Intersections of Group Divisible Triple Systems

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Abstract. We determine those triples (g, u, k) for which there exists a pair of group divisible designs with block size 3, each on the same u groups of size g, having exactly k blocks in common.

1. Introduction

If g, u are positive integers, a *group divisible triple system with* u *groups of size* g, abbreviated here as $GD(g, u)$, is a triple $(X, \mathbf{G}, B)$, where X is a set of size gu, $\mathbf{G}$ is a partition of X into u cells of size g each, called *groups*, and $\mathbf{B}$ is a collection of 3-subsets of X, called *blocks*, enjoying the following two properties:

(1) If $G \in \mathbf{G}$, and $B \in \mathbf{B}$ then $|G \bigcap B| \leq 1$ and
(2) If $x, y \in X$, and x, y are not together in the same group, then x, y are together contained in exactly one block.

In a $GD(g, u)$, it is easy to see that there are exactly $b(g, u) = \frac{1}{6}g^2u(u-1)$ blocks, and each element of X is contained in exactly $\frac{1}{2}g(u-1)$ blocks. As these numbers must be integers, necessary conditions for the existence of a $GD(g, u)$ are

$$(*) \qquad \begin{array}{l} 2\,|g(u-1) \\ 3\,|gu(u-1). \end{array}$$

Also, $u \neq 2$.

Hanani [4] has shown that these necessary conditions are in fact sufficient for the existence of a $GD(g, u)$.

We call a pair (g, u) of positive integers *admissible* provided that (*) holds, and of course $u \neq 2$.

Our goal here is to determine the set $I(g, u)$: it is defined to be the set of all k for which there exists a pair of $GD(g, u)$, on the same set of u groups of size g, with exactly k blocks in common. Note that if (g, u) is not admissible, then $I(g, u) = \emptyset$.

Let $S(t)$ denote the set of non-negative integers less than or equal to t, with the exception of $t-1, t-2, t-3$, and $t-5$. Let $J(g, u) = S(b(g, u))$ if (g, u) is admissible, otherwise $J(g, u) = \emptyset$.

Here is our result:

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Main Theorem. *Let g, u be positive integers. Then $I(g,u) = J(g,u)$, except*

$$\begin{aligned} I(1,9) &= J(1,9)\backslash\{5,8\} \\ I(2,4) &= J(2,4)\backslash\{1,4\} \\ I(3,3) &= J(3,3)\backslash\{1,2,5\} \\ I(4,3) &= J(4,3)\backslash\{5,7,10\}. \end{aligned}$$

The rest of this paper is devoted to proving the above theorem.

2. Preliminaries

Some cases of the main theorem have already been settled.

If (g, u) is not admissible, then the theorem is trivially true.

A $GD(1,u)$ is a Steiner triple system of order u. The intersection problem for Steiner triple systems was settled by C.C. Lindner and A. Rosa in [7].

A $GD(2,u)$ is a Steiner triple system of order $2u+1$, from which a point x has been deleted. The blocks containing x (the *flower* at x) become the u groups. The flower intersection problem was solved by D.G. Hoffman and C.C. Lindner in [5].

A $GD(g,3)$ is a latin square of order g in disguise. The intersection problem for latin squares was solved by H.L. Fu in [3].

We leave to the reader the boring proof that $I(g,u) \subseteq J(g,u)$.

We conclude this section with some definitions.

If u is a positive integer, let

$$\delta(u) = \begin{cases} 1 & \text{if} \quad u \equiv 1 \quad \text{or} \quad 3 \pmod 6 \\ 2 & \text{if} \quad u \equiv 0 \quad \text{or} \quad 4 \pmod 6 \\ 3 & \text{if} \quad u \equiv 5 \pmod 6 \\ 6 & \text{if} \quad u \equiv 2 \pmod 6. \end{cases}$$

Note that the necessary conditions (*) are equivalent to

(*′) $$\delta(u) \mid g.$$

We define the *multiplier*, $\mu(g,u)$, to be the quotient $g/\delta(u)$.

Let a, b be non-negative integers, let A, B be sets of non-negative integers. Then

$[a,b]$ is the set of all integers x in the range $a \leq x \leq b$.

$a + B = \{a + b \mid b \in B\}$.

$A + B = \bigcup_{a \in A}(a + B)$.

$aB = \{ab \mid b \in \mathbf{B}\}$.

