Smallest Transversals of Small 3-graphs

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Abstract. A smallest transversal of a k-graph (or k-uniform hypergraph) is any smallest set of vertices that intersects all edges. We investigate smallest transversals of small (up to ten vertex) 3-graphs. In particular, we show how large the smallest transversal of small 3-graphs can be as a function of the number of edges and vertices. Also, we identify all 3-graphs with up to nine vertices that have largest smallest transversals.

This work is related to a problem of Turán, and to the covering problem. In particular, extremal 3-graphs correspond to covering designs with blocks of size $n-3$.

1. Introduction.

1.1 Overview

By a *k-graph*, also known as a *k-uniform hypergraph*, we mean an ordered pair (V, E) such that V is a finite set and E is a set of k-element subsets, or k-sets, of V. The elements of V are the *vertices* of the k-graph, and the elements of E are the *edges* of the k-graph; we reserve the letters n and m for the number of vertices and edges, respectively.

A *transversal* of a k-graph is any set of vertices that intersects all the edges. We let $\tau(H)$ denote the smallest size of a transversal of a k-graph H, and $t(n, m, k)$ denote the largest value of $\tau(H)$, over all k-graphs H with n vertices and m edges. Determining $t(n, m, k)$ is equivalent to a problem proposed by Turán [T], and to the covering problem, as we shall explain shortly.

With respect to fixed values of k and n, a value m is said to be *critical* or *extremal* if $t(n, m-1, k) = t(n, m, k) - 1$. A k-graph is critical if it has a critical number of edges.

Values of $t(n, m, 3)$ for $n \leq 10$ can be determined from Table 1, as follows. Since $t(n, m, k)$ is non-decreasing in m, it suffices to list only those triples n, m, $t(n, m, 3)$ for which m is critical. We denote by $m(n, k, t)$ the critical m, such that $t(n, m, k) = t$. Thus, $m(n, k, t)$ is the smallest possible number of edges of a k-graph with n vertices and smallest transversal of size t.

The purpose of this paper is threefold. First, we present the values of $t(n, m, k)$ for $k = 3$ and $n \leq 10$, summarized in Table 1. In the rest of Section 1 we give a brief survey, and show how the entries of Table 1 can be derived from the literature, either directly or with simple arguments.

Second, in Section 2 we determine all non-isomorphic critical 3-graphs with $n \leq 9$. (Two 3-graphs are isomorphic if the vertices of one can be relabelled so that the edge sets are the same.) With only a few exceptions, these results are all new.

Third, there are two entries in Table 1 which are known but for which no proofs have appeared. In Section 3 we supply two such proofs.

1.2 Related problems and previous work

The third problem proposed by Turán in his 1961 list of research problems [T] asks for the smallest number m of p-sets of an n-set, such that every q-set of the n-set contains at least one p-set. Considering the n-set complements of the p-sets, this is equivalent to asking for the smallest number of p-sets of an n-set, such that every $(n-q)$-set of the n-set is non-intersecting with at least one p-set; in other words, a smallest transversal of the p-sets has at least $n-q+1$ vertices. Thus Turán's problem asks for $m(n,p,n-q+1)$.

An equivalent formulation of Turán's problem, usually referred to as the *covering problem*, asks for $C(n,b,r)$, the smallest number of b-sets of an n-set, such that every r-set of the n-set (where $b < r$, as opposed to $p < q$) is contained in at least one b-set. (The equivalence is obtained when the b-sets and r-sets are the n-set complements of the p-sets and q-sets, respectively.) Thus, $C(n,b,r) = m(n,n-b,r+1)$. In the context of this problem, each b-set is a *block*, and the collection of b-sets is a *covering* or a *covering design* or a *blocking set*.

An early result in this area is due to Katona, Nemetz and Simonovits [KNS], who determined $C(n,n-3,n-4) = m(n,3,n-3)$ for $n \leq 9$. Guy [G] communicated that Vera Sós Turán and independently M. Simonovits established $C(n,n-3,n-4)$ for $n \leq 12$, although apparently no proofs have been published for $n \geq 10$.

Todorov [To] established the inequalities $t(n,m,3) \leq (n+m)/4$ and $t(n,m,3) \leq (2n+m)/6$, recently rediscovered by Chvátal and McDiarmid [CM]. Other results (in most cases, the particular instance of a more general theorem) are:

$C(6,3,2) = m(6,3,3) = 6$,	due to Fort and Hedlund [FH];
$C(7,4,2) = m(7,3,3) = 5$,	due to Mills [M1];
$C(7,4,3) = m(7,3,4) = 12$,	due to Kalbfleisch and Stanton [KS] and independently Swift [Sw];
$C(8,5,2) = m(8,3,3) = 4$,	due to Stanton and Kalbfleisch [SK]; and
$C(10,7,5) = m(10,3,6) = 20$	appears in Mills [M2] without proof.

Results mentioned so far (except for the six equalities shown immediately above) give only lower bound arguments for Table 1 entries, as upper bounds for entries in Table 1 (with one exception) follow from the construction due to Turán, described by Ringel in [R]. We will elaborate on this shortly.

There are also results that identify certain critical 3-graphs. A 3-graph with n-vertices, m edges, and smallest transversal size t is said to be of *type* $[n, m, t]$. Stanton, Allston, Wallis and Cowan [SAWC] found the four covering designs whose complements yield the critical 3-graphs of type $[7\ 5\ 3]$; Stanton [S] showed the uniqueness of the covering design whose complement is the only critical 3-graph of type $[8\ 4\ 3]$. In Section 2 we will identify all other critical 3-graphs with fewer than ten vertices.

Finally, Brown [B] showed that there are at least $j-1$ non-isomorphic 3-graphs of type $[3j, j(j-1)(2j-1), 3j-3]$; later Kostochka [K] showed there are at least 2^{j-2} such 3-graphs. In particular, there are at least two 3-graphs of type $[9, 30, 6]$. In Section 2 we will see that there are only two.

1.3 Lower bounds

In this subsection we show how results mentioned above establish the lower bounds of the values of $m(n, 3, t)$ shown in Table 1. The subscripts in the entries of Table 1 refer to the following arguments. (For example, from (B) it follows that $m(n, 3, \lfloor (n+m)/4 \rfloor + 1) \geq m+1$; setting $n = 9$ and $m = 6$ gives the bound $m(9, 3, 4) \geq 7$.)

(A)	$t(n, m, k) \leq m$	trivial
(B)	$t(n, m, 3) \leq (n+m)/4$	[To] [CM]
(C)	$t(n, m, 3) \leq (2n+m)/6$	[To] [CM]
(V)	$t(n, m, k) \leq 1 + t(n-1, m - \lceil km/n \rceil, k)$	see below
(Z)	$t\left(n, \binom{n}{k} - 1, k\right) \leq n - k$	see below.

To see that (Z) holds, note that some set of k vertices is not an edge, and that the complement of this set is a transversal.

To see that (V) holds, remove a vertex of largest degree, and argue by induction. (The *degree* of a vertex is the number of edges that contain it.) For example, to show that $t(9, 29, 3) \leq 5$, consider a 3-graph H with $n = 9$ and $m = 29$. The average vertex degree is $87/9$, so some vertex v has degree at least 10. The 3-graph $H - v$ has $n = 8$ and $m \leq 19$; assuming the validity of entries in Table 1 for $n = 8$, it follows that $t(9, 19, 3) = 4$. Thus $H - v$ has a transversal T of size 4, and so $T + v$ is a transversal of H of size 5.

The lower bound for entry $(7, 4)$, namely $m(7, 3, 4) \geq 12$, is established in [KS]. Their proof uses the fact that there is a unique 3-graph of type $[6\ 6\ 3]$ and involves some case analysis.

The lower bound for entry $(10, 6)$, namely $m(10, 3, 6) \geq 20$, follows from $C(10, 7, 5) \geq 20$, given without proof in [M2]. The lower bound for entry $(10, 7)$, namely $m(10, 3, 7) \geq 45$, follows from $C(10, 7, 6) \geq 45$, credited in [G] to Vera Sós Turán and independently M. Simonovits. Proofs of these two results will be presented in Section 3.

1.4 Upper bounds

The upper bounds for all entries $m(n,3,t)$ of Table 1 can be established by exhibiting 3-graphs with the appropriate values of n, m and t.

