) do not apply in this case.

If P is finite, any two circles have the same number $n+1$ of points, and n is called the order of $\mathcal{B}$. There are n^2+1 points and $n(n^2+1)$ circles in a Möbius plane of order n; there are n^2+n points, n^3 circles, and $n+1$ parallel classes in a Laguerre plane of order n; there are $(n+1)^2$ points, $n(n^2-1)$ circles and $n+1$ parallel classes of either type in a Minkowski plane of order n.

For every point $p \in P$ there is an internal incidence structure, whose point set consists of all points of P not parallel to p and whose set of lines consists of all circles containing p (without the point p), and all parallel classes not passing through p; this is an affine plane, the derived affine plane $\mathcal{A}_p$ at p. We call the projective closure of $\mathcal{A}_p$ the derived projective plane $\mathcal{P}_p$ at p. If $\mathcal{B}$ has order n then the derived affine plane and derived projective plane are also of order n. A circle K not passing through p induces an oval in $\mathcal{P}_p$ by

- $K \subset P\backslash\{p\}$ (Möbius plane); in particular the infinite line of $\mathcal{P}_p$ (with respect to $\mathcal{A}_p$) is an exterior line to this oval.
- $(K\backslash|p|) \cup \{\omega\}$, where ω denotes the infinite point of lines that come from parallel classes (Laguerre plane); in particular the infinite line of $\mathcal{P}_p$ (with respect to $\mathcal{A}_p$) is a tangent to this oval at ω.
- $(K\backslash(|p|_+ \cup |p|_-)) \cup \{\omega_+, \omega_-\}$, where ω_+ and ω_- denote the infinite point of lines that come from $(+)$-parallel classes or $(-)$-parallel classes, respectively, (Minkowski plane); in particular, the infinite line of $\mathcal{P}_p$ (with respect to $\mathcal{A}_p$) is a secant to this oval.

According to the celebrated theorem of Segre [10] an oval in a finite desarguesian projective plane of odd order is a quadric. Chen and Kaerlein proved in [3] by simply counting the possibilities of quadrics having a given tangent at a given point or passing through two given points that a finite Laguerre plane or Minkowski plane of odd order having at least one desarguesian derived projective plane is miquelian. Thas [11] achieved a similar result for finite Möbius planes of odd order $q \notin \{11, 23, 59\}$ using a completely different method. In particular, a Benz plane of odd order ≤ 7 is miquelian. However, there are non-desarguesian projective planes of order 9, and a computer search in [7] confirmed recently that there are only 4 nonisomorphic projective planes of order 9. Besides the desarguesian plane $\mathcal{D}_9$ of order 9, these are the translation plane $\mathcal{T}$ (Hall plane), its dual $\mathcal{T}^*$, and the Hughes plane $\mathcal{H}$ of order 9 (a brief description of these planes will be given in Chapter 2).

In [5] Denniston proved that none of the three non-desarguesian projective planes of order 9 gives rise to a Möbius plane of order 9 although these planes contain many ovals. In this note we answer the corresponding question for Laguerre planes and Minkowski planes of order 9. As in [5] we exploit the explicit determination of all ovals in the 3 non-desarguesian projective planes of order 9 given in [4] or [8]. We prove:

Theorem A. *No derived projective plane of a Laguerre plane of order 9 is isomorphic to one of the non-desarguesian projective planes $\mathcal{T}$, $\mathcal{T}^*$, or $\mathcal{H}$ of order 9.*

Besides the miquelian Minkowski plane of order 9 there is a nonisomorphic Minkowski plane $\mathcal{M}^\#$ which is constructed quite similar to the miquelian model over the planar Dickson near-field of order 9 (a description of this plane will be given in 4.6). Similar as for Laguerre planes of order 9 we ask whether $\mathcal{M}^\#$ is the only Minkowski plane that can be constructed from the three non-desarguesian projective planes of order 9. We prove:

