SOME CONSTRUCTIONS OF BALANCED INCOMPLETE BLOCK ROW-COLUMN DESIGNS

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Abstract. BIBRC (balanced incomplete block with nested rows and columns) designs were introduced by Singh and Dey (1979) and these designs were mostly obtained by trial and error. Agrawal and Prasad (1983) gave some systematic methods of construction of these designs. We provide further systematic and general methods of construction of BIBRC designs in the present note.

1. Introduction.

A balanced incomplete block design with nested rows and columns (BIBRC) with parameters v, b, r, p, q, λ is an arrangement of v treatments in b blocks each of size pq $(pq < v)$ such that:

(i) each treatment occurs at most once in a block;

(ii) each treatment occurs in r blocks and every pair of treatments occurs in λ^* blocks;

(iii) each block can be arranged into an array of p rows and q columns so that for a given pair of treatments (i, j)

$$(p-1)\lambda_{r(i,j)} + (q-1)\lambda_{c(i,j)} - \lambda_{e(i,j)} = \lambda$$

where $\lambda_{r(i,j)}$, $\lambda_{c(i,j)}$ and $\lambda_{e(i,j)}$ denote the number of blocks in which i and j occur in the same row, same column, and elsewhere, respectively, and λ is a constant independent of the pair of treatments chosen.

If

$$\lambda_{r(i,j)} = \lambda_r,\ \lambda c(i,j) = \lambda_c \forall (i,j)$$

then

$$p\,\lambda_r + q\,\lambda_c - \lambda^* = \lambda.$$

These designs were introduced by Singh and Dey (1979); most of their examples were constructed by trial and error. Agrawal and Prasad (1983) gave some systematic methods of construction of these designs. In this note, we provide further systematic and general methods of construction of BIBRC designs using differences. Das and Kulshreshtha (1968) gave an easy method of construction of a balanced incomplete block design (BIBD) using several initial blocks as follows:

Let v be a prime power and let the v elements of $GF(v)$ denote the v treatments of the design. If I is an arbitrarily chosen initial block of $k(k < v)$ distinct treatments, then the $(v-1)$ initial blocks $I, xI, x^2 I, \ldots, x^{v-2} I$, when developed mod $\{v, P(x)\}$, x being a primitive root and $P(x)$, a minimum polynomial of $GF(v)$, yield a BIBD for v treatments in $b = v(v-1)$ blocks. When v is odd prime or prime power, the number of blocks can be reduced to $b = \binom{v}{2}$ by taking only the inital blocks $I, xI, x^2 I, \ldots, x^{(\frac{v-3}{2})} I$.

It is also known (Saha (1980)) that if $m(2 \leq m < v-1)$ divides $v-1$ and the cosets C_i are defined as

$$C_i = x^i C_1,\ C_1 = \{x^{jm} \mid 0 \leq j \leq S-1\}$$

$i = 0, 1, \ldots, m-1$, $S = \frac{v-1}{m}$, then, the m initial blocks, $\{C_i\}$ generate a BIBD. Note that the result holds trivially for $m = 1$. We use these results to construct BIBRC designs in Section 2. In Section 3 we give illustrations.

2. Some constructions of BIBRC designs.

In this section we prove a lemma and three theorems. In each theorem, v, the number of treatments, is a prime or prime power; in Theorem 2.2 and Theorem 2.3, v must also be odd. The treatments are represented by the elements of $GF(v)$. Let x denote a primitive element and $P(x)$, a minimum polynomial of $GF(v)$.

Lemma 2.0. *Let p, q be two positive integers such that $LCM(p, q) = pq$. Then the integers $(0, 1, 2, \ldots, pq-1)$ can always be written in a $p \times q$ array as an addition (mod pq) table with $(0, q, 2q, \ldots, (p-1)q)$ as the column margin and $(0, p, 2p, \ldots, (q-1)p)$ as the row margin. Thus, the rows of the array form a partition of all the pq integers $(0, 1, 2, \ldots, pq-1)$. And similarly the columns of the array also form a partition of the integers $(0, 1, 2, \ldots, pq-1)$.*

Proof: Clearly two entries belonging to the same row or same column of the addition table are unequal since $p > 0, q > 0$. If two entries belonging to different rows and different columns of the addition table become equal, then, a multiple of p will equal a multiple of q, that is,

$$ap = bq, \quad \left.\begin{array}{l} a \in \{1, 2, \ldots, (q-1)\} \\ b \in \{1, 2, \ldots, (p-1)\} \end{array}\right\}. \tag{1}$$

Since $LCM(p, q) = pq$, we must have

$$p = bq = pqz \tag{2}$$

where z is any positive integer. Obviously then, the two positive integers a, b satisfying (2) are such that $a \geq q$ and $b \geq p$ and, so, such a pair of a, b will never satisfy (1). Hence, the lemma follows. ∎

Theorem 2.1. *Let A be an arbitrarily chosen $p \times q$ array of $k = pq(< v)$ distinct symbols of $GF(v)$. Then the initial blocks $A, xA, x^2A, \ldots, x^{v-2}A$, when developed* mod $\{v, P(x)\}$ *generate a BIBRC design with the parameters $v, b = v(v-1), r = (v-1)pq, p, q, \lambda = pq(p-1)(q-1)$.*

Corollary 2.1.1. *If v is odd prime or prime power, then the initial blocks A, $xA, x^2A, \ldots, x^{(\frac{v-3}{2})}A$, written in $p \times q$ arrays and developed* mod $\{v, P(x)\}$ *generate a BIBRC design with parameters $v, b = v(v-1)/2, r = (v-1)pq/2$, $p, q, \lambda = pq(p-1)(q-1)/2$.*

Proof: The proofs of Theorem 2.1 and its corollary follow from the results of Das and Kulshreshtha (1968) as described in Section 1, and the observation that $\lambda_r = pq(q-1), \lambda_c = pq(p-1)$ and $\lambda^* = pq(pq-1)$. ∎

In the following we give other methods of construction of BIBRC designs in lesser number of blocks than those of Theorem 2.1. Here the elements of A are not arbitrarily chosen. They depend on x, a primitive root of $GF(v)$.

Theorem 2.2. *Let v be an odd number which is a prime or prime power and $v-1$ be divisible by $m(\geq 1)$, and p is a positive integer such that $\frac{v-1}{m} = pq$. Consider a $p \times q$ array A as an addition table(* mod v*) with the entries $(0, mp, 2mp, \ldots, v-mp-1)$ as the row margin and entries $(0, m, 2m, \ldots, m(p-1))$ as the column margin. The entries of the array A_0 are then obtained from the addition table A by replacing every element e by x^e. Then, the mp initial blocks A_0, $A_1 = xA_0, A_2 = x^2A_0, \ldots, A_{mp-1} = x^{mp-1}A_0$ when developed* mod $\{v, P(x)\}$, *generate a BIBRC design with parameters*

$$v, b = mpv, \ r = p(v-1), p, q, \ \lambda = p(p-1)(q-1)$$

where $k = pq = \frac{v-1}{m}$.

Proof: Let x be a primitive root of $GF(v)$. Clearly, $x^{v-1} = 1$. Let the non-zero elements $GF(v)$ be $x^0, x, x^2, \ldots, x^{v-2}$. Consider the arrays $A_0, A_1, \ldots, A_{mp-1}$ as the mp initial blocks each of size pq. Here, A_0 is a subgroup of the multiplicative group of the nonzero elements of $GF(v)$ and $A_1, \ldots, A_{mp-1}$ are the cosets of A_0. Hence, by Saha (1980) these, when developed mod $\{v, P(x)\}$, generate BIBD $(v, b^* = mpv, \ r^* = p(v-1), \ k = pq, \ \lambda^* = p(pq-1))$. Next, consider the rows of $A_0, A_1, \ldots, A_{mp-1}$ as the mp^2 initial blocks each of size q. By a similar argument as above, the first rows of $A_0, A_1, \ldots, A_{mp-1}$ as the mp initial blocks, when developed mod $\{v, P(x)\}$, generate a BIBD (Saha (1980)). Similarly, the second rows of $A_0, A_1, \ldots, A_{mp-1}$, when developed mod $v, P(x)$, generate a BIBD. Thus, all the rows of $A_0, A_1, \ldots, A_{mp-1}$ give a BIBD with parameters

$$v, b_1 = mp^2v, \ r_1 = p(v-1), \ k_1 = q, \ \lambda_r = p(q-1).$$

Next, consider the columns of $A_0, A_1, \ldots, A_{mp-1}$ as the mpq initial blocks each of size p. By Das and Kulshreshtha (1968), all these when developed together mod $\{v, P(x)\}$, yield a BIBD with parameters

$$v, b_2 = mpqv,\ r_2 = p(v-1),\ k_2 = p,\ \lambda_c = p(p-1).$$

Thus, $A_0, A_1, \ldots, A_{mp-1}$, when developed mod $\{v, P(x)\}$, generate a BIBRC design with parameters

$$\begin{aligned} v, b = mpv,\ r &= p(v-1), p, q \quad \text{and} \\ \lambda &= p\lambda_r + q\lambda_c - \lambda^* \\ &= p(p-1)(q-1). \end{aligned}$$

∎

Remark: It is to be noted that the first m arrays $A_0, A_1, \ldots, A_{m-1}$ are sufficient to prove that these when developed mod $\{v, P(x)\}$, yield a BIBD $(v, b^*, r^*, k, \lambda^*)$ and the mp rows of $A_0, A_1, \ldots, A_{m-1}$ when developed mod $\{v, P(x)\}$, generate BIBD $(v, b_1, r_1, k_1, \lambda_r)$. But all the mpq column blocks of $A_0, \ldots, A_{mp-1}$ are required to generate a BIBD $(v, b_2, r_2, k_2, \lambda_c)$.

If p and q are relatively prime to each other, that is, $LCM(p, q) = pq$, then the number of blocks can be considerably reduced. We state and prove the relevant result in Theorem 2.3.

Theorem 2.3. *Let A_0^* be the multiplication table with $(x^0 x^{mp}, x^{2mp}, \ldots, x^{v-mp-1})$ as the row margin and $(x^0 x^{mq}, x^{2mq}, \ldots, x^{v-qm-1})$ as the column margin. Then the m initial blocks $A_0^*, A_1^* = xA_0^*, \ldots, A_{m-1}^* = x^{m-1}A_0^*$, when developed* mod $\{v, P(x)\}$, *generate a BIBRC design with parameters*

$$v', b' = mv,\ r' = p(v'-1), p, q,\ \lambda' = (p-1)(q-1).$$

Proof: The result follows on the same lines as indicated in Theorem 2.2 with an appeal to Lemma 2.0. ∎

3. Illustrations.

In this section we give three examples to illustrate Theorem 2.1, Theorem 2.2, and Theorem 2.3.

Example 2.1: Let $v = 7,\ p = q = 2,\ x = 3$ is a primitive root of $GF(7)$.

$$\text{Let } A = \begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix}. \quad \text{Then } xA = \begin{matrix} 3 & 2 \\ 6 & 1 \end{matrix},\ x^2A = \begin{matrix} 2 & 6 \\ 4 & 3 \end{matrix}.$$

These 3 blocks A, xA, x^2A, when developed mod 7, yield a BIBRC design with parameters $v = 7,\ b = 21,\ r = 12,\ p = q = 2,\ \lambda = 2$.

Example 2.2: Let $v = 13$, $m = 3$, $p = q = 2$. $x = 2$ is a primitive root of $GF(13)$. Consider

$$A_0 = \begin{matrix} x^0 & x^6 \\ x^3 & x^9 \end{matrix} = \begin{matrix} 1 & 12 \\ 8 & 5 \end{matrix}, A_1 = \begin{matrix} 2 & 11 \\ 3 & 10 \end{matrix}$$

$$A_2 = \begin{matrix} 4 & 9 \\ 6 & 6 \end{matrix}, A_3 = \begin{matrix} 8 & 5 \\ 12 & 1 \end{matrix}, A_4 = \begin{matrix} 3 & 10 \\ 11 & 2 \end{matrix}$$

$$A_5 = \begin{matrix} 6 & 7 \\ 9 & 4 \end{matrix}.$$

These 6 blocks, when developed mod 13, generate a BIBRC design with parameters $v = 13$, $b = 78$, $r = 24$, $p = q = 2$, $\lambda = 2$.

Example 2.3: Let $v' = 19$, $m = 3$, $p = 2$, $q = 3$. $x = 2$ is a primitive root of $GF(19)$.

$LCM(p, q) = pq = 6$. Consider

$$A_0^\star = \begin{matrix} x^0 & x^6 & x^{12} \\ x^9 & x^{15} & x^3 \end{matrix} = \begin{matrix} 1 & 7 & 11 \\ 18 & 12 & 8 \end{matrix}$$

$$A_1^\star = \begin{matrix} 2 & 14 & 3 \\ 17 & 5 & 16 \end{matrix}, \quad A_2^\star = \begin{matrix} 4 & 9 & 6 \\ 15 & 10 & 13 \end{matrix}.$$

These 3 initial blocks, when developed mod 19, generate a BIBRC design with parameters $v' = 19$, $b' = 57$, $r' = 36$, $p = 2$, $q = 3$, $\lambda' = 2$.

References

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