The following construction of a 3-graph for arbitrary values of n and t, which we will refer to as $T(n,t)$, is due to Turán (see [R]). Partition the n vertices into $r = n - t$ sets $S_0, S_1, \ldots, S_{r-1}$ such that $S_j = \lceil (j+1)n/r \rceil - \lceil jn/r \rceil$ for $j = 0, 1, \ldots, r-1$. The edges are all sets of vertices $\{a, b, c\}$ such that either

$$a \in S_j \quad b \in S_j \quad c \in S_j \qquad \text{for some } j = 0, 1, \ldots, r-1 \text{ or}$$
$$a \in S_j \quad b \in S_j \quad c \in S_{(j+1) \bmod r} \text{for some } j = 0, 1, \ldots, r-1.$$

To show that $\tau(T(n,t)) = t$, see [R]. Examples of $T(n,t)$ are given explicitly in Section 3. For all entries $m(n,3,t)$ of Table 1 except $(n,t) = (9,5)$, the 3-graph $T(n,t)$ establishes an upper bound for $m(n,3,t)$. For example, $T(10,6)$ is constructed from sets S_0, S_1, S_2, S_3, of sizes 3, 2, 3, 2, respectively, and has $m = 20$ and $\tau = 6$. Thus $t(10,20,3) \geq 6$, and so $m(10,3,6) \leq 20$.

We call vertices x and y of a 3-graph H *twins* if xab is an edge exactly when yab is an edge, for all distinct vertices a, b in $H - \{x, y\}$. A *twin-class* of a 3-graph is a maximal (with respect to inclusion) non-empty set of twins. For example, the twin-classes of $T(n,t)$ are the sets S_j. Observe that if both xy and yz are twins then xz are twins; thus, twin-classes are equivalence classes.

A 3-graph that establishes $m(9,3,5) \leq 12$ is the affine plane consisting of 9 points and four groups of three mutually parallel lines. (In an affine plane, every two points are in exactly one line, and every pair of non-parallel lines intersect in exactly one point.) An explicit representation of this hypergraph, which we denote AP_9, is $V = abcdefghi$, and $E = abc\ def\ ghi\ adg\ beh\ cfi\ aei\ bfg\ cdh\ afh\ bdi\ ceg$.

Since $acegi$ is a transversal, $\tau(AP_9) \leq 5$. To see that $\tau(AP_9) > 4$, let F be a set of five vertices. There are 54 sets F consisting of two intersecting lines, and 72 sets F consisting of a line and two points of a parallel line. Since the total number of sets F is $\binom{9}{5} = 126$, it follows that every set F contains at least one line. Thus, every set $V - F$ of four vertices misses at least one line.

In fact, Brouwer and Schrijver [BS] showed a much more general result, namely, that the size of a smallest transversal of the d-dimensional affine space with q^d points is $d(q-1)+1$. The preceding proof that $\tau(AP_9) = 5$ has been included because it is so short.

Since $\tau(AP_9) = 5$, $t(9,12,3) \geq 5$ and so $m(9,3,5) \leq 12$.

Table 1: Values of $m(n,3,t)$

	$t=1$	2	3	4	5	6	7	8
$n=3$	1_{ABZ}							
4	1_A	4_{BCZ}						
5	1_A	3_B	10_Z					
6	1_A	2_{AB}	6_{BC}	20_Z				
7	1_A	2_A	5_B	12	35_Z			
8	1_A	2_A	4_B	8_{BC}	20_V	56_Z		
9	1_A	2_A	3_{AB}	7_B	12_{CV}	30_V	84_Z	
10	1_A	2_A	3_A	6_B	10_{BC}	20	45	120_Z

2. Critical 3-graphs.

In this section we find all non-isomorphic critical 3-graphs with $n \leq 9$. Recall that two k-graphs are isomorphic if the vertices of one can be relabelled so that the edge sets are the same; we use the symbol $\cong$ to denote isomorphism. Recall also that a k-graph H is critical if $\tau(H) = t(n,m,k)$ and $t(n,m-1,k) = t(n,m,k) - 1$. Finally, recall that by a 3-graph of type $[n\ m\ \tau]$ we mean a 3-graph with n vertices, m edges, and smallest transversal τ.

Since the only k-graphs discussed in the rest of the paper are 3-graphs, we abbreviate notation from this point on by writing $t(n,m)$ and $m(n,t)$ for $t(n,m,3)$ and $m(n,3,t)$, respectively. Also, we will write sets *without* using braces. For example, $abcdef$ and $abc\ \ abd\ cde\ cdf\ aef\ bef$ represent the respective vertex and edge sets of some 3-graph.

Consider a critical 3-graph H. If $m = \tau$, then no two edges intersect, and H is uniquely determined up to isomorphism. If $m = \binom{n}{3}$ then all sets of three vertices are edges, and H is uniquely determined. Thus, in the rest of this section we consider only those 3-graphs for which $\tau < m < \binom{n}{3}$.

Observe that for $n \leq 6$ the only critical 3-graphs with $\tau < m < \binom{n}{3}$ have type $[5\ 3\ 2]$ or $[6\ 6\ 3]$. It is not difficult to verify that there is only one 3-graph of each of these types, namely Turán 's $T(5,2)$ and $T(6,3)$, respectively.

2.1 Critical 3-graphs with $n = 7$

2.1.1 Type [7 5 3]

Stanton, Allston, Wallis and Cowan [SAWC] found that there are exactly four $(7,4,2)$ covering designs with five blocks. Taking the complements of these blocks gives the critical 3-graphs of type $[7\ 5\ 3]$, shown in the table below.

Observe that $C(7,3) \cong T(7,3)$: put $S_0 = bc$, $S_1 = ag$, $S_2 = de$, $S_3 = f$.

All 3-graphs of type [7 5 3]			
Name	Edges	Degree sequence	Twin-classes
$A(7,3)$	$abc\ def\ abg\ acg\ bcg$	$3^4 1^3$	$abcg\ def$
$B(7,3)$	$abc\ def\ adg\ bcg\ efg$	$3\ 2^6$	$bc\ ef$
$C(7,3)$	$abc\ def\ adg\ aeg\ bcg$	$3^2 2^4 1$	$ag\ bc\ de$
$D(7,3)$	$abc\ def\ adg\ beg\ cfg$	$3\ 2^6$	— — —

2.1.2 Type [7 12 4]

Consider a critical 3-graph H of type [7 12 4]. Let $V = abcdefg$. No vertex has degree greater than six, since $t(6,5) = 2$. Since the average degree is $36/7$, some vertex (say g) has degree six. Thus $H - g$ has type [6 6 3], and so is isomorphic to $T(6,3)$. We may assume that the edges of $H - g$ are $abc\ abd\ cde\ cdf\ aef\ bdf$.

Let P be the set of pairs of vertices that are in edges with g. Let N be the triples of $H - g$ that are *not* edges, that is, $N = abe\ abf\ acd\ ace\ acf\ ade\ adf\ bcd\ bce\ bcf\ bde\ bdf\ cef\ def$. Observe that

$$\text{every triple of } N \text{ must contain a pair of } P \qquad (1)$$

(otherwise, the complement of the triple in $V - g$ is a transversal of H of size three, contradiction).

Case 1. No vertex of $H - g$ is in more than two pairs of P. Thus each vertex of $H - g$ is in exactly two pairs of P. Recall that $H - g$ is $T(6,3)$ with $S_0 = ab$, $S_1 = cd$, $S_2 = ef$.

Case 1.1 There are vertices $y \in S_j$, $x \in S_{j+1}$, $z \in S_{j+2}$ such that $xy \in P$, $xz \in P$. By symmetry, we may assume that $x = a$, $y = e$, $z = c$. Thus the two pairs of P that contain a are ac and ae. Now $abf \in N$, $ab \notin P$, $af \notin P$, so by 1 $bf \in P$. Also, $adf \in N$, so by 1 $df \in P$. The triples of N that do not yet contain pairs of P are $bcd\ bce\ bde\ cef$. The only two possible pairs that can 'hit' these four triples are bd and ce. Thus $P = ac\ ae\ bd\ bf\ ce\ df$. Call the resulting 3-graph $A(7,4)$.

Case 1.2 There are vertices $xy \in S_j$, $z \in S_{j+1}$ such that $xy \in P$, $xz \in P$. By symmetry, we may assume that $x = a$, $y = b$, $z = c$. Thus the two pairs of P that contain a are ab and ac. Now $ade \in N$, so by 1 $de \in P$. Also, $adf \in N$, so by 1 $df \in P$. The triples of N that do not yet contain pairs of P are $bcd\ bce\ bcf\ cef$. There are three choices for the two remaining pairs of P that satisfy 1: $bc\ ce$, $bc\ cf$, and $bc\ ef$. However, in the first two cases some vertex of $H - g$ is in more than two pairs of P. Thus $P = ab\ ac\ bc\ de\ df\ ef$. Call the resulting 3-graph $B(7,4)$.

It is not difficult to show that if the hypothesis of Case 1.1 does not hold, then the hypothesis of Case 1.2 does. This concludes Case 1.