Theorem B. *No derived projective plane of a Minkowski plane $\mathcal{M}$ of order 9 is isomorphic to the dual translation plane $\mathcal{T}^*$ or the Hughes plane $\mathcal{H}$. If at least one derived projective plane of $\mathcal{M}$ is isomorphic to the translation plane $\mathcal{T}$, then $\mathcal{M}$ is isomorphic to $\mathcal{M}^\#$ (and each derived projective plane is isomorphic to $\mathcal{T}$).*

Together with the result of Denniston [5] on Möbius planes of order 9, the two theorems can be summarized into

Theorem C. *A Möbius plane or a Laguerre plane of order 9 is miquelian. A Minkowski plane of order 9 is either miquelian or isomorphic to $\mathcal{M}^\#$.*

2. Three non-desarguesian projective planes of order 9 and their ovals.

2.1. The translation plane $\mathcal{T}$ (Hall plane)

For detailed information about this plane we refer to [9, Section 4], [8, Section 2], [8, Section 2], or [1]. Since $\mathcal{T}$ occurs as derived projective plane of a Minkowski plane, we include a brief description of this plane.

In the Galois field $F = GF(9)$ of order 9 let

$$\sigma: F \to F: x \mapsto x^3$$

denote the Frobenius automorphism and let $F^2 = \{x^2 \mid x \in F\}$ be the set of squares in F. The field F can be described as the quadradic field extension of $F_0 = GF(3)$ obtained by adjoining an element i, $i^2 = -1$; the members of F then can be written in the form $a + bi$ where $a,b \in F_0$, and F is a 2-dimensional vector space over F_0 with basis $\{1, i\}$. In this notation σ becomes $a + bi \mapsto a - bi$. We keep the addition of F and define a new multiplication $\circ$ by

$$a \circ b = \begin{cases} ab & \\ & \text{for} \\ a\sigma(b) & \end{cases} \begin{matrix} a \in F^2 \\ \\ a \notin F^2. \end{matrix}$$

Then $(F,+,\circ)$ is a near-field, the Dickson near-field of type $\{3,2\}$. Its automorphism group $\Phi = \text{Aut}\,(F,+,\circ)$ consists of all F_0-linear mappings of the 2-dimensional vector space F over F_0 that fix all members of $F_0 \cdot 1$; hence, Φ comprises 6 automorphisms. In particular, $\sigma \in \Phi$.

Let $A = F \times F$ and let $L_a = \{\{(x, m \circ x + t) \mid x \in F\} \mid m,t \in F\} \cup \{\{(c,y) \mid y \in F\} \mid c \in F\}$; then (A, L_a) is an affine plane (compare [1]) and all mappings

$$(x,y) \mapsto (x+a, y+b)(a,b \in F)$$

are collineations of (A, L_a) each of which fixes one parallel bundle linewise. Thus, the projective closure is a translation plane of order 9 having the infinite line as translation line.

The whole collineation group Γ of $\mathcal{T}$ has order $2^8 \cdot 3^5 \cdot 5 = 311,040$. In the given representation Γ is the semidirect product of the translation group with the stabilizer Γ_o of $o = (0,0)$ where Γ_o is generated by the mappings

$$(x,y) \mapsto (r \circ x, s \circ y)(r,s \in F\{0\})$$
$$(x,y) \mapsto (y,x)$$
$$(x,y) \mapsto (x+y, x-y)$$
$$(x,y) \mapsto (\varphi(x), \varphi(y))(\varphi \in \Phi = \text{Aut}(F,+,\circ))$$

In particular, Γ fixes the infinite translation line. The points on the translation line can be partitioned into five special pairs which are invariant under every collineation, that is, a collineation maps a special pair of points onto a special pair. In the given description of $\mathcal{T}$ the special pairs are $\{(0),(\infty)\}$, $\{(m),(-m)\}$, $m \in F\backslash\{0\}$, where (m) denotes the infinite point on the line $y = m \circ x$ for $m \ddagger \infty$ or the Y-axis for $m = \infty$.