The set $a * B$ is defined inductively as follows:

$0 * B = \{0\}$, and $(a+1) * B = (a * B) + B$.

Note that $a * B$ consists of all sums $b_1 + b_2 + \cdots + b_a$, where each $b_i \in B$.

Note also that $S(m) + S(n) = S(m+n)$ if $m, n \geq 8$, and $m * S(n) = S(mn)$ if $n \geq 8$.

If X is a finite set, then $|X|$ denotes its cardinality.

Recall that $(Q, \cdot)$ is a *quasigroup* if $\cdot$ is a binary operation on Q, and for each $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have a unique solution $x, y \in Q$.

(X, B) is a *PBD (pairwise balanced design of index 1)* if B is a collection of subsets of X called *blocks*, each with at least two elements, with the property that any two distinct elements of X are together contained in exactly one block of B. (Our constructions of PBD's will sometimes produce "blocks" of size ≤ 1. We will implicitly ignore them.)

Now to work!

3. Constructions

Construction A Let $(T, \mathbf{B})$ be a PBD. Suppose for each $B \in \mathbf{B}$, that $(G \times B, \mathbf{G}_B, \mathbf{B}_B)$ is a $GD(|G|, |B|)$ with $\mathbf{G}_B\{G \times \{x\}| \times x \in B\}$. Then $(G \times T, \bigcup_{B \in \mathbf{B}} \mathbf{B}_B)$ is a $GD(|G|, |T|)$ on groups $\mathbf{G}_T = \{G \times \{x\} | x \in T\}$.

Corollary 1: Suppose for all i that the PBD $(T, \mathbf{B})$ has λ_i blocks of size i. Then

$$\sum_i \lambda_i * I(g, i) \subseteq I(g, |T|).$$

∎

Construction B: Let $(X, \mathbf{G}, B)$ be a $GD(g, u)$, and for each $B \in \mathbf{B}$, let $(M \times B, \mathbf{G}_B, \mathbf{B}_B)$ be a $GD(m, 3)$ with $\mathbf{G}_B = \{M \times \{x\} | x \in B\}$. Then $(M \times X, \mathbf{G}', \bigcup_{B \in B} \mathbf{B}_B)$ is a $GD(mg, u)$, where $\mathbf{G}' = \{M \times G | G \in \mathbf{G}\}$.

Corollary 2: If (g, u) is admissible, then $b(g, u) * I(m, 3) \subseteq I(mg, u)$. ∎

Before we give the next constructions, we need to define certain quasigroups.

A *quasigroup frame with u holes of size g*, denoted $QF(g, u)$, is a triple $(Q, \cdot, \mathbf{H})$, where $(Q, \cdot)$ is a quasigroup, and $\mathbf{H}$ is a partition of Q into u cells of size g, called *holes*, so that each hole is a sub-quasigroup. (i.e., if $H \in \mathbf{H}$, and $x, y \in H$, then $x \cdot y \in H$.) A necessary condition for the existence of a $QF(g, u)$ is that $u \neq 2$. That the condition is also sufficient can be shown by taking the direct product of an idempotent quasigroup (one satisfying $x \cdot x = x$) of order u with a quasigroup of order g.

A $CQF(g, u)$ (a commutative quasigroup frame) is a $QF(g, u)$ satisfying $x \cdot y = y \cdot x$ whenever x, y are *not* in the same hole. Necessary conditions here are that $u \neq 2$, and $2 | g(u-1)$.

These conditions are also sufficient, we sketch a proof. If u is odd, we take an appropriate direct product of a $CQF(1, u)$ with a commutative quasigroup of order g. If $u \neq 2$ is even, then g is even. There is a $CQF(2,u)$, see for example [2]. We take an appropriate direct product with a commutative quasigroup of order $g/2$.