Case 2. Some vertex of $H - g$ (say a) is in at least three pairs of P. Since a has degree three in $H - g$, and since no vertex has degree greater than six, a is in exactly three pairs of P. Thus $H - a$ is type [6 6 3] and is thus isomorphic to $T(6,3)$. The fact that $H - ag$ has edges cde cdf bef implies that the edges of $H - a$ are bcg bdg cde cdf bef efg (a and g are twins in H). The pairs of P that do not contain a are bc bd ef. Now by 1

since $abe \in N$ and $be \notin P$, at least one of $ab \in P$, $ae \in P$
since $abf \in N$ and $bf \notin P$, at least one of $ab \in P$, $af \in P$
since $acd \in N$ and $cd \notin P$, at least one of $ac \in P$, $ad \in P$
since $ace \in N$ and $ce \notin P$, at least one of $ac \in P$, $ae \in P$
since $acf \in N$ and $cf \notin P$, at least one of $ac \in P$, $af \in P$
since $ade \in N$ and $de \notin P$, at least one of $ad \in P$, $ae \in P$
since $adf \in N$ and $df \notin P$, at least one of $ad \in P$, $af \in P$.

It follows that the remaining pairs of P are ab ac ad, ac ae af, or ad ae af. The latter two cases yield isomorphic 3-graphs (swapping c and d gives an isomorphism). Call the 3-graphs resulting from the first two cases $C(7,4)$ and $D(7,4)$, respectively. The results of this section are summarized in the following table.

Observe that $C(7,4) \cong T(7,4)$: put $S_0 = abg$, $S_1 = cd$, $S_2 = ef$.

All 3-graphs of type [7 12 4]			
	Edges	Degrees	Twin-classes
$A(7,4)$	abc abd cde cdf aef bef acg aeg bdg bfg ceg dfg	$6\ 5^6$	— — —
$B(7,4)$	abc abd cde cdf aef bef abg acg bcg deg dfg efg	$6\ 5^6$	ab ef
$C(7,4)$	abc abd cde cdf aef bef abg acg adg bcg bdg efg	$6^3 5^2 4^2$	abg cd ef
$D(7,4)$	abc abd cde cdf aef bef acg aeg afg bcg bdg efg	$6^2 5^4 4$	ag ef

2.4 Critical 3-graphs with $n = 8$

2.4.1 Type [8 4 3]

Stanton [S] showed that there is a unique $(8,5,2)$ covering design with block size five. Taking the complement gives the unique 3-graph of type [8 4 3], namely $T(8,3)$.

2.4.2 Type [8 8 4]

Consider a 3-graph of this type. No vertex has degree greater than three, since $t(7,4) = 2$. This together with the fact that the average degree is exactly three implies that every vertex has degree three. Let $V = abcdefgh$. $H - h$ has type [7 5 3].

Case 1. $H - h \cong A(7,3)$.
We may assume that $H - h$ has edges abc abd acd bcd efg. Since $H - g$ has type [7 5 3], it follows that $H - g$ is isomorphic to one of $A(7,3)$, $B(7,3)$, $C(7,3)$ or $D(7,3)$. But $H - g$ has at least four vertices of degree three, namely $abcd$, so $H - g$ must be $A(7,3)$, and so the edges of $H - g$ are the edges of $H - gh$ plus the edge efh. Repeating this argument for $H - f$ and $H - e$, it follows that egh and fgh are edges of H. Thus H has edges abc abd acd bcd efg efh egh fgh. Call this 3-graph $A(8,4)$.

Case 2. $H - h \cong B(7,3)$.
We may assume that $H-h$ has edges abc adg bcg def efg. Thus $H-ah$ has edges bcg def efg. It is a routine but tedious task to verify that no 3-graph isomorphic to $H - ah$ is obtained by deleting a vertex from $A(7,3)$, $C(7,3)$, or $D(7,3)$; thus $H - a \cong B(7,3)$. This is possible in only one way: $H - a$ has edges bch dgh bcg def efg (observe that a and h are twins in H). But now g has degree four, contradiction. Thus there is no 3-graph H with $H - h \cong B(7,3)$.

Case 3. $H - h \cong C(7,3)$.
We may assume that $H - h$ has edges abc abd cde cdf efg. $H - g$ is isomorphic to one of $A(7,3)$, $B(7,3)$, $C(7,3)$ or $D(7,3)$. Since $H - gh$ has two vertices of degree three and at least one of degree two, $H - g$ must be $C(7,3)$. This is possible only if $H - g$ has edges abc abd cde cdf efh (observe that g and h are twins in H). Since $H - g$ and $H - h$ are determined, all remaining edges contain g and h. Since every vertex has degree three, this happens only if the remaining edges are agh bgh. Thus H has edges abc abd cde cdf efg efh agh bgh. Call this 3-graph $B(8,4)$.

Case 4. $H - h \cong D(7,3)$.
We may assume that $H - h$ has edges abc def adg beg cfg. $H - a$ is isomorphic to one of $A(7,3)$, $B(7,3)$, $C(7,3)$ or $D(7,3)$. It is a routine but tedious task to verify that a 3-graph isomorphic to $H - ah$ can extend only to $D(7,3)$, and in only three ways: the edges of $H - a$ that contain h are bdh ceh, bch dgh, or bfh cdh. Now recall that every vertex of H has degree three. This observation eliminates the second of the above three cases, since g would have degree four. Also, it implies that the last edge of H must be afh in the first case, and aeh in the third case. Thus H has edges abc adg def beg cfg bdh ceh afh or edges abc adg bef beg cfg bfh cdh aeh. These two 3-graphs are isomorphic (swapping a with d, b with e, and c with f gives an isomorphism). Call this 3-graph $C(8,4)$.

The results of this section are summarized in the following table.

Observe-that $B(8,4) \cong T(8,4)$: put $S_0 = ab$, $S_1 = cd$, $S_2 = ef$, $S_3 = gh$.

All 3-graphs of type [8 8 4]			
	Edges	Degrees	Twin-classes
$A(8,4)$	*abc abd acd bcd efg efh egh fgh*	3^8	*abcd efgh*
$B(8,4)$	*abc abd cde cdf efg efh agh bgh*	3^8	*ab cd ef gh*
$C(8,4)$	*abc adg def beg cfg bdh ceh afh*	3^8	---

2.4.3 Type [8 20 5]

No vertex has degree greater than eight, since $t(7,11) = 3$. Since the average degree is $60/8$, some vertex has degree eight. In fact, since $(60-24)/5 > 7$, at least four vertices have degree eight. Let $V = abcdefgh$ and let h be a vertex of degree eight. $H-h$ is one of $A(7,4), B(7,4), C(7,4)$ or $D(7,4)$. Relabel vertices of $V-h$ so that $H-gh$ is $T(6,3)$ and has edges *abc abd cde cdf aef bef*.

In all cases that follow, N is those triples of $V-h$ that are not edges of $H-h$, and P is those pairs of $V-h$ that are in edges with h. Observe that every triple of N must contain a pair of P (otherwise, the complement of the triple in $V-h$ is a transversal of H of size four).

Case 1. $H-h \cong A(7,4)$.

We may assume that the edges of $H-h$ are *abc abd cde cdf aef bef acg aeg bdg bfg ceg dfg*. Thus the triples of N are *abe abf abg acd ace acf ade adf adg afg bcd bce bcf bcg bde bdf beg cdg cef cfg def deg efg*.

By the symmetry of $H-h$, and the fact that at least two of $V-gh$ have degree eight, we may assume that a has degree eight. Thus $H-a$ is one of $A(7,4)$, $B(7,4)$, $C(7,4)$ or $D(7,4)$. Since $H-ah$ has $n = 6$ $m = 7$ and no twin-classes, it follows that$H-a \cong A(7,4)$. Observe that g is the only vertex of $H-ah$, such that there are three other vertices (namely *bdf*), each two of which is in an edge with g. It follows that g is the vertex of degree six in $H-a$. The edges of $H-a$ that contain h must be *bch bdh cgh efh egh*; the remaining three edges of H must contain a and h. But then *cfg* is a triple of N that does not contain a pair of P, contradiction. So $H-h$ cannot be isomorphic to $A(7,4)$.

Case 2. $H-h \cong C(7,4)$, and $H-v \ncong A(7,4)$ for any $v \in V$.

We may assume that the edges of $H-h$ are *abc abd cde cdf aef bef aeg afg beg bfg cdg efg* and so the triples of N are *abe abf abg acd ace acf acg ade adf adg bcd bce bcf bcg bde bdf bdg cef ceg cfg def deg dfg*.

Case 2.1 At least one of *ab* has degree eight.

By symmetry, we may assume that a has degree eight, so $H-a$ is one of $B(7,4)$, $C(7,4)$ or $D(7,4)$. Since *efg* is a twin-class in $H-ah$, $H-a$ must be $C(7,4)$. This can happen only if the edges of $H-a$ that contain h are *bch bdh efh egh fgh*. Let Q be the vertices of $V-ah$ that are in edges of H with ah. Since $abe \in N$

and $be \notin P$, at least one of ab ae is in P, so at least one of be is in Q. Similarly, Q contains at least one of each of bf bg cd ce cf cg de df dg. It follows that Q must be bcd. Thus $P = ab\ ac\ ad\ bc\ bd\ ef\ eg\ \ fg$. Call the resulting 3-graph H_1.