It is easy to see, that

$$\{(x, x^{-1}) \mid x \in F, x \ddagger 0\} \cup \{(0),(\infty)\}$$

is an oval $\mathcal{O}$ in $\mathcal{T}$ (here x^{-1} denotes the inverse of x in the original multiplication $\cdot$ of F). According to [4, Section 8] or [8, 2.4] the orbit $\Gamma(\mathcal{O})$ of the oval $\mathcal{O}$ under the full collineation group Γ comprises all ovals in $\mathcal{T}$. Moreover, the stabilizer of $\mathcal{O}$ consists of 32 collineations and there are $2^3 \cdot 3^5 \cdot 5 = 9720$ ovals in $\mathcal{T}$. As $\mathcal{O}$ contains the infinite points (0) and (∞), every oval intersects the infinite line in a special pair of points.

2.2. The dual translation plane $\mathcal{T}^\star$

This plane is obtained by dualisation of the translation plane $\mathcal{T}$. All results mentioned in 2.1 carry over accordingly to $\mathcal{T}^\star$. In particular, the collineation group $\Gamma^\star$ of $\mathcal{T}^\star$ fixes the translation point τ. Corresponding to special pairs of points on the translation line in $\mathcal{T}$, the lines through τ can be partitioned into five special pairs of lines.

All ovals in $\mathcal{T}^*$ are obtained by dualisation of ovals in $\mathcal{T}$, and they are all in the same orbit. No oval contains the point τ and to every oval there are precisely two tangents through τ and these tangents form a special pair of lines.

2.3. The Hughes plane $\mathcal{H}$

For detailed information about this plane we refer to [9, Section 5], [8, Section 3], or [6]. We list some properties of $\mathcal{H}$ as far as they are needed in the proof of our theorems.

This plane of order 9 contains a distinguished Baer subplane $\mathcal{J} = (R, \mathcal{R})$ of order 3. In particular, $\mathcal{J}$ is desarguesian. Points of this subplane will be called real points and the remaining points will be called complex points. Similarly, lines that pass through two real points are called real lines and those lines that pass through precisely one real point will be called complex lines. By definition of a Baer subplane, every complex line contains precisely one real point and dually through every complex point passes precisely one real line.

Every collineation of $\mathcal{H}$ fixes $\mathcal{J}$ and the full collineation group $PGL(3,3)$ of the desarguesian plane $\mathcal{D}_3$ of order 3 is induced on $\mathcal{J}$. Furthermore, the collineation group Γ of $\mathcal{H}$ has a subgroup of order six whose members fix every real point and Γ has order $6 \cdot 5616 = 33,696$. There are two orbits of ovals in $\mathcal{H}$ (see [4, Section 8] or [8, 3.3]). One type of ovals, which we accordingly call real ovals, have 4 real points and 6 complex points. The other orbit consists of ovals that have 10 complex points (and no real points); we call these ovals complex ovals. Since a real oval is stabilized by 48 collineations, there are 702 of them in $\mathcal{H}$; similarly, there are 2106 complex ovals in $\mathcal{H}$ as each such oval is stabilized by 16 collineations.

Obviously, the 4 real points of a real oval form an oval in $\mathcal{J}$. As $\mathcal{J}$ is desarguesian, these real points describe a quadric by the Theorem of Segre. In the desarguesian projective plane $\mathcal{D}_3$ of order 3 there are 234 quadrics (there are 18 quadrics through a given point with a given tangent at that point and there are $13 \cdot 4$ flags; since a quadric has 4 points, each quadric appears four times in the above count of flags and so there are $18 \cdot 13 = 234$ quadrics in $\mathcal{D}_3$), so each such quadric extends to 3 different real ovals in $\mathcal{H}$.