Construction C: Let $(Q,\cdot,H)$ be a $QF(g,u)$. For $1 \leq i \leq 3$, let $(Q \times \{i\}, G_i, B_1)$ be a $GD(g,u)$ with $\mathbf{G}_i\{H \times \{i\} | H \in \mathbf{H}\}$. Then $(Q \times \{1,2,3\}, \mathbf{G}, \mathbf{B})$ is a $GD(3g,u)$, where $\mathbf{G} = \{H \times \{1,2,3\} | H \in \mathbf{H}\}$, and

$$\mathbf{B} = \mathbf{B}_0 \bigcup \mathbf{B}_1 \bigcup \mathbf{B}_2 \bigcup \mathbf{B}_3, \quad \text{where}$$
$$\mathbf{B}_0 = \{\{(x,1),(y,2),(x\cdot y,3)\} | x,y \in Q, x,y \quad \text{not in the same hole.}\}$$

Corollary 3: Let $(Q,\cdot,H),(Q,\#H)$ be $QF(g,u)$'s, let $k = |\{(x,y) \in Q^2 | x$ and y are not in the same hole, $x\cdot y = x\# y\}|$. Then $k+3 * I(g,u) \subseteq I(3g,u)$. ∎

Construction D: Let $(Q,\cdot,\mathbf{H})$ be a $CQF(g,u)$, let $(Q \times \{0\}, \mathbf{G}, \mathbf{B})$ be a $GD(g,u)$ with $\mathbf{G} = \{H \times \{0\} | H \in \mathbf{H}\}$. Then $(Q \times \{0,1\}, \mathbf{G}', \mathbf{B}')$ is a $GD(2g,u)$, where $\mathbf{G}' = \{H \times \{0,1\} | H \in \mathbf{H}\}$, and $\mathbf{B}' = \mathbf{B} \bigcup \{\{(x,1),(y,1),(x\cdot y,0)\} | \{x,y\} \subseteq Q, x,y$ not in the same hole$\}$.

Corollary 4: Let $(Q,\cdot,H),(Q,\#,H)$ be a $CQF(g,u)$, let $k = |\{\{x,y\} \subseteq Q|$ x and y are not in the same hole, $x\cdot y = x\# y.\}|$. Then $k + I(g,u) \subseteq I(2g,u)$. ∎

It behooves us to find several convenient values of k to be used in corollaries 3 and 4. So let $(Q,\cdot,H)$ be a $QF(g,u)$, and let π be a permutation on Q satisfying $\pi H = H$ for all $H \in \mathbf{H}$. (i.e., $\{\pi(h) | h \in H\} = H$.) Define the binary operation $\#$ on Q by $x\# y = \pi(x\cdot y)$. Then $(Q,\#,H)$ is again a $QF(g,u)$. The parameter k depends only on the number of fixed points of π, and this can be any number in the set $u * ([0,g-2] \bigcup \{g\})$. Each fixed point contributes $g(u-1)$ pairs $(x,y) \in Q^2$ in Corollary 3. We have proved the following:

Sublemma 1. *If* $k \in g(u-1)(u * ([0,g-2] \bigcup \{g\}))$, *then* k *satisfies the hypotheses of Corollary 3, provided* $u \neq 2$. ∎

If $(Q,\cdot,H)$ is a $CQF(g,u)$, then in the construction above, so is $(Q,\#,H)$.

Sublemma 2. *If there is a* $CQF(g,u)$, *and if* $k \in \frac{1}{2}g(u-1)(u * ([0,g-2] \bigcup \{g\}))$, *then* k *satisfies the hypotheses of Corollary 4.* ∎

Sublemma 3. *(C.K. Fu, [2]) If* $g = 2 \neq u$, *then the set* $F(u)$ *of those* k *satisfying the hypothesis of Corollary 4 is given by* $F(u) = S(2u(u-1))$, *except* $1,2,3,5,6 \notin F(3)$, *and* $3,7,11,15,16,17,20 \notin F(4)$. ∎

4. Some Pairwise Balanced Designs

The $PBD(X,\{X\})$ is called *trivial*. All others are *non-trivial*.

Lemma 1. *If $u = 7$, or $u \geq 9$, there is a non trivial PBD of order/ u with no blocks of size 2.*

Proof: Let $0 \leq s \leq t$, $s \neq 2, t \neq 2, 6$. Then there is a pair of orthogonal latin squares of order t. It is well known that this is equivalent to a PBD of order $4t$, with 4 pairwise disjoint blocks of size t, and all the other blocks of size 4. Deleting $t - s$ points from one of the blocks of size t gives a PBD of order $3t + s$ with blocks of size $t, s, 4$ and 3. This leaves only the values $u \in \{7, 11, 14, 17, 23\}$. If $u = 4\ell - 1$, the complete graph $K_{2\ell}$ on 2ℓ vertices can be one-factored, let $F_1, F_2, \ldots, F_{2\ell-1}$ be the one-factors. Let $\{\infty_i | 1 \leq i \leq 2\ell - 1\}$, be a block of $2\ell - 1$ additional points. For each $1 \leq i \leq 2\ell - 1$, take the blocks $\{\infty_i, x, y\}$, $x, y \in F_i$.