Case 2.2 At least one of cd has degree eight.

By symmetry, we may assume that c has degree eight, so $H - c$ is one of $B(7,4)$, $C(7,4)$ or $D(7,4)$. Since efg is a twin-class in $H - ch$, $H - c$ must be $C(7,4)$. This can happen only if the edges of $H - c$ that contain h are abh deh dfh dgh. Let Q be the vertices of $V - ch$ that are in edges of H with ch. Since $acd \in N$ and $ad \notin P$, at least one of ac cd is in P, so at least one of ad is in Q. Similarly, Q contains at least one of ae af ag bd be bf bg ef eg fg. It is a routine exercise to verify that Q must be one of $abfg$ $abeg$ $abef$ $defg$. By the symmetry of efg, the 3-graphs corresponding to the first three cases are isomorphic. Thus $P = ab\ de\ df\ dg\ ac\ bc\ cf\ cg$, or $P = ab\ de\ df\ dg\ cd\ ce\ cf\ cg$. Call the resulting 3-graphs and H_2 and H_3 respectively. Observe that $H_3 \cong H_1$.

Case 2.3 None of $abcd$ has degree eight.

Thus each of efg has degree eight. Throughout Case 2.3 we let Q be the set of vertices of $V - gh$ that are in edges of H with gh.

Case 2.3.1 $H - g \cong C(7,4)$.

Observe that this can happen in three ways, and that the number of edges that contain h and vertices $abcdef$, respectively, is 113322, 221133, 332211. But in the second case e has degree nine and in the third case a has degree eight, so the first case must hold. Thus the edges of $H - g$ that contain h are abh cdh ceh cfh deh dfh. Since $acg \in N$ and $ac \notin P$, at least one of ah ch is in P, so at least one of ac is in Q. Similarly, Q contains at least one of ad bc bd. Q must be either ab or cd. Since c does not have degree eight, Q must be ab, and so $P = ab\ ag\ bg\ cd\ ce\ cf\ de\ df$. Call the resulting 3-graph H_4.

Case 2.3.2 $H - g \cong D(7,4)$.

Let $uv = ab$, $wx = cd$, $yz = ef$. Then the number of edges that contain h and vertices $uvwxyz$, respectively, is 212232, 223221, 322122. But in the first case y (that is, e or f) has degree nine and in the third case u (that is, a or b) has degree eight, so the second case must hold. We may relabel vertices so that $w = c$ and $y = e$. Thus the edges of $H - g$ that contain h are abh ach bch ceh deh dfh. Since $adg \in N$ and $ad \notin P$, at least one of ad is in Q. Similarly, Q contains at least one of each of bd cf, so Q must be cd or df. Since c does not have degree eight, Q is df and $P = ab\ ac\ bc\ ce\ de\ df\ dg\ fg$. Call the resulting 3-graph H_5.

Case 2.3.3 $H - g \cong B(7,4)$.

By symmetry (namely, since ab cd ef are each twins in $H - h$) there are only three subcases to consider.

Subcase 2.3.3.1 The edges of $H - g$ that contain h are abh ach bch deh dfh efh. Since $adg \in N$ and $ad \notin P$, at least one of ad is in Q. Likewise, at least one of each of $bd\ ce\ cf$ is in Q, so Q must be cd and $P = ab\ ac\ bc\ cg\ de\ df\ dg\ ef$. Call the resulting 3-graph H_6. Observe that $H_6 \cong H_5$.

Subcase 2.3.3.2 The edges of $H - g$ that contain h are $abh\ afh\ bfh\ cdh\ ceh\ deh$. Arguing as in the previous subcase, Q must contain at least one of each of $ac\ ad\ bc\ bd\ cf\ df$. Thus $Q = cd$ and $P = ab\ af\ bf\ cd\ ce\ cg\ de\ dg$. Call the resulting 3-graph H_7. Observe that H_7 is isomorphic to H_4.

Subcase 2.3.3.3 The edges of $H - g$ that contain h are $aeh\ afh\ bch\ bdh\ cdh\ efh$. Arguing as in the previous subcase, Q must contain at least one of each of $ab\ ac\ ad\ ce\ cf\ de\ df$. But this is not possible, since Q contains only two vertices.

Case 3. $H - h \cong B(7,4)$, and $H - v \ncong A(7,4)$, $H - v \ncong B(7,4)$, for all $v \in V$.
We may assume that the edges of $H - h$ are $abc\ abd\ cde\ cdf\ aef\ bef\ abg\ acg\ bcg\ deg\ dfg\ efg$ and so the triples of N are $abe\ abf\ acd\ ace\ acf\ ade\ adf\ adg\ aeg\ afg\ bcd\ bce\ bcf\ bde\ bdf\ bdg\ beg\ bfg\ cdg\ cef\ ceg\ cfg\ def$.

Case 3.1 At least one of ab has degree eight.
By symmetry, we may assume that a has degree eight. Throughout Case 3.1, let Q be the set of vertices of $V - ah$ that are in edges of H with ah.

Case 3.1.1 $H - a \cong B(7,4)$.
$H - ah$ can extend to $H - a \cong B(7,4)$ in only one way (see Appendix 1), namely the edges of $H - a$ that contain h are $bch\ bdh\ bgh\ cgh\ efh$. But then Q must contain at least one of each of $be\ bf\ cd\ ce\ cf\ de\ df\ dg\ eg\ fg$, which is not possible.

Case 3.1.2 $H - a \cong D(7,4)$.
$H - ah$ can extend to $H - a \cong D(7,4)$ in only one way (see Appendix 1), namely the edges of $H - a$ that contain h are $bdh\ beh\ bfh\ cgh\ efh$. Q must contain at least one of each of $cd\ ce\ cf\ de\ df\ dg\ eg\ fg$. Thus $Q = cdg$ or $Q = def$ and thus $P = ac\ ad\ ag\ bd\ be\ bf\ cg\ ef$ or $P = ad\ ae\ af\ bd\ be\ bf\ cg\ ef$. Call the resulting 3-graphs H_8 and H_9 respectively.

Case 3.2 At least one of ef has degree eight.
By symmetry, we may assume that e has degree eight. Throughout Case 3.2, let Q be the set of vertices of $V - eh$ that are in edges of H with eh.

Case 3.2.1 $H - e \cong B(7,4)$.
$H - eh$ can extend to $H - e \cong B(7,4)$ in only one way (see Appendix 1), namely the edges of $H - e$ that contain h are $afh\ bfh\ cdh\ dgh\ fgh$. But then Q must contain at least one of each of $ab\ ac\ ad\ ag\ bc\ bd\ bg\ cf\ cg\ df$, which is not possible.

Case 3.2.2 $H - e \cong D(7,4)$.
$H - eh$ can extend to $H - e \cong D(7,4)$ in two ways (see Appendix 1), namely the edges of $H - e$ that contain h are either $adh\ afh\ bdh\ bfh\ cgh$ or else $afh\ bfh\ cdh\ cgh\ dgh$.

In the first case, Q must contain at least one of each of $ab\ ac\ ag\ bc\ bg\ cf\ df$. Thus Q must be abf and so $P = ad\ ae\ af\ bd\ be\ bf\ cg\ ef$. Call the resulting 3-graph H_9.

In the second case, Q must contain at least one of each of $ab\ ac\ ad\ ag\ bc\ bd\ bg\ cf\ df$. Thus $Q = abf$ and so $P = ae\ af\ be\ bf\ cd\ cg\ dg\ ef$. Call the resulting 3-graph H_{10}.

Case 3.3 None of $abef$ have degree eight.

Then each of $cdgh$ has degree eight. In particular, $H - c$ must be $B(7,4)$ or $D(7,4)$. But from Appendix 1 it follows that $H - ch$ does not extend to $D(7,4)$, and $H - ch$ extends to $B(7,4)$ only if the edges of $H - c$ that contain h are $abh\ agh\ bgh\ deh\ dfh$. But then Q must contain at least one of each of $ad\ ae\ af\ bd\ be\ bf\ dg\ ef\ eg\ fg$, which is not possible.

Case 4. $H - v \cong D(7,4)$, for each $v \in V$ with degree eight.

In particular, $H - h \cong D(7,4)$. We may assume that the edges of $H - h$ are $abe\ abd\ cde\ cdf\ aef\ bef\ abg\ acg\ bcg\ ceg\ deg\ dfg$ and so the triples of N are $abe\ abf\ acd\ ace\ acf\ ade\ adf\ adg\ aeg\ afg\ bcd\ bce\ bcf\ bde\ bdf\ bdg\ beg\ bfg\ cdg\ cef\ cfg\ def\ efg$.

Case 4.1 At least one of ab has degree eight.