3. Proof of Theorem A.

Throughout this chapter $\mathcal{L}$ denotes a Laguerre plane of order 9. Then $\mathcal{L}$ has $10 \cdot 9 = 90$ points and $9 \cdot 9 \cdot 9 = 729$ circles. As mentioned in the introduction circles not passing through the point x of derivation induce ovals in the derived projective plane $\mathcal{P}$ at x and all these ovals contain the point ω and have the infinite line W as a tangent at ω. So there must be $8 \cdot 9 \cdot 9 = 648$ ovals in $\mathcal{P}$ that come from circles of $\mathcal{L}$. In the sequel we examine up to isomorphism all possibilities of flags (ω, W) in each of the 3 non-desarguesian projective planes of order 9. Using the determination of ovals in these planes we decide whether there are enough ovals through ω and having tangent W at ω to constitute a Laguerre plane.

3.1. Suppose that a derived projective plane of $\mathcal{L}$ is isomorphic to the translation plane $\mathcal{T}$: Since the translation line T of $\mathcal{T}$ is a secant to any oval in $\mathcal{T}$, this line cannot be the line W (by definition W is a tangent to ovals that come from circles of $\mathcal{L}$). If $\omega \in \mathcal{T}$, then each oval in $\mathcal{T}$ which contains ω passes also through the second point ω' of the special pair containing ω; since T stems from a parallel class in $\mathcal{L}$, there must be, however, ovals through ω and points $\in T\{\omega, \omega'\}$ (see axiom (B1)). If finally $\omega \notin T$, then T comes from a circle of $\mathcal{L}$; hence, there must be an oval for which T is a tangent (see axiom (B2)) contrary to the fact that T is a secant to all ovals in T.

3.2. Suppose that a derived projective plane of $\mathcal{L}$ is isomorphic to the dual translation plane $\mathcal{T}^*$: Since there passes no oval through the translation point τ of $\mathcal{T}^*$, this point cannot be the point ω. Similarly, if W does not pass through τ, the point τ is a finite point and comes from a point of $\mathcal{L}$; however, by the Laguerre axiom (B1) there must be an oval through τ — a contradiction. If finally $\tau \in W$, then each oval through ω having W as tangent at ω must also have the second line W' through τ of the special pair containing W as tangent; however, by the Laguerre axiom (B1) the line W' must be a secant to some oval.

We now suppose that a derived projective plane of $\mathcal{L}$ is isomorphic to the Hughes plane $\mathcal{H}$: We maintain the notation of 2.3 and denote the Baer subplane of real points and real lines of $\mathcal{H}$ by $\mathcal{Y} = (R, \mathcal{R})$; the plane $\mathcal{Y}$ is isomorphic to the desarguesian projective plane $\mathcal{D}_3$ of order 3. We distinguish 3 cases:

- ω is a real point;
- ω is a complex point and W is a real line;
- ω is a complex point and W is a complex line.

3.3. Suppose that ω is a real point. In this case all ovals that come from circles of $\mathcal{L}$ are real. The intersection of such an oval with R defines a quadric in the projective plane $\mathcal{Y} \cong \mathcal{D}_3$. In $\mathcal{D}_3$ there are precisely 234 quadrics (compare 2.3). As a circle is uniquely determined by 3 of its points, each oval that comes from a circle is uniquely determined by its real points and, hence, there are at most 234 circles. However, in $\mathcal{L}$ there are 648 circles that appear as ovals in a derived projective plane — a contradiction.

3.4. Suppose that ω is a complex point and W is a real line. In this case W is the unique real line through ω; any other line through ω (= parallel classes) intersects R in precisely one point. Thus, the points in $R\backslash W$ are pairwise nonparallel and any three points in $R\backslash W$ can be uniquely joined by a circle. Furthermore, to each circle K having 3 real points and any two real points x, y with $x \in K$ there is precisely one circle tangential to K at x through y; this circle induces a real line or a real oval. Hence, the circles that induce real ovals determine a Möbius plane of order 3 on $R\backslash W$ (plus an additional point, such that $R\backslash W$ is the derived affine plane at this point). We choose two different points $x, y \in R\backslash W$. In $\mathcal{L}$ there are 9 circles through x, y; thus, there are 8 real ovals through x, y in $\mathcal{H}$. These ovals are uniquely determined by their real part. However, in an inversive plane of order 3 there are only 4 circles through two points — a contradiction.