For $u = 14$, take the following blocks: $\{9, 10, 11, 12, 13\}$, $\{0, 4, 8, 11\}$, $\{1, 5, 6, 12\}$, $\{2, 3, 7, 13\}$, $\{5, 7, 11\}$, $\{1, 3, 11\}$, $\{2, 6, 11\}$, $\{0, 7, 12\}$, $\{3, 8, 12\}$, $\{2, 4, 12\}$, $\{0, 5, 13\}$, $\{1, 8, 13\}$, $\{4, 6, 13\}$, $\{0, 1, 2, 9\}$, $\{3, 4, 5, 9\}$, $\{6, 7, 8, 9\}$, $\{0, 3, 6, 10\}$, $\{1, 4, 7, 10\}$, $\{2, 5, 8, 10\}$.

For $u = 17$, begin with a $GD(4, 4)$. Add a new point, extending each of the 4 groups to a block of size 5. ∎

Lemma 2. *If $u \equiv 0, 1 (mod 3)$, $u \geq 7$, there is a non-trivial PBD of order u with each block size $\equiv 0, 1 (mod 3)$, and at least one block of size ≥ 4.*

Proof: Most values of u follow from Lemma 9 of [5]. All that remains is $u \in \{10, 12, 13\}$. For $u = 10$, add a point to the groups of a $GD(3, 3)$. A $GD(4, 3)$ does $u = 12$. A projective plane of order 3 does $u = 13$. ∎

Lemma 2′. *If $u \equiv 0, 1 (mod 3)$, $u \geq 7$, there is a non-trivial PBD of order u with each block size $\equiv 0, 1 (mod 3)$.*

Proof: By lemma 2, we need only do $u \in \{7, 9\}$, but there are Steiner triple systems of those orders. ∎

Lemma 3. *If $u \geq 7$ is odd, there is a non trivial PBD of order u with all block sizes odd.*

Proof: As in Lemma 1, if $0 \leq s \leq t, t \neq 2, 6$, there is $PBD(X, B)$ of order $3t + s$. B contains three blocks B_1, B_2, B_3 of size t, and a block B_4 of size s; the remaining blocks have size 3 or 4. We construct a design on $(X \times \{0, 1\}) \bigcup \{\infty\}$ as follows: For $1 \leq i \leq 4$, $(B_i \times \{0, 1\}) \bigcup \{\infty\}$ is a block. For every other block B of $\mathbf{B}$, take the blocks of a $GD(2, |B|)$ with groups $\{x\} \times \{0, 1\}$, $x \in B$.

This takes care of all u except $u \in \{11, 13, 15, 17\}$.

The constructions in Lemma 1 for $u = 11$ and 17 have all block sizes odd. There are Steiner triple systems of order 13 and 15. ∎

5. Proof of the Main Theorem

Throughout this section, (g,u) is an admissible pair, with $g \geq 3$, $u \geq 4$, as $g < 3$ or $u < 4$ has been settled.

In particular, $I(m,3) = J(m,3)$ for all $m \geq 5$, so Corollary 2 shows $I(g,u) = J(g,u)$ whenever $\mu(g,u) \geq 5$, since $(\delta(u), u)$ is admissible for $u \geq 2$.

Hence we need only consider $g \in \{3,4,6,8,9,12,18,24\}$, and $\mu(g,u) \leq 4$.

For $g = 3$ (so u is odd) we will use a special method. It is well known that a commutative idempotent quasigroup of order u (i.e., a $CQF(1,u)$) exists if and only if u is odd. For such u, let $C(u) = \{k|$ there is a pair $(Q,\cdot)$, $(Q,\#)$ of commutative, idempotent quasigroups of order u, with $k = |\{\{x,y\} \subseteq Q | x \neq y, x\cdot y = x\# y\}|$. T. Webb showed in [10] that $C(u) = S(\frac{1}{2}u(u-1))$, except that $C(5) = \{2,10\}$.