By symmetry, we may assume that a has degree eight, so $H - a \cong D(7,4)$. $H - ah$ extends to $D(7,4)$ in only one way (see Appendix 1), namely if the edges of $H - a$ that contain h are $bch\ bdh\ bgh\ cgh\ efh$. But then Q, the vertices that are in edges with ah, has only three vertices yet must contain at least one of each of $be\ bf\ cd\ cf\ de\ df\ dg\ eg\ fg$, which is not possible.

Case 4.2 Vertex d has degree eight.

$H - dh$ extends to $H - d \cong D(7,4)$ in two ways (corresponding to $D - d$ and $D - e$ in Appendix 1), namely if the edges of $H - d$ that contain h are $abh\ ceh\ cfh\ egh\ fgh$, or else $abh\ afh\ bfh\ ceh\ egh$. Let Q be the vertices that are in edges with dh.

In the first case, Q must contain at least one of each of $ac\ ae\ af\ ag\ bc\ be\ bf\ bg\ cg\ ef$, which is not possible. In the second case, cfg is a triple of N that contains no pair of P, contradiction.

Case 4.3 Vertex e has degree eight.

$H - eh$ extends to $H - e \cong D(7,4)$ in two ways (corresponding to $D - d$ and $D - e$ in Appendix 1), namely if the edges of $H - e$ that contain h are either $afh\ bfh\ cdh\ cgh\ dgh$, or else $adh\ afh\ bdh\ bgh\ cgh$. In the first case Q must contain at least one of each of $ab\ ac\ ad\ ag\ bc\ bd\ bg\ cf\ df\ fg$, which is not possible. In the second case bcf is a triple of N that contains no pair of P, contradiction.

Case 4.4 None of $abde$ has degree eight.

Then each of $cfgh$ has degree eight. In particular, f has degree eight, and so $H - f \cong D(7,4)$. This can happen only if the edges of $H - f$ that contain h

are *aeh beh cdh dgh*. Then Q must contain at least one of each of *ab ac ad ag bc bd bg ce cg de eg*. Thus Q is one of *abcd abeg acdg bcdg*. But in each case, either d or e has degree eight, contradiction. This concludes Case 4. Observe that no 3-graph H satisfies the hypothesis of Case 4.

The results of this section are summarized in the following table. Observe that $T(8,5) \cong H_1 \cong H_3$, $U(8,5) \cong H_4 \cong H_7$, $V(8,5) \cong H_5 \cong H_6 \cong H_{10}$, $W(8,5) \cong H_2$, $X(8,5) \cong H_9$ and $Y(8,5) \cong H_8$.

All 3-graphs of type [8 20 5]			
	Edges	Degrees	*Twin − classes*
$T(8,5)$	*abc abd abe abf acd ace acf agh bcd bce bcf bgh cgh def deg deh dfg dfh efg efh*	$8^6\ 6^2$	*abc def gh*
$U(8,5)$	*abc abd abe acd ace adh aeh afg bcd bce bfg bfh bgh cfg cfh cgh def deg deh fgh*	$8^4\ 7^4$	*bc de fg*
$V(8,5)$	*abc abd abe acd ace afg afh agh bcd bce bch bfg bgh cfg cgh def deg deh dfh efh*	$8^4\ 7^4$	*bc de*
$W(8,5)$	*abc abd abe acd ace afg afh agh bcd bce bfg bfh bgh cfg cgh def deg deh dfh efh*	$8^4\ 7^4$	*ab de fh*
$X(8,5)$	*abf abg abh acd ace ade afg bcd bce bfg cfg cfh cgh def deg deh dfh dgh efh egh*	$8^5\ 7^2\ 6$	*de fg*
$Y(8,5)$	*abf abg abh acd ace ade afg agh bcd bce bfg bgh cfg cfh cgh def deg deh dfh efh*	$8^4\ 7^4$	*de fh*

2.5 Critical 3-graphs with n = 9

2.5.1 Type [9 7 4]

No vertex of H has degree greater than three, since $t(8,3) = 2$. Since the average degree is $21/9$, some vertex has degree three. In fact, since $(21 - 6)/7 > 2$, there are at least three vertices with degree three. Deleting any of these vertices leaves the unique critical 3-graph of type [8 4 4]. Call this 3-graph $A(8,3)$.

Let $V = abcdefghi$, and let i have degree three. We may assume that the edges of $V - i$ are *abc abd cde fgh*. Observe that the twin-classes of $V - i$ are *ab cd fgh*. Let N be all quintuples of $V - i$ that do not contain an edge of $H - i$. Thus $N =$ *abefg abefh abegh acdfg acdfh acdgh acefg acefh adegh adefg adefh adegh bcdfg bcdfh bcdgh bcefg bcefh bcegh bdefg bdefh bdegh*.

Let P be all pairs of $V - i$ that are in an edge of H with i. Observe that every quintuple of N contains a pair of P (otherwise, the complement of the quintuple in $V - i$ is a transversal of H of size three, contradiction.)

Case 1. At least one of *fgh* has degree three in H.
By symmetry, we may assume that h has degree three. Since $H - h \cong A(8,3)$ and has edges *abc abd cde*, the remaining edge of $H - h$ must be *fgi*. Now the remaining two edges of H contain *hi*. Let Q be the two vertices in edges with *hi*.

Observe that $fg \in P$ and that the remaining two pairs in P contain h. Since $abefh \in N$ it follows that at least one of $abef$ is in Q. Similarly, at least one of each of $abeg$ $acdf$ $acdg$ $acef$ $aceg$ $adef$ $adeg$ $bcdf$ $bcdg$ $bcef$ $bceg$ $bdef$ $bdeg$ is in Q. It is a routine task to verify that Q must be ab, de, or fg. Call the resulting 3-graphs H_1, H_2, and H_3, respectively. Thus

H_1 has edges abc abd ahi bhi cde fgh fgi,

H_2 has edges abc abd cde dhi ehi fgh fgi,

H_3 has edges abc abd cde fgh fgi fhi ghi.

Case 2. Vertex e has degree three in H.
Since $H - e \cong A(8,3)$ and contains edges abc abd fgh, the remaining edge of $H - e$ must be cdi. The remaining two edges of H contain ei. Let Q be the two vertices in edges with ei.

Observe that $cd \in P$, that the remaining two pairs in P contain e, and that $abefg \in N$. Thus Q contains at least one of each of $abfg$. Similarly, Q contains at least one of each of $abfh$ $abgh$ $acfg$ $acfh$ $acgh$ $adfg$ $adfh$ $adgh$ $bcfg$ $bcfh$ $bcgh$ $bdfg$ $bdfh$ $bdgh$. It follows that $Q = ab$. Thus H has edges abc abd aei bei cde cdi fgh. Call this 3-graph H_4.

Case 3. At least one of cd and none of $efgh$, has degree three in H.
By symmetry, we may assume that c has degree three. $H - c \cong A(8,3)$ and contains edges abd fgh. Since e does not have degree three, the remaining edges of $H - c$ must be eix and iyz, where xyz is abd or fgh. By the symmetry of ab and fgh, there are only three cases to consider, depending on whether x is d, one of ab, or one of fgh. Let Q contain the vertex in an edge with ci.

Case 3.1 $x = d$, $y = a$, $z = b$.
Thus P contains de and ab. But then Q must contain at least one of each of $adfg$ $adfh$ $adgh$ $aefg$ $aefh$ $aegh$ $bdfg$ $bdfh$ $bdgh$ $befg$ $befh$ $begh$, which is not possible.

Case 3.2 $x = a$, $y = b$, $z = d$.
Thus P contains ae and bd. But then Q must contain at least one of each of $adfg$ $adfh$ $adgh$ $befg$ $befh$ $begh$, which is not possible.

Case 3.3 $x = f$, $y = g$, $z = h$.
Thus P contains ef and gh. But then Q must contain at least one of each of $adfg$ $adfh$ $bdfg$ $bdfh$. Thus Q contains f, but then f has degree three, contradiction.

Case 4. None of $cdefgh$ have degree three in H.
Both ab have degree three in H. Thus $H - a \cong A(8,3)$ and contains edges cde fgh. Since neither c nor d has degree three, and since b is in at most one edge with h, the remaining edges of $H - a$ must be bxi yzi, where $xyz = fgh$. By

symmetry of fgh, we may assume that $x = f, y = g, z = h$. Thus P contains bf and gh. Let Q contain the vertex in an edge with ai. Observe that Q must contain at least one of each of $cdfg$ $cdfh$ $cefg$ $cefh$ $defg$ $defh$. But then Q contains f, and f has degree three, contradiction.

The results of this section are summarized in the following table. Observe that $H_1 \cong A(9,4)$, $H_2 \cong B(9,4)$, $H_3 \cong C(9,4)$, $H_4 \cong D(9,4)$. Also observe that $H_2 \cong T(9,4)$: put $S_0 = fg$, $S_1 = hi$, $S_2 = ab$, $S_3 = cd$, $S_4 = e$.