3.5. Suppose that ω is a complex point and W is a complex line. Then there is precisely one real line L through ω. Because W is complex, L and W are distinct; as L passes through ω, the line L comes from a parallel class of $\mathcal{L}$. We choose two real points $x \in L \cap R$ and $y \in R\backslash L$. Then x and y are nonparallel and in $\mathcal{L}$ there are 9 circles through x and y; thus, there must be 8 ovals in $\mathcal{H}$ through the two points and these ovals must be real. In $\mathcal{Y}$ the intersection $L \cap R$ is a tangent to the real part of these ovals. We count the number of quadrics in $\mathcal{D}_3$ that pass through two given points and have at one of them a given line as tangent. To do so we coordinatize $\mathcal{D}_3$ in such a way that x becomes the infinite point of the Y-axis, the tangent becomes the infinite line, and y becomes the point $(0,0)$. The quadrics we are interested in then are parabolae and have the special form $\{(t, a\cdot t^2 + b\cdot t) \mid t \in GF(3)\} \cup \{x\}$, where $a, b \in GF(3)$, $a \neq 0$. Hence, there are only 6 quadrics, and since its real points determine the whole circle, there can be only 6 ovals in $\mathcal{H}$ through x and y coming from circles, contradicting the fact that there are 8 such ovals.

3.6. We have shown that no derived projective plane of $\mathcal{L}$ can be isomorphic to $\mathcal{T}$, $\mathcal{T}^*$, or $\mathcal{H}$. If at least one derived projective plane of $\mathcal{L}$ is desarguesian, then $\mathcal{L}$ is miquelian according to [3].

4. Proof of Theorem B

Throughout this chapter $\mathcal{M}$ denotes a Minkowski plane of order 9. Then $\mathcal{M}$ has $10 \cdot 10 = 100$ points and $10 \cdot 9 \cdot 8 = 720$ circles. As mentioned in the introduction circles not passing through the point x of derivation induce ovals in the derived projective plane $\mathcal{P}$ at x and all these ovals contain the two points ω_+ and ω_- on the infinite line W (thus, W must be a secant to any oval that comes from a circle of $\mathcal{M}$). So there must be $9 \cdot 9 \cdot 8 = 648$ ovals in $\mathcal{P}$ that come from circles of $\mathcal{M}$. In the sequel we examine up to isomorphism all possibilities of pairs (ω_+, ω_-) in each of the 3 non-desarguesian projective planes of order 9. Using the determination of ovals in these planes we decide whether there are enough ovals through ω_+ and ω_- to constitute a Minkowski plane.

4.1 Suppose that a derived projective plane of $\mathcal{M}$ is isomorphic to the dual translation plane $\mathcal{T}^*$: Since there passes no oval through the translation point τ of $\mathcal{T}^*$ this point can be neither the point ω_+ nor ω_-. Similarly, if W does not pass through τ, the point τ is a finite point and comes from a point of $\mathcal{M}$, and by the Minkowski axiom (B1) there must be an oval through τ — a contradiction. If finally $\tau \in W$, $\tau \neq \omega_+, \omega_-$, then we consider the second line W' through τ of the special pair containing W; by the Minkowski axiom (B2) the line W' must be a tangent to some oval $\mathcal{O}$. But then W is also a tangent to $\mathcal{O}$ and $\mathcal{O}$ cannot stem from a circle of $\mathcal{M}$ — a contradiction.