Lemma 4. $3 * C(u) \subseteq I(3,u)$

Proof: For $i \in Z_3$ let $(Q, \overset{\circ}{i})$ be a commutative idempotent quasigroup of order u. We construct a $GD(3,u)$ on the set $Q \times z_3$. The groups are $\{q\} \times z_3$, $q \in Q$. For each $i \in z_3$, and for each $\{x,y\} \subseteq Q$, $x \neq y$, $\{(x\imath), (y,i), (x \overset{\circ}{i} y, i+1)\}$ is a block. ∎

The construction in Lemma 4 is essentially due to R.C. Bose, [1].

For $u \geq 7$, Lemma 4 gives $I(3,u) = J(3,u)$, but gives only $\{6,14,22,30\} \subseteq I(3,5)$. See the appendix for the rest of $I(3,5)$. Now let $g = 4$, so $u \equiv 0,1 \pmod 3$.

If $u = 4$, then corollary 4, with $g = 2$, $u = 4$, together with sublemma 3, at $u = 4$, show $I(4,4) \supseteq J(4,4) \backslash \{19,23,25,28\}$. Corollary 2, with $m = 2$, shows $I(4,4) \supseteq 4[0,8]$. We need only show $19,23,25 \in I(4,4)$, these cases are in the appendix.

The above case $(g,u) = (4,4)$ more or less illustrates our attack on a given admissible pair (g,u).

We first use corollary 2 with $m = \mu(g,u)$. If $\mu(g,u) = 4$, this gives $I(g,u) \supseteq J(g,u) \backslash \{b(g,u) - 9, b(g,u) - 6\}$. If $\mu(g,u) = 3$, this at least shows all multiples of three in $J(g,u)$ are in $I(g,u)$. If $\mu(g,u) = 2$, we get only the multiples of 4. If $\mu(g,u) = 1$, corollary 2 is of no use.

If $\mu(g,u) \in \{2,4\}$, we then use corollary 4, with sublemma 2, or sublemma 3 if $g = 4$. If $\mu(g,u) = 3$, we use corollary 3, with sublemma 1.

Finally, any remaining cases are relegated to the appendix.

Returning to $g = 4$, the next case is $u = 6$. Here $\mu = 2$, and corollary 4, together with sublemma 3 shows $I(4,6) = J(4,6)$.

If $\mu \in \{7,9\}$, then $\mu = 4$, so corollary 2 with $m = 4$ does all but two intersection numbers. These are picked up by corollary 4, with $k = 2u(u-1)$.

For $u \geq 10$, we proceed by induction on u, using corollary 1 and lemma 2.

The above case $g = 4$ illustrates our attack for any fixed g. We solve the problem for enough small values of u, so that we can use corollary 1, in conjunction with lemma 1 if $g \equiv 0 \pmod 6$, lemma 2′ if $g \equiv 2$ or $4 \pmod 6$, and lemma 3 if $g = 3 \pmod 6$. (We only needed lemma 2 when $g = 4$, as $I(4,3) \neq J(4,3)$.) We proceed to $g = 6$, so $u \geq 4$.

At $u = 4$, corollary 2 with $m = 3$ shows all the multiples of 3 in $J(6,4)$ are in $I(6,4)$. Corollary 3 together with sublemma 1 gives all the even numbers of $J(6,4)$ except 62 and 68. The appendix shows that 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 62, 65, 68 $\in I(6,4)$.

At $u = 5$, corollary 4 with sublemma 2 shows $I(6,5) = J(6,5)$. At $u = 6$, corollary 3 with sublemma 1 shows $I(6,6) = J(6,6)$.

We open a can of worms at $u = 8$. None of our corollaries, sublemmas, or lemmas are of any use. So it's time for some ad-hoc tom-foolery!