All 3-graphs of type [9 7 4]			
	Edges	Degrees	Twin-classes
$A(9,4)$	*abc abd ahi bhi cde fgh fgi*	$3^4 2^4 1$	*ab cd fg hi*
$B(9,4)$	*abc abd cde dhi ehi fgh fgi*	$3^3 2^6$	*ab fg hi*
$C(9,4)$	*abc abd cde fgh fgi fhi ghi*	$3^4 2^4 1$	*ab cd fghi*
$D(9,4)$	*abc abd aei bei cde cdi fgh*	$3^6 1^3$	*ab cd ei fgh*

2.5.2 Type [9 12 5]

No vertex of H has degree greater than four, since $t(8,7) = 3$. Since the average degree is exactly four, every vertex has degree four. Thus for each $v \in V, H - v$ is type [8 8 4] and so is one of $A(8,4)$, $B(8,4)$ or $C(8,4)$.

Every pair of vertices is in at least one edge (otherwise, some pair of vertices xy intersects eight edges; since $t(7,4) = 2$, a transversal of $H - xy$ together with xy is a transversal of H of size four, contradiction). Since the number of pairs of vertices is equal to the sum of the degrees of the vertices (namely thirty-six), it follows that no pair of vertices is in more than one edge. Thus $H - v$ can not be $A(8,4)$ or $B(8,4)$.

Let $V = abcdefghi$. Since $H - v$ must be $C(8,4)$ for all $v \in V$, we may assume that the edges of $H - i$ are abc adg afh beh bfg cdh ceg def. Now $H - h$ is $C(8,4)$ only if the edges of $H - h$ that contain i are either afi bei cdi or else aei bdi cfi. But in the former case af be cd are each in more than one edge of H, contradiction. Thus the latter case holds.

Now the edges of $H - a$, except for those containing hi, are known to be bdi beh bfg cdh ceg cfi def. It follows that $H - a$ is $C(8,4)$ only if the edge of $H - a$ containing hi is ghi. Thus H is unique up to isomorphism. Observe that this 3-graph is the affine plane AP_9, with four sets of parallel lines of three points (every two points are in exactly one line).

All 3-graphs of type [9 12 5]			
	Edges	Degrees	*Twin – classes*
$A(9,5)$	*abc adg aei afh bdi beh bfg cdh ceg cfi def ghi*	4^9	— — —

2.5.3 Type [9 30 6]

No vertex of H has degree greater than ten, since $t(8,19) = 4$. Since the average degree is exactly ten, every vertex has degree ten. Thus for each vertex $v \in V$,

$H - v$ is type [8 20 5] and so is one of $T(8,5)$, $U(8,5)$, $V(8,5)$, $W(8,5)$, $X(8,5)$ or $Y(8,5)$. Observe that

every pair of vertices is in at least two edges. (2)

(otherwise, some pair of vertices xy intersects at least nineteen edges; since $t(7,11) = 3$, a transversal of $H - xy$ together with xy is a transversal of H of size five, contradiction). Also,

every pair of vertices is in at most four edges. (3)

(Assuming the contrary, let rs be such a pair. Then $H - r$ has $n = 8$ $m = 20$ and a vertex s with degree at most five. But no such 3-graph has $\tau = 5$, so $\tau(H - r) = 4$, and a transversal of $H - r$ together with r is a transversal of H of size five, contradiction.) Let $V = abcdefghi$.

Case 1. $H - v \cong T(8,5)$ for some $v \in V$.
We may assume that $v = i$ and that $S_0 = fgh$, $S_1 = abc$ and $S_2 = de$. Thus the edges of $H - i$ are abc abd abe acd ace afg afh agh bcd bce bfg bfh bgh cfg cfh cgh def deg deh fgh.

Observe that each of df dg dh ef eg eh is in only one edge of $H - i$. Thus (2) implies that dfi dgi dhi efi egi ehi are edges of H. Also, since fg fh gh are each in four edges of $H - i$, none of fgi fhi ghi is an edge of H. $H - ai$ is $T(7,4)$, and now $H - a$ can be one of $T(8,5)$, $U(8,5)$, $V(8,5)$, $W(8,5)$, $X(8,5)$ or $Y(8,5)$ only if the edges of $H - a$ that contain i are bci dei dfi dgi efi egi ehi (see Appendix 2). $H - bi$ is $T(7,4)$. The edges of $H - b$ that contain i must be aci dei dfi dgi dhi efi egi ehi. Also, $H - ci$ is $T(7,4)$ and the edges of $H - c$ that contain i must be abi dei dfi dgi dhi efi egi ehi. Thus the ten edges of H that contain i are abi aci bci dei dfi dgi dhi efi egi ehi. Observe that $H \cong T(9,6)$: set $S_0 = fgh$, $S_1 = abc$, $S_2 = dei$.

Case 2. $H - v \cong U(8,5)$ for some $v \in V$.
We may assume that $v = i$ and that the edges of $H - i$ are as in the conclusion of Section 2.4.3 (Type [8 20 5]), namely abc abd abe acd ace adh aeh afg bcd bce bfg bfh bgh cfg cfh cgh def deg deh fgh.

Observe that af ag df dg ef eg are each in only one edge of $H - i$ and that fg is in four edges. By (2), afi agi dfi dgi efi egi must be edges of H. By (3), fgi is not an edge of H.

Now $H - hi$ is $T(7,4)$ and $H - h$ must be one of $T(8,5)$, $U(8,5)$, $V(8,5)$, $W(8,5)$, $X(8,5)$, or $Y(8,5)$. By the previous constraints, this can happen only if $H - h \cong U(8,5)$, and the edges of $H - h$ that contain i are afi agi bci dei dfi dgi efi egi (see Appendix 2).

Similarly, $H - bi$ is $B(7,4)$, $H - b$ must be $X(8,5)$, and the edges of $H - b$ that contain i are afi agi chi dei dfi dgi efi egi. Finally, since i has degree ten, the remaining triple of H must be bhi.

Thus the ten edges of H that contain i are afi agi bci bhi chi dei dfi dgi efi egi. Call the resulting 3-graph $U(9,6)$.

Case 3. $H - v \cong V(8,5)$ for some $v \in V$.
We may asume that $v = i$ and that the edges of $H-i$ are as in the conclusion of Section 2.4.3 (Type [8 20 5]), namely abc abd abe acd ace afg afh agh bcd bce bch bfg bgh cfg cgh def deg deh dfh efh. Observe that each of bf cf df ef is in only one edge of $H - i$ and that bc is in four edges. By (2), bfi cfi dfi efi must be edges of H. By (3), bci is not an edge of H.

Now $H - ai$ is $B(7,4)$ and $H - a$ is one of $T(8,5) \ldots Y(8,5)$. But the above constraints imply that this is not possible (see Appendix 2). Thus no 3-graph H satisfies the hypothesis of Case 3.

Case 4. $H - v \cong W(8,5)$ for some $v \in V$.
We may assume that $v = i$ and that the edges of $H - i$ are as in the conclusion of Section 2.4.3 (Type [8 20 5]), namely abc abd abe acd ace afg afh agh bcd bce bfg bfh bgh cfg cgh def deg deh dfh efh. Observe that each of cf ch dg eg is in only one edge of $H - i$ and that fh is in four edges. By (2), cfi chi dgi egi must be edges of H. By (3), fhi is not an edge of H.

Now $H - bi$ is $D(7,4)$ and $H - b$ is one of $T(8,5) \ldots Y(8,5)$. But the above constraints imply that this is not possible (see Appendix 2). Thus no 3-graph H satisfies the hypothesis of Case 4.

Case 5. $H - v \cong X(8,5)$ for some $v \in V$.
We may assume that $v = i$ and that the edges of $H - i$ are as in the conclusion of Section 2.4.3 (Type [8 20 5]), namely abf abg abh acd ace ade afg bcd bce bfg cfg cfh cgh def deg deh dfh dgh efh egh. Observe that each of ah bd be bh is in only one edge of $H - i$ and that de is in four edges. By (2), ahi bdi bei bhi must be edges of H. By (3), dei is not an edge of H.

Now $H - hi$ is $B(7,4)$ and $H - h$ is one of $T(8,5) \ldots Y(8,5)$. But the above constraints imply that this is only possible if $H - h$ is $U(8,5)$ and the edges of $H - h$ that contain i are afi agi bci bdi bei cdi cei fgi (see Appendix 2: note that vertices $abcdefg$ of $B(7,4)$ correspond respectively to vertices $fgbcdea$ here). Thus the hypothesis of Case 2 holds.

