# CHROMATIC RELATEDNESS OF GRAPHS

Gerhard Benadé, Izak Broere

Department of Mathematics
Rand Afrikaans University
Johannesburg
SOUTH AFRICA

**Abstract.** The $F$-free chromatic number $\chi(M:-F)$ of a graph $M$ is defined as the least number of classes in a partition of the vertices of $M$ such that $F$ does not occur as an induced subgraph in the subgraph induced by any of the colour classes. Two graphs $G$ and $H$ are called chromatically related if, for each positive integer $k$, there exists a graph $M_k$ such that $\chi(M_k:-G) = \chi(M_k:-H) = k$, and distantly related whenever a chain of such relatednesses exists between them. Using a basic theorem of Folkman [3], we show that every two graphs on at least two vertices are distantly related.

## 1. Introduction.

All graphs considered in this paper are finite and simple, and have at least two vertices whenever they are used to define colourings as described below. We use the notation $G[V]$ for the subgraph of $G$ induced by $V \subseteq V(G)$ and we write $H \leq G$ if $H$ is an induced subgraph of $G$. For other notation and undefined concepts we refer the reader to [2]. For a given graph $F$, $-F$ is the class of *F-free* graphs, that is, graphs that do not contain $F$ as an induced subgraph. A partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ is an *F-free colouring* of $G$, also called a *$-F$ n-colouring*, if $G[V_i]$ is $F$-free for each $i$; $G$ is called *$-F$ n-colourable* if such a partition exists. The smallest number of colour classes in an $F$-free colouring of $G$ is the *F-free chromatic number* of $G$, denoted by $\chi(G:-F)$.

1.1 Definition: Let a positive integer $n$ be given. Then two graphs $G$ and $H$ are called *n-chromatically related*, denoted by $G \asymp_n H$, if there exists a graph $M_n$ such that $\chi(M_n:-G) = \chi(M_n:-H) = n$. If $G \asymp_n H$ for each $n \geq 1$, we call $G$ and $H$ *chromatically related* and write $G \asymp H$.

These relations are not equivalence relations, since they are not transitive. For instance, we will see that $K_3 \asymp K_4$ and $K_4 \asymp K_5$, but $K_3 \not\asymp K_5$. The following concept, however, does yield an equivalence relation: It is simply the transitive closure of $\asymp$.

1.2 Definition: Let $k$ be any integer and $G$ and $H$ be graphs. Then $G$ and $H$ are called *k-distantly related*, denoted by $G \sim_k H$, if there exist graphs $R_0, R_1, \ldots, R_k$ such that

$$G \cong R_0 \asymp R_1 \asymp \ldots \asymp R_k \cong H.$$

The graphs $G$ and $H$ are called *distantly related*, denoted by $G \sim H$, if $G \sim_k H$ for some integer $k$.

Note that, since the above definitions use the concept of chromatic number defined relative to a graph, all the graphs concerned must have at least two vertices since it does not make sense to prohibit the occurrence of $K_1$ as subgraph. The following result is an immediate consequence of the definition.

**1.3 Proposition.** *For every two graphs $F$ and $G$, $\chi(F\colon -G) = \chi(\overline{F}\colon -\overline{G})$.*

Hence, every statement involving the chromatic number of a graph can be used to obtain a corresponding statement about the complement of the graