A Geometric Characterisation of Miquelian Inversive Planes of Even Order

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Abstract. We obtain a new characterisation, by a configuration theorem, of the miquelian geometries among the finite inversive (= Möbius) planes of even order. The main tool used is a characterisation due to J. Tits of elliptic ovoids in three-dimensional projective space.

1. Introduction.

In this paper, we shall prove

Theorem. *Let I be a finite inversive plane of even order. Then I is miquelian if and only if the following theorem (T) holds in I:*

(T) Let C_i, $0 \leq i \leq 3$, be four circles of I such that C_i and C_{i+1} are tangent for all i (addition in the suffix is modulo 4). Then the four points of tangency are concyclic.

It may be interesting to compare and contrast the statement (T) above with the statement of the theorem of Miquel whence miquelian inversive planes derive their name (see [5]).

In [4] Dembowski proved that if I is an inversive plane of even order s then I is isomorphic to the incidence system of points and non-trivial plane sections of some ovoid [5, p. 48] of the three-dimensional projective space $PG(3, s)$ of order s; in consequence, s must be a power of two. To realize the power of this theorem, note that while the services of a Cray supercomputer had to be requisitioned to settle the question of existence of an affine plane of order ten, the existence question of its putative extension, namely, a $3-(101, 11, 1)$ design, is settled completely theoretically by this theorem.

There are two known classes of ovoids of $PG(3, s)$, namely, the elliptic ovoids (that is, non-degenerate quadrics of Witt index 1, see [5, pp. 43-49]) and the Tits ovoids [9]. Correspondingly there are only two known classes of finite inversive planes, namely, the miquelian and the Suzuki-Tits inversive planes [5, pp. 273-275].The outstanding question is whether there are any others. We hope that the above theorem may be of help in obtaining new results in this direction. Like the recent progress in [1] and [2] on this classification question, here also we take Dembowski's theorem as our starting point. Our proof also depends crucially on a beautiful characterisation of elliptic ovoids due to Tits: an ovoid of $PG(3, s)$,

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$s = 2^e$, is an elliptic ovoid if (and only if) all its non-trivial plane sections are conics (non-degenerate quadrics in $PG(2, s)$). It may be noted that in [6] Glynn has obtained (among many other things) a substantial refinement of Tits' theorem: for the same conclusion, it suffices to know that all the non-trivial sections by the planes through any given tangent line (to the ovoid) are conics. Using Glynn's theorem in place of Tits', a corresponding stronger version of our main theorem can easily be obtained. We have retained the weaker version for the sake of clarity and elegance.

For $s = 2^e$, $W(s)$ is the geometry of a linear complex of lines in $PG(3, s)$. That is, its points are the points of $PG(3, s)$ while its lines are the totally isotropic lines of $PG(3, s)$ with respect to a given non-degenerate symplectic form. In [3] $W(s)$ was characterised as the unique regular generalized quadrangle of order (s, s). For a point x of $W(s)$ the *star* at x is the union of the lines of $W(s)$ through x. The star at x is a plane in the ambient projective space. Conversely, each plane in $PG(3, s)$ is the star at a uniquely determined point in the plane. Indeed, $x \mapsto$ star at x is the polarity of $PG(3, s)$ induced by the given symplectic form.

An *ovoid* [8, p. 19] of $W(s)$ is a point-set meeting each line of $W(s)$ in a unique point. It is easy to see that each ovoid of $W(s)$ is an ovoid in the ambient projective space in the sense of [5, p. 48]. We say that an ovoid of $W(s)$ is an *elliptic* (respectively, *Tits*) *ovoid* of $W(s)$ if it is an elliptic (respectively, Tits) ovoid in the ambient $PG(3, s)$.

For any ovoid θ of $W(s)$, $s = 2^e$, let $I(\theta)$ denote the incidence system whose points are the points of $W(s)$ on θ, whose blocks (circles) are the points of $W(s)$ off θ, and in which incidence is collinearity in $W(s)$. It is easy to see that $I(\theta)$ is an inversive plane of order s (see [7, p. 126] and the introductory discussion in [1]). In [2] it was noted that the following proposition paraphrases Dembowski's theorem:

Proposition 0. *Let I be an inversive plane of even order s. Then I is isomorphic to $I(\theta)$ for some ovoid θ of $W(s)$. $I(\theta)$ is miquelian (respectively, Suzuki-Tits) if and only if θ is an elliptic (respectively, Tits) ovoid of $W(s)$.*

Recall [5, p. 253] that a *pencil* in an inversive plane I of order s is a set of s mutually tangent circles through some point x. The point x is called the *carrier* of the pencil. A pencil with carrier x corresponds to one of the $s + 1$ parallel classes of lines in the affine plane of order s obtained by contracting I at x. Since there are $s^2 + 1$ choices for the point x, it follows that I has $(s + 1)(s^2 + 1)$ pencils. For any pencil P with carrier x, let us put $\gamma_P = P \cup \{x\}$. Let $W(I)$ denote the incidence system whose points are the points and circles of I, whose blocks are the sets γ_P as P range over all the pencils of I, and in which incidence is set-theoretic "belonging".