Let $k \in J(6,8)$. We will produce two $GD(6,8)$'s with k blocks in common, on the set $V \bigcup Z_{30}$, where $V = \{\infty_i | 1 \leq i \leq 18\}$. The groups are $\mathbf{G}_1 \bigcup \mathbf{G}_2$, where $\mathbf{G}_1 = \{\{\infty_{6i+j} | 1 \leq j \leq 6\} | i \in \{0,1,2\}\}$, and $\mathbf{G}_2 = \{\{i + 5j | 0 \leq j \leq 5\} | 0 \leq i \leq 4\}$. It is possible to write $k = \alpha + 15\beta + 30\gamma$, where $\alpha \in J(6,3) = I(6,3)$, $\beta \in [0,16] \bigcup \{18\}$, and $\gamma \in \{0,1\}$. Let $\mathbf{C}_1, \mathbf{C}_2$ be the blocks of a pair of $GD(6,3)$'s with groups $\mathbf{G}_1$ having α blocks in common. Let $\mathbf{D}_1 = \{\{i, i+1, i+3\} | i \in Z_{30}\}$. If $\gamma = 1$, let $\mathbf{D}_2 = \mathbf{D}_1$; if $\gamma = 0$ let $\mathbf{D}_2 = \{\{i, i+2, i+3\} | i \in Z_{30}\}$. Let G be the graph on vertices Z_{30} obtained by removing from the complete graph on Z_{30} the edges $\{i, i+j\}$, $i \in Z_{30}$, $j \in \{1,2,3,5,10,15\}$. By a theorem of Stern and Lenz [8], G can be 1-factored, let F_i, $1 \leq i \leq 18$, be the 1-factors. Let $E_1 = \{\{\infty_i, x, y\} | i \in [1,18], \{x,y\} \in F_i\}$. Let π be a permutation of $[1,18]$ with β fixed points, let $E_2 \{\{\infty_{\pi(i)}, x, y\} | i \in [1,18], \{x,y\} \in F_i\}$. Then $(V \bigcup Z_{30}, \mathbf{G}_1 \bigcup \mathbf{G}_2, \mathbf{C}_i \bigcup \mathbf{D}_i \bigcup E_i)$, $i = 1,2$ are two $GD(6,8)$'s with exactly k blocks in common. Hence $I(6,8) = J(6,8)$.

Corollary 1, together with lemma 1, shows that $I(6,u) = J(6,u)$ for all the remaining values of u.

We have now passed the point where the appendix is needed. Since we have described the general approach, we will only sketch the rest of the proof.

At $g = 8$, $u \equiv 0, 1 \pmod 3$, only the cases $u = 4, 6$ are needed before corollary 1, with lemma 2′, takes over.

At $g = 9$, only $u = 5$ is needed before corollary 1, with lemma 3 takes over.

The only other thing that needs checking is $(g,u) \in \{12,18,24\} \times \{4,5,6,8\}$. This completes the proof of the main theorem. ∎

6. Hill Climbing

The traditional approach to constructing combinatorial designs using computers often employs backtracking algorithms to search exhaustively through all the possibilities. For designs of even moderate size this approach can be slow, but for larger designs it quickly becomes unfeasible.

An alternative but perhaps less frequently applicable technique is to use a so-called hill-climbing algorithm. Although such algorithms are not guaranteed to construct a required design, (even when such a design is known to exist), they do have the advantage of being very fast, and they can therefore be applied to larger designs. Computational experience has shown that hill-climbing algorithms can very often be used to find combinatorial designs. For more details about hill-climbing algorithms and when they can usefully be applied, we refer the reader to an excellent treatment by D. Stinson in [8].

In the present paper we have applied a hill-climbing technique to obtain examples of particular intersection numbers for some of the smaller (and more awkward) designs. Our basic approach was to use a hill-climbing algorithm to obtain a design of the required size we then deleted a random collection of triples from the completed design, and finally used hill-climbing to construct yet another design of the same size. By deleting an appropriate number of triples in the second step, one can guarantee that the two designs have many triples in common. Repeated application of the above technique quickly provided all the required intersection numbers.

Suppose we wish to construct a group divisible triple system $GD(g, u)$. Then, using the same notation as in section 1, such a triple system contains $g^2 u(u-1)/6$ blocks (each consisting of 3 points) satisfying conditions 1) and 2) of section 1. The following hill-climbing algorithm can be used to construct (or complete) the required design:

Algorithm A:

```
BEGIN
        store any "given" triples (that form a partial design);
        WHILE (more triples are needed to complete the design) DO
        BEGIN
                (* "live" means has not yet occurred with all possible points with
                        which it can occur *)
                find a random "live point" j;
                pick two random "mates", m and n, to form a triple (j, m, n)
                IF the pair (m, n) has already occurred in the partial design
                      THEN delete the (unique) triple that already contains m and n;
                add the new triple (j, m, n) to the partial design;
        END;
END;
```

If B is used to denote the set of blocks (or triples) in the design (or partial design); then a more mathematical decription of the same algorithm is the following:

Algorithm B:

```
BEGIN
      B:=set of "given" triples;
      Number Of Triples:= cardinality of B;
      WHILE (Number of Triples < g^2 u(u - 1)/6)
      BEGIN
            IF (there is a triple BO such that B U {BO} is a partial system)
            THEN BEGIN
                        (* add the triple BO to the partial design *)
                        Number Of Triples: Number Of Triples 1;
                        B:= B ∪ BO;
                  END
            ELSE BEGIN
                        find triples B1 in B, and BO not in B such that
                          B U {BO} \ {B1} is a partial system;
                        (* swap the triples BO and B1 *)
                        B :- B ∪ {BO} \ {B1};
                  END;
      END;
END;
```

One of the most useful features of the hill-climbing algorithm, (as described above), is that at no stage during its execution does the number of blocks in the partial design decrease. The continual (and time consuming) backtracking that is inherent in the backtracking algorithm is therefore completely avoided by the hill-climbing algorithm. If we define an iteration to be the process of either adding an extra block to the partial design or swapping two blocks, then by a careful implementation of the above algorithm we can also make the time taken per iteration independent of both g and u This of course is highly desirable when dealing with designs of larger size.

The implementation of the hill-climbing algorithm used in the present paper fully exploits the ideas of D. Stinson in [8]. For the sake of both clarity and completeness, our implementation of those ideas is outlined below.

During one of the steps of Algorithm A (above) a random "live point" is picked. The term "live" is used here to mean any point that has not yet occurred in the (partial) design with all the points with which it could occur. One way to efficiently choose "live" points is to maintain a list of currently "live" points. Since we are attempting to choose a random "live" point this list does not need to be ordered. An auxiliary table is however maintained that indicates where in this list each "live" point occurs. The list and auxiliary table are used during updating operations as follows: If a "live" point "dies", it is removed from the list of "live" points, and the last point in the list is moved to occupy its place. The auxiliary table is then updated accordingly. If as a consequence of a block-swapping move (see above algorithms) a "dead" point becomes a "live" point, then it is added to the end of the list, and the auxiliary table is once again updated. Using the above tables one can choose a random "live" point by first picking a random integer between 1 and the number of "live" points, and then selecting the point that occurs at that position in the unordered list of "live" points.

Once we have obtained a random "live" point, j, we then need to pick (once again at random) two other points, m and n, (refered to as "mates") that have not yet occurred with j in the partial design. To facilitate the choice of the "mates", we maintain for each "live" point a table of other points that have not yet occurred with that point. These tables, (one for each "live" point), are stored in a two dimension array, and an auxiliary array is used to maintain an index that can be used to reference each table. The tables together with the index array are updated in much the same way that the "live" point list and auxiliary table were maintained.

Having picked two "mates", m and n, for the "live" point j, we now need to determine if the pair (m, n) has already occurred in a triple of the partial design. To simplify and speedup this test we maintain a two dimensional array that at position $[m, n]$ contains the location of any triple in which both the points m and n have already occurred. With this information we now decide whether the triple (j, m, n) can be used to extend the (partial) design, or whether we must first delete the previous triple containing m and n, before adding the new triple (j, m, n).

Finally, in order to enable the hill-climbing algorithm to be used when constructing complete designs from a given (and fixed) partial design we also maintain a list of the original triples of the partial design that must not be removed.

7. Appendix

This appendix contains precise descriptions of pairs of small designs for which not all possible cases of intersection numbers are covered by the more general theory developed elsewhere in this paper. In particular (using the notation of section 1) we exhibit details of the "missing cases" as shown in the table.

For the sake of brevity the following coding scheme has been employed when displaying a pair of designs (of the same size) that have the required number of blocks in common (i.e. they exhibit a "missing" intersecting number). We first label the points in the designs as follows:

GD(g, u)	Intersection numbers
GD(3, 5)	0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 23, 24, 26.
GD(4, 4)	19, 23, 25.
GD(6, 4)	1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, 61, 62 65, 68.

Thus the letters AEI fully describe one (possible) triple in $GD(3,5)$, whereas the letters AEC do not represent an allowed triple, because the triple they describe contains two points from the same level, and this violates condition (1) of section 1. Using this coding scheme we are now able to give examples of pairs of designs with "awkward" intersection numbers; we first list the triples that the two designs have in common, and we then give two lists of triples that when added to the common triple each provide a complete design. These two lists are referred to as the first and second variations respectively.

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