Case 6. $H - v \cong Y(8,5)$ for some $v \in V$.
We may assume that $v = i$ and that the edges of $H - i$ are as in the conclusion of Section 2.4.3 (Type [8 20 5]), namely abf abg abh acd ace ade afg agh bcd bce bfg bgh cfg cfh cgh def deg deh dfh efh. Observe that each of bd be dg eg is in only one edge of $H - i$ and that de is in four edges. By (2), bdi bei dgi egi must be edges of H. By (3), dei is not an edge of H.

Now $H - ai$ is $D(7,4)$ and $H - a$ is one of $T(8,5) \ldots Y(8,5)$. But the above constraints imply that this is not possible (see Appendix 2). Thus no 3-graph H satisfies the hypothesis of Case 6.

The results of this section are summarized in the following table. Observe that $U(9,6) - a \cong U(8,5)$, $U(9,6) - b \cong X(8,5)$, $U(9,6) - d \cong X(8,5)$, $U(9,6) - f \cong X(8,5)$, $U(9,6) - h \cong U(8,5)$, $U(9,6) - i \cong U(8,5)$, and that $T(9,6) - v \cong T(8,5)$ for all $v \in T(9,6)$.

All 3-graphs of type [9 30 6]			
	Edges	Degrees	Twin-classes
$T(9,6)$	*abc abd abe abf acd ace acf agh agi ahi bcd bce bcf bgh bgi bhi cgh cgi chi def deg deh dei dfg dfh dfi efg efh efi ghi*	10^9	*abc def ghi*
$U(9,6)$	*abc abd abe acd ace adh aeh afg afi agi bcd bce bci bfg bfh bgh bhi cfg cfh cgh chi def deg deh dei dfi dgi efi egi fgh*	10^9	*bc de fg*

This concludes the catalogue of all (non-trivial) critical 3-graphs with nine vertices.

4. Two proofs.

In this last section we supply proofs that $t(10,19) \leq 5$ and $t(10,44) \leq 6$, from which it follows that $m(10,6) \geq 20$ and $m(10,7) \geq 45$. As noted in Section 1, proofs of these results have not appeared before.

We first present two lemmas.

Lemma 1. *Let H be a 3-graph of type [9 13 5]. Then H is isomorphic to AP_9 plus one edge. (We call this 3-graph AP_9^+.)*

Proof: No vertex of H has degree greater than five, since $t(8,7) = 3$. Since the sum of the degrees is $39 = 4 \times 9 + 3$, there are at least three vertices of degree five. For any such vertex v, $H - v$ is type [8 8 4] and so is one of $A(8,4)$, $B(8,4)$, or $C(8,4)$.

Let $V = abcdefghi$, and let vwx be three vertices of degree five.

Case 1. $H - v \cong A(8,4)$.
We may assume that $v = i$ and that the edges of $H - i$ are *abc abd acd bcd efg efh egh fgh*. By the symmetry of $H - i$ we may assume that $w = h$. Thus $H - h$ is one of $A(8,4)$ $B(8,4)$ $C(8,4)$. Since $H - h$ already contains edges *abc abd acd bcd efg*, $H - h$ must be $A(8,4)$ and have edges *abc abd acd bcd efg efi egi fgi*. Now e and f each have degree five. The edges of $H - e$ and $H - f$ must be *abc abd acd bcd fgh fgi fhi ghi* and *abc abd acd bcd egh egi ehi ghi* respectively. But now H has fourteen edges, contradiction.

Case 2. $H - v \cong B(8,4)$.
We may assume that $v = i$ and that the edges of $H - i$ are *abc abd ade cdf efg efh agh bgh*. By the symmetry of $H - i$, we may assume that $w = h$. Thus

$H-h$ is one of $A(8,4)$ $B(8,4)$ $C(8,4)$. Since $H-h$ already contains edges abc abd cde cdf efg, $H-h$ must be $B(8,4)$ and have edges abc abd cde cdf efg efi agi bgi. Now g has degree five. $H-g$ must be $B(8,4)$ and have edges abc abd cde cdf efh efi ahi bhi. But now H has thirteen edges, and yet $abef$ is a transversal of size four, contradiction.

Case 3. $H-y \cong C(8,4)$, for every vertex y of degree 5.
We may assume that $v=i$ and that the edges of $H-i$ are abc adg afh beh bfg cdh ceg def. Thus the set N of quadruples of $V-i$ that contain no edges of $H-i$ is $abde$ $abdf$ $abdh$ $abef$ $abeg$ $abgh$ $acde$ $acdf$ $acef$ $aceh$ $acgh$ $adeh$ $aefg$ $aegh$ $bcde$ $bcdf$ $bcdg$ $bcef$ $bcfh$ $bcgh$ $bdeg$ $bdfh$ $bdfg$ $cdfg$ $cefh$ $cfgh$ $degh$ $dfgh$ $efgh$.

Let P be the pairs of vertices of $V-i$ that are in an edge of H with i. Observe that every quadruple of N must contain a pair of P, for otherwise the complement of the quadruple in $V-i$ is a transversal of H of size four, contradiction.

Observe that there is an automorphism of $H-i$ that maps each vertex of gh to the other. Also, for any two vertices of $abcdef$, there is an automorphism that maps one to the other. There are two cases to consider.

Case 3.1 At least one of $abcdef$ has degree five in H.
By the aforementioned symmetry, we may assume that a has degree five. There are two ways that $H-ai$ can extend to $H-a \cong C(8,4)$. Let Q be the vertices of $V-ai$ that are in edges of H with ai.

Case 3.1.1 The edges of $H-a$ that contain i are bci dgi fhi.
Since $abde$ is in N and none of bd be de are in P, at least one of ab ad ae is in P, that is, at least one of bde is in Q. Similarly, at least one of each of bdf bdh bef beg bgh cde cdf cef ceh cgh deh efg egh is in Q. It is a routine exercise to check that (since Q contains only two vertices) this is not possible.

Case 3.1.2 The edges of $H-a$ that contain i are bdi cfi ghi.
At least one of each of bef beg cde ceh deh efg are in Q. Thus Q must be one of be ce de ef eg eh. The resulting six 3-graphs are all isomorphic to AP_9 plus one edge (the edges of $H-i$ together with the edges aei bdi cfi ghi yields AP_9).

Case 3.2 At least one of gh has degree five in H.
By the aforementioned symmetry, we may assume that h has degree five. There are two ways that $H-hi$ can extend such that $H-h \cong C(8,4)$. Let Q be the vertices of $V-hi$ that are in edges of H with hi.

Case 3.2.1 The edges of $H-h$ that contain i are adi bfi cei.
This is not possible, since the quadruple $abeg$ of N contains no pair of P.

Case 3.2.2 The edges of $H-h$ that contain i are aei bdi cfi.

At least one of each of abg acg bcg deg dfg efg is in Q. Thus Q must be one of ab bg cd dg eg fg. The resulting six 3-graphs are all isomorphic to AP_9 plus one edge (the edges of $H - h$ together with edges aei bdi cfi ghi gives AP_9).

This concludes the proof of Lemma 1. ∎

Lemma 2. *Let xy be vertices of a 3-graph H such that $H - x$ is isomorphic to either AP_9 or AP_9^+ and $H - y$ is isomorphic to either AP_9 or AP_9^+. Let P_x be the pairs of vertices in edges of Y that contain x. Let P_y be the pairs of vertices in edges of X that contain y. Then $P_x = P_y$.*

Proof of Lemma 2: The edge of AP_9^+ that can be deleted to leave AP_9 is the only edge that intersects three of the other edges in two vertices. Call this edge U. Deleting a vertex not in U from AP_9^+ leaves a 3-graph isomorphic to $C(8,4)$ plus one edge (namely U), and U intersects at least two of the edges of $C(8,4)$ in two vertices.

If $H - xy$ is isomorphic to $AP_9 - v = C(8,4)$ then the lemma follows from checking that $C(8,4)$ extends to AP_9 in only one way.

If $H - xy$ is isomorphic to $C(8,4)$ plus an edge, then the "extra edge" is the only edge to intersect at least two of the other edges in two vertices. Call this edge U. Since U intersects at least two of the edges of $C(8,4)$ in two vertices, U cannot be an edge of AP_9 with the edges of this copy of $C(8,4)$ (at most one of these two edges could be the "extra edge", so U would still intersect at least one other edge of AP_9 in two vertices, contradicting the fact that edges of AP_9 interect in at most one vertex). Thus $H - xy$ extends to AP_9^+ in only one way, namely U must be the "extra edge" of AP_9^+, and so Lemma 2 holds. ∎

Theorem 1. $t(10,19) \leq 5$.

Proof: By contradiction. Let H be a 3-graph of type [10 19 6]. No vertex of H has degree greater than seven, since $t(9,11) = 4$. Since $2 \times 7 + 8 \times 5 < 57$, which is the sum of the degrees, there are at least three vertices of degree six or seven. Removing a vertex of degree six leaves a 3-graph of type [9 13 5], which must be AP_9^+.