Then Proposition 0 may be rephrased as follows:

Proposition 1. *Let I be an inversive plane of even order s. Then $W(I)$ is isomorphic to $W(s)$ and the point-set θ of I is an ovoid of $W(I)$. I is miquelian (respectively, Suzuki-Tits) if and only if θ is an elliptic (respectively, Tits) ovoid of $W(I)$.*

Proof: By Proposition O, we may take $I = I(\theta)$ where θ is an ovoid of $W(s)$. By definition of $I(\theta)$, the point-set of $W(s)$ consists of the points and circles of I, and, in particular, the point-set θ of I is an ovoid of $W(s)$. Take any line γ of $W(s)$. Since γ is tangent to θ, γ consists of a unique point x of I and s circles of I. Take any two of these circles. Since $W(s)$ is a generalised quadrangle, the only points of $W(s)$ collinear with both these circles are the remaining $s - 1$ points on γ. In particular, the only point of I which is collinear with both these circles is the point x. That is, the only point of I which is incident in I with both these circles is the point x. Thus, the s circles on γ are pairwise tangent at x; hence, they constitute a pencil P of I with carrier x. We thus have $\gamma = P \cup \{x\}$. Since the number $(s+1)(s^2+1)$ of pencils of I equals the number of lines of $W(s)$, all the pencils of I arise thus. Hence, $W(s) = W(I)$. The rest is immediate from Proposition 0.

In [2] we defined a *conic* of $W(s)$ to be a conic in some plane of the ambient $PG(3, s)$ such that no two points of the conic are collinear in $W(s)$. In [2] it was shown that there are $s^6 - s^2$ conics of $W(s)$ and they are isomorphic under the action of the automorphism group $Sp(4, s)$ of $W(s)$; further, conics of $W(s)$ may equivalently be defined as the non-trivial plane sections of the elliptic ovoids of $W(s)$.

Recall that a ruling of a hyperbolic quadric (that is, non-degenerate quadric of Witt index two, see [5, pp. 43-49]) of $PG(3, s)$ is a line of $PG(3, s)$ contained in the quadric. Any hyperbolic quadric h has two parallel classes of rulings, each of which partitions h. Rulings in each parallel class meet all the rulings in the other. In other words, these rulings constitute a pair of opposite reguli [5, p. 220]. Conversely, the union of the lines in any regulus is a hyperbolic quadric. This defines a two-to-one correspondence between reguli and hyperbolic quadrics of $PG(3, s)$.

We define a *hyperboloid* of $W(s)$, $s = 2^e$, to be a hyperbolic quadric in the ambient $PG(3, s)$ all whose rulings are lines of $W(s)$. Given any two disjoint lines γ_1, γ_2 of $W(s)$, the lines of $W(s)$ meeting both of γ_1, γ_2 form a regulus and the union of the lines in this regulus is a hyperboloid of $W(s)$. This property is dual to the property of "regularity" of $W(s)$ as a generalised quadrangle [8, p. 4] and it holds since $W(s)$ is regular [8, p. 77] and self-dual [8, p. 43] for even s. Thus, any two disjoint lines of $W(s)$ are together contained in a unique hyperboloid of $W(s)$. Clearly the intersection of a hyperboloid h of $W(s)$ with any plane is a conic or cross (that is, the union of two intersecting lines) of $W(s)$. Indeed, if x is any point, then the intersection of h with the star at x is a cross or a conic according as x is on or off h. Thus, the number of conics of $W(s)$ contained in a given hyperboloid of $W(s)$ equals $(s+1)\,(s^2+1) - (s+1)^2 = s^3 - s$.

In view of Theorem 3.2.1 in [8, p. 43], an alternative description of $W(s)$, $s = 2^e$, is as the points and lines contained in a fixed non-degenerate quadric Ω in $PG(4, s)$. In terms of this description, the elliptic ovoids and hyperboloids of $W(s)$ are the sections of Ω by the hyperplanes of $PG(4, s)$. Thus viewed, the following Lemma (which is an unpublished observation of B. Bagchi and N.S.N. Sastry) becomes quite transparent. But the counting argument presented below is technically simpler.