4.2. Suppose that a derived projective plane of $\mathcal{M}$ is isomorphic to the Hughes plane $\mathcal{H}$: We maintain the notation of 2.3 and denote the Baer subplane of real points and real lines of $\mathcal{H}$ by $\mathcal{Y} = (R, \mathcal{R})$. We distinguish 3 cases:

- ω_+ or ω_- is a real point;
- ω_+ and ω_- are complex points and $W = \omega_+\omega_-$ is a complex line;
- ω_+ and ω_- are complex points and $W = \omega_+\omega_-$ is a real line.

4.3. Suppose that ω_+ or ω_- is a real point. Then all ovals that are induced from circles are real. In the desarguesian projective plane $\mathcal{Y}$ of order 3 there are precisely $4 \cdot 18 = 72$ quadrics passing through a fixed point. Since each such quadric extends to 3 real ovals of $\mathcal{H}$, there are $3 \cdot 72 = 216$ ovals in $\mathcal{H}$ that pass through a fixed real point contrary to 648 ovals that must be induced from circles of $\mathcal{M}$.

4.4. Suppose that ω_+ and ω_- are complex points and $W = \omega_+\omega_-$ is a complex line. By definition, there is precisely one real line L_+ and L_- through ω_+ and ω_- respectively. Let p be the real point $L_+ \cap L_-$. This point is finite and comes from a point of $\mathcal{M}$. Thus, there is an oval $\mathcal{O}$ passing through p. The real points $\mathcal{O} \cap R$ of $\mathcal{O}$ describe a quadric in $\mathcal{Y}$ which has two tangents at p (namely, $L_+ \cap R$ and $L_- \cap R$) — a contradiction.

4.5. Suppose that and ω_+ and ω_- are complex points and $W = \omega_+\omega_-$ is a real line. Then the finite real points $R\backslash W$ are pairwise nonparallel, and $\mathcal{M}$ induces a Möbius plane of order 3 on $R\backslash W$. We choose two points $p, q \in R\backslash W$. The circles through p and q in $\mathcal{M}$ induce one real line and 7 real ovals in $\mathcal{H}$. However, in the miquelian Möbius plane of order 3 there are precisely 4 circles through 2 given points, that is, there are 3 real ovals and one real line through p, q. Since each circle through p, q is uniquely determined by its real points, this gives a contradiction.

4.6. Before we consider the last case where one derived projective plane is isomorphic to the translation plane $\mathcal{T}$, we give a description of the Minkowski plane $\mathcal{M}^{\#}$ mentioned in the introduction. We maintain the notation of 2.1. The point space of $\mathcal{M}^{\#}$ is $(F \cup \{\infty\}) \times (F \cup \{\infty\})$, where $\infty \notin F = GF(9)$. Two points (x, y), (u, v) are (+)-parallel if and only if $x = u$, and they are (−)-parallel if and only if $y = v$. Circles are of the form

$$\{(x, m \circ x + t) \mid x \in F\} \cup \{(\infty, \infty)\} \quad (m, t \in F, m \neq 0),$$

or of the form

$$\{(x, a \circ (x - b)^{-1} + c) \mid x \in F, x \neq b\} \cup \{(b, \infty), (\infty, c)\} \quad (a, b, c \in F, a \neq 0),$$

where $^{-1}$ refers to the inverse in the usual multiplication of the field GF(9). Obviously, the derived projective plane at (∞, ∞) is isomorphic to $\mathcal{T}$, the infinite line being the translation line of $\mathcal{T}$ and the points ω_+ and ω_- form a pair of special points on the translation line of $\mathcal{T}$.