Let abc be vertices of degree six or seven in H. Thus each of $H - a$, $H - b$, and $H - c$ is isomorphic to one of AP_9 or AP_9^+. Let ABC be the edges of $H - a$, $H - b$, and $H - c$ respectively that induce AP_9. By Lemma 2, c is in an edge with the same pairs of vertices of $C - a = A - c$ that a is in an edge with. Note that there are four such pairs. Again by Lemma 2, c is in an edge with the same pairs of vertices of $C - b = B - c$ that b is in an edge with. Now observe that the four pairs of vertices of C that are in an edge with a are distinct from the four pairs of vertices of C that are in an edge with b (because no two edges of AP_9 have two vertices in common). Thus there are at least eight pairs of vertices of C that are in an edge with c. But c has degree at most seven, contradiction. This concludes the proof of the theorem, namely that $t(10,19) \leq 5$. ∎

Theorem 2. $t(10,44) \leq 6$.

Proof: By contradiction. Let H be a 3-graph of type [10 44 7]. No vertex of H has degree greater than fourteen, since $t(9,29) = 5$. Since the average degree is $132/10 > 13$, some vertex has degree fourteen. Since $118/9 > 13$, another vertex has degree fourteen. Deleting either of these two vertices leaves a 3-graph of type [9 30 6], namely $T(9,6)$ or $U(9,6)$. Let $V = abcdefghij$ and let vw be two vertices in V with degree fourteen.

Case 1. $H - v \cong T(9,6)$

We may assume that $v = j$ and that the edges of $H-j$ are as listed in the conclusion of Section 2.5.3 (Type [9 30 6]). By the symmetry of $T(9,6)$, we may assume that $w = a$.

Thus $H - aj \cong T(8,5)$, and so $H - a$ must be $T(9,6)$. Furthermore, since $T(8,5)$ extends to $T(9,6)$ in only one way, the edges of $H - a$ that contain j must be bcj bdj bej bfj cdj cej cfj ghj gij hij. Now b and c have degree at least (and thus exactly) fourteen.

Since $H - bj \cong T(8,5)$, $H - b$ must be $T(9,6)$ and the edges of $H - b$ that contain j must be acj adj aej afj cdj cej cfj ghj gij hij.

Since $H - cj \cong T(8,5)$, $H - c$ must be $T(9,6)$ and the edges of $H - c$ that contain j must be abj adj aej afj bdj bej bfj ghj gij hij. But now j has degree at least fifteen, contradiction.

Case 2. $H - v \cong U(9,6)$.

We may assume that $v = j$ and that the edges of $H - j$ are as listed in the conclusion of Section 2.5.3 (Type [9 30 6]). It is a routine exercise to verify that for any two vertices of ahi, there is an automorphism of $U(9,6)$ that maps one vertex to the other, and that this also holds for any two vertices of $bcdefg$. Thus by relabelling vertices if necessary, we may assume that $w = i$ or $w = c$.

Case 2.1 $w = i$

Thus $H-i \cong U(9,6)$ and $H-ij \cong U(8,5)$. Since $U(8,5)$ extends in only one way to $U(9,6)$, the edges of $H - i$ that contain j must be afj agj bcj bhj chj dej dfj dgj efj egj. The remaining edges contain ij.

Case 2.1.1 At least one of dij eij is an edge of H.

By symmetry, we may assume that dij is an edge. But now d has degree fourteen, so $H-d \cong U(9,6)$. The edges of $H-d$ that contain j must be abj acj ahj bcj efj egj ehj eij fij gij. But now j has degree at least seventeen, contradiction.

Case 2.1.2 At least one of fij gij is an edge of H.

By symmetry, we may assume that fij is an edge. But now f has degree fourteen, so $H-f \cong U(9,6)$. The edges of $H-f$ that contain j must be agj aij bcj bhj cgj chj dej dij eij ghj. But now j has degree at least fifteen, contradiction.

Case 2.1.3 None of dij eij fij gij are edges of H.

Thus aij bij cij hij are edges of H. But now H has $m = 44$, and yet $abcdeh$ is a transversal of size six, contradiction.

Case 2.2 $w = c$
Thus $H - c \cong U(9,6)$. Since $H - ci \cong X(8,5)$, which extends in only one way to $U(9,6)$, the edges of $H - c$ that contain j must be abj adj aej bdj bej bij fgj fhj ghj hij. Now b has degree fourteen, so $H - b \cong U(9,6)$, and the edges of $H - b$ that contain j must be acj adj aej cdj cej cij fgj fhj ghj hij. But now H has $m = 44$, and yet $adeghi$ is a transversal of size six, contradiction.

This concludes the proof of the theorem, namely that $t(10,44) \leq 6$. ∎

4. Appendices.

4.1 Appendix 1

The following shows all isomorphisms of $B(7,4) - v$ and $D(7,4) - v$. Recall that $B(7,4)$ has edges abc abd abg acg aef bcg bef cde cdf deg dfg efg, and $D(7,4)$ has edges abc abd abg acg aef bcg bef cde cdf ceg deg dfg.

Observe that $B - a \cong B - b \cong D - a \cong D - b$, that $B - e \cong B - f \cong D - d \cong D - e$, that $B - g \cong D - c \cong D - g$, and that $B - a$ $B - c$, $B - d$, $B - e$, $B - g$, and $D - f$ are all pairwise non-isomorphic.

3-graph	m	Degree sequence	Twin-classes
$B - a$	7	$4^4\,32$	ef
$B - b$	7	$4^4\,32$	ef
$B - c$	7	$4^3 3^3$	ab ef
$B - d$	7	$4^3 3^3$	abg ef
$B - e$	7	$4^4\,32$	ab cg
$B - f$	7	$4^4\,32$	ab cg
$B - g$	6	3^6	ab cd ef
$D - a$	7	$4^4\,32$	cg
$D - b$	7	$4^4\,32$	cg
$D - c$	6	3^6	ab dg ef
$D - d$	7	$4^4\,32$	ab cg
$D - e$	7	$4^4\,32$	ab cg
$D - f$	8	$5^2 4^2 3^2$	ab cg
$D - g$	6	3^6	ab cd ef

4.2 *Appendix 2*

The following table shows all different ways in which a 3-graph of type [7 12 4] can extend to a 3-graph of type [8 20 5]. Recall that $T(7,4)$ has edges *abc abd abe acd ace afg bcd bce bfg cfg dfg efg*, $B(7,4)$ has edges *abc abd abg acg aef bcg bef cde cdf deg dfg efg*, and $D(7,4)$ has edges *abc abd abg acg aef bcg bef cde cdf ceg deg dfg*.

3-graph on V	Edges added (with h)	3-graph on $V + h$
$T(7,4)$	*ab ac bc de df dg ef eg*	$T(8,5)$
$T(7,4)$	*af ag bf bg cf cg de fg*	$T(8,5)$
$T(7,4)$	*ad ae bf bg cf cg de fg*	$U(8,5)$
$T(7,4)$	*af ag bc de df dg ef eg*	$U(8,5)$
$T(7,4)$	*af ag bc bg cg de df ef*	$V(8,5)$
$T(7,4)$	*af ag bf bg cg de df ef*	$W(8,5)$
$B(7,4)$	*ab ag bg cd ce cf de df*	$U(8,5)$
$B(7,4)$	*ae af be bf cd cg dg ef*	$V(8,5)$
$B(7,4)$	*ad ae af bd be bf cg ef*	$X(8,5)$
$B(7,4)$	*ad ae af bc bd bg cg ef*	$Y(8,5)$
$D(7,4)$	*ae af be bf cd dg ef fg*	$V(8,5)$
$D(7,4)$	*ae af be bf cd cg dg ef*	$W(8,5)$
$D(7,4)$	*ae be bf cd cf df dg fg*	$X(8,5)$
$D(7,4)$	*ad ae af bc bd bg cg ef*	$X(8,5)$
$D(7,4)$	*ab ae be cd cf df dg fg*	$Y(8,5)$
$D(7,4)$	*ad ae af bd be bf cg ef*	$Y(8,5)$

4.3 *Appendix 3*

Each of $T(8,5), U(8,5), X(8,5)$ extends in only one way to $T(9,6), U(9,6), U(9,6)$, respectively. Assume that the edges of $T(8,5)$, $U(8,5), X(8,5)$ are as listed in the concluding table of Section 2.4.3 (Type [8 20 5]). These three 3-graphs can extend to a 3-graph of type [9 30 6] only as shown below.

3-graph ($V = abcdefgh$)	Edges added (with i)	Resulting 3-graph ($V = abcdefghi$)
$T(8,5)$	*abi aci bci dei dfi dgi dhi efi egi ehi*	$T(9,6)$
$U(8,5)$	*afi agi bci bhi chi dei dfi dgi efi egi*	$U(9,6)$
$X(8,5)$	*afi agi ahi bci bdi bei bhi cdi cei fgi*	$U(9,6)$

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