Lemma. *Let $s = 2^e$. Then the intersection between any hyperboloid of $W(s)$ and any elliptic quadric of $W(s)$ is a conic of $W(s)$.*

Proof: Let h be any hyperboloid of $W(s)$. Any ovoid θ of $W(s)$ meets any line of $W(s)$ in a unique point, and h is the disjoint union of $s+1$ such lines, whence $|h \cap \theta| = s + 1$. Hence, if θ contains one of the $s(s^2-1)$ conics in h then $h \cap \theta$ equals this conic. Since by [2] each conic of $W(s)$ is contained in $s/2$ elliptic ovoids of $W(s)$, this yields a total of $s^2(s^2-1)/2$ elliptic ovoids of $W(s)$ meeting h in some conic of $W(s)$. But the number $s^2(s^2-1)/2$(= the index of $PGL(2, s^2).2$ in $Sp(4, s)$) is the total number of elliptic ovoids of $W(s)$. Hence, the result. ∎

Proof of the Theorem: Let I be a miquelian inversive plane of even order s. By Proposition 1, the point-set θ of I is an elliptic ovoid of $W(I) = W(s)$. Let C_i, $0 \le i \le 3$, be four circles of I such that $C_i \cap C_{i+1} = \{x_i\}$ for all i. If the four points x_i are not distinct then there is nothing to prove. So assume they are distinct. Hence, so are the circles C_i. Let P_i be the unique pencil of I, with carrier x_i, containing C_i. Consider the line $\gamma_i = P_i \cup \{x_i\}$ of $W(I)$. Thus, γ_i is the line of $W(I)$ joining C_i and C_{i+1}. The generalised quadrangle $W(I)$ has no triangles. Since the points C_i of $W(I)$ are distinct, and since both the lines γ_1, γ_3 meet γ_0 and γ_2. it follows that γ_0 and γ_2 (as also γ_1 and γ_3) are disjoint lines of $W(I)$. Let h be the unique hyperboloid of $W(I)$ containing γ_0 and γ_2, Then all four lines γ_i are rulings of h. Hence, the four points x_i are in h and therefore in $h \cap \theta$. But by the Lemma $h \cap \theta$ is a conic of $W(I)$. A fortiori, $h \cap \theta$ is contained in the star at some point C of $W(I)$ outside the ovoid θ. Thus, C is a circle of I and it is collinear in $W(I)$ with the four points x_i. That is, these four points are incident in I with the circle C; hence, they are concyclic. Thus, (T) holds in I.

To prove the converse, assume (T) holds in the inversive plane I of even order s. By Proposition 1, the point-set θ of I is an ovoid of $W(I) = W(s)$, and it suffices to show that θ is an elliptic ovoid of $W(I)$. Let C be any circle of I. Fix three distinct points x_0, x_1, x_2, in C .Then $\{x_0, x_1, x_2\}$ is a triad of $W(I)$ (that is, a set of three points no two of which are collinear). Since $Sp(4, s)$ is transitive on triads, there is a hyperboloid of $W(s)$ containing any given triad. (Indeed, a counting argument shows that each triad is in $s/2$ hyperboloids of $W(s)$.) Fix a hyperboloid h of $W(I)$ containing x_0, x_1, x_2. Let γ_0 and γ_2 be the rulings through x_0 and x_2, respectively) in one of the two reguli of h, and let γ_1 be the ruling through x_1 in the other regulus. Let $\gamma_0 \cap \gamma_1 = \{C_1\}$ and $\gamma_1 \cap \gamma_2 = \{C_2\}$. Since C_1, C_2, are collinear in $W(I)$ with $x_1 \in \theta$, they are circles of I tangent at x_1. Now take an arbitrary point $x_3 \neq x_0, x_1, x_2$, in $h \cap \theta$. Let γ_3 be the ruling through x_3 in the regulus of h containing γ_1. Let $\gamma_2 \cap \gamma_3 = \{C_3\}$ and $\gamma_3 \cap \gamma_0 = \{C_0\}$. Then $C_i, 0 \leq i \leq 3$, are circles of I satisfying the hypothesis of (T) and $x_i, 0 \leq i \leq 3$, are the points of tangency. Hence, by the assumed validity of (T), x_3 is concyclic with x_0, x_1, x_2. That is, $x_3 \in C$. Hence, $h \cap \theta \subseteq C$. Since both C and $h \cap \theta$ have size $s + 1$ (see the second sentence in the proof of the Lemma), it follows that $h \cap \theta = C$. Hence, $C \subseteq h$. Also, by definition of $W(I)$, C is contained in the star π at the point C of $W(I)$. Hence, $C \subseteq h \cap \pi$. But the intersection $h \cap \pi$ of the hyperboloid h and the plane π is a cross or a conic. Since a cross of $W(I)$ contains at most two points of the ovoid θ, it can not contain the subset C of θ of size $s + 1$. Hence, $h \cap \pi$ is a conic containing the circle C. Since a conic and a circle have the same size, C must be a conic. Thus, all the non-trivial plane sections of the ovoid θ are conics. Hence, by [10], θ must be an elliptic ovoid. Hence, I is miquelian. This completes the proof. ∎

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