4.7. Suppose that a derived projective plane of $\mathcal{M}$ is isomorphic to the translation plane $\mathcal{T}$: we distinguish two main cases whether $\{\omega_+, \omega_-\}$ is a special pair of points on the translation line (this situation occurs in $\mathcal{M}^\#$) or whether it is not. We first show that the latter case cannot occur. If neither ω_+ nor ω_- lies on the translation line T, then T stems from a circle of $\mathcal{M}$. However, by the axiom (B2) of a Minkowski plane, this line must be a tangent to some oval — a contradiction to the fact that all ovals in $\mathcal{T}$ have T as secant. If one of the two points ω_+ is on T and the other is not, say $\omega_+ \in T$ and $\omega_- \notin T$ then T comes from a (+)-parallel class of $\mathcal{M}$. Hence, by the Minkowski axiom (B1), there passes an oval of $\mathcal{T}$ through ω_+ and any point of $T\backslash\{\omega_+\}$ contradicting the fact that each oval in $\mathcal{T}$ intersects T in a special pair of points. For the same reason ω_+ and ω_- must form a special pair of points if they both lie on T.

4.8. Suppose that $\{\omega_+, \omega_-\}$ is a special pair of points on the translation line T of $\mathcal{T}$. Without loss of generality we may assume that ω_+ is the infinite point of the Y-axis and ω_- is the infinite point of the X-axis. These two points are on 1944 ovals in $\mathcal{T}$ which are of the form

$$\mathcal{O}(a,b,c,\varphi) = \{(x, a \circ \varphi((\varphi^{-1}(x-b))^{-1}) + c) \mid x \in F\backslash\{b\}\} \cup \{\omega_+, \omega_-\}$$

where $a, b, c \in F$, $a \neq 0$, and $\varphi \in \Phi = \mathrm{Aut}(F, +, \circ)$. Since $\mathcal{O}(a,b,c,\varphi \circ \sigma) = \mathcal{O}(a,b,c,\varphi)$ we may assume that $Im(\varphi(i)) = i$. In coordinates φ then has the form $\varphi(u+vi) = u + dv + vi$ for some $d \in F_0$ and we, thus, write $\mathcal{O}(a,b,c,d) = \mathcal{O}(a,b,c,\varphi)$. The oval $\mathcal{O}(a,b,c,d)$ has the horizontal line $y = c$ and the vertical line $x = b$ as tangent at ω_+ and ω_- respectively, and we denote the set of all such ovals by $\mathcal{O}_{b,c}$. In particular, a circle of the Minkowski plane through the infinite points (b, ∞) and (∞, c) must be found among $\mathcal{O}_{b,c}$. Moreover, the bundle of circles through (b, ∞) and (c, ∞) in $\mathcal{M}$ induces a bundle $\mathcal{K}_{b,c} \in \mathcal{O}_{b,c}$ of ovals in $\mathcal{T}$ such that any two different ovals in this bundle have no finite points of $\mathcal{T}$ in common, and any two ovals of different bundles $\mathcal{K}_{b,c}$ and $\mathcal{K}_{b',c'}$, $(b,c) \neq (b',c')$ have at most two finite points in common (besides the infinite points ω_+ and ω_-; compare the axiom (B1) of a Minkowski plane). These two properties are carried over to the images $\gamma(\mathcal{K}_{b,c})$ of $\mathcal{K}_{b,c}$ under a collineation γ of $\mathcal{T}$ even if ovals in $\gamma(\mathcal{K}_{b,c})$ are not induced by circles of $\mathcal{M}$ (as γ may not extend to an automorphism of the Minkowski plane). Furthermore, $\mathcal{O}(a,b,c,d)$ contains the point $(b+1, a+c)$ independently of d. As there is precisely one circle through the points (b,∞), (∞, c), $(b+1, a+c)$, there is precisely one $d = d(a,b,c) \in F_0$ such that $\mathcal{O}(a,b,c,d)$ comes from a circle of $\mathcal{M}$ (for all $a, b, c \in F$, $a \neq 0$). Up to isomorphism we may further assume that $\mathcal{O}(1,0,0,0)$ stems from a circle of $\mathcal{M}$, that is, $d(1,0,0) = 0$.

We consider the bundle $\mathcal{K}_{0,0}$ and claim that $d = 0$ for all circles in this bundle. Candidates for the ovals in $\mathcal{K}_{0,0}$ are of the form $\mathcal{O}(a,0,0,d)$. As seen before, for each $a \in F\backslash\{0\}$ there is precisely one $d_a = d(a,0,0) \in F_0$ such that $\mathcal{O}(a,0,0,d_a) \in \mathcal{K}_{0,0}$. By assumption $d_1 = 0$. For $a = -1 + \varepsilon i$, $\varepsilon = \pm 1$, we find $d_a = 0$; otherwise $(1 + \varepsilon d_a + \varepsilon i, -1 - \varepsilon d_a + d_a i) \in \mathcal{O}(1,0,0,0) \cap \mathcal{O}(a,0,0,d_a)$. It then follows that $d_a = 0$ for $a = \varepsilon i$, $\varepsilon = \pm 1$, for otherwise $(1 - \varepsilon d_a - \varepsilon i, 1 - (d_a + \varepsilon) i) \in \mathcal{O}(-1 + \varepsilon i, 0,0,0) \cap \mathcal{O}(a,0,0,d_a)$. Similarly $d_1 = 0$ as $(1 + d_{-1} + i, 1 - d_{-1} - i) \in \mathcal{O}(-1 + i, 0,0,0) \cap \mathcal{O}(-1,0,0,d_{-1})$ if $d_{-1} \neq 0$. We finally consider the case $a = 1 + \varepsilon i$, $\varepsilon = \pm 1$. If $d_a = \varepsilon$ then $(i, i) \in \mathcal{O}(a,0,0,d_a) \cap \mathcal{O}(-1,0,0,0)$, and $(i, -\varepsilon) \in \mathcal{O}(a,0,0,d_a) \cap \mathcal{O}(-\varepsilon i, 0,0,0)$ if $d_a = -\varepsilon$. This shows that $d_a = 0$ in these cases too. Using the translation group of $\mathcal{T}$ we infer that all ovals in a bundle $\mathcal{K}_{b,c}$ have the same parameter $d \in F_0$, that is, $d = d(a,b,c)$ does not depend on a but possibly on b and c. We, therefore, write $d = d(b,c)$ in the sequel. With this notation we already know that $d(0,0) = 0$.

Since $\mathcal{O}(1,0,0,0) \cap \mathcal{O}(1,1,0,d)$ contains the three points $(-1,-1)$, $(di, -di)$, $(-1 + di, 1 + di)$ if $d \neq 0$, the oval $\mathcal{O}(-1,1,0,d)$, $d \neq 0$, cannot stem from a circle. Hence, $d(1,0) = 0 = d(0,0)$ and with the help of translations of $\mathcal{T}$ we find $d(b+1,c) = d(b,c)$ for all $b, c \in F$. Similarly, $\mathcal{O}(1,0,0,0) \cap \mathcal{O}(-1,0,1,d)$ contains the three points $(-1,-1)$, $(-di, di)$, $(1 + di, -1 + di)$ if $d \neq 0$, so $d(0,1) = 0$ and, therefore, $d(b, c+1) = d(b,c)$ for all $b, c \in F$. In particular, $d(1,1) = d(1,0) = d(0,0) = 0$.

Since a collineation of the form $(x,y) \mapsto (r \circ x, s \circ y)$, $r, s \in F\{0\}$, maps $\mathcal{O}(a,b,c,d)$ onto $\mathcal{O}(a', r \circ b, s \circ c, d)$ where $a' = s \circ a \circ \varphi((\varphi^{-1}(r'))^{-1})$ and $r' \circ r = 1$, we infer from $d(1,0) = 0$ that $d(b,0) = 0$ for all $b \in F\{0\}$. Similarly, we obtain $d(0,c) = d(0,1) = 0$ for all $c \in F \backslash \{0\}$ and $d(b,c) = d(1,1) = 0$ for all $b, c \in F \backslash \{0\}$. This proves that $d = 0$, that is, $\phi = id$, and, thus, $\mathcal{M}$ is isomorphic to the Minkowski plane $\mathcal{M}^{\#}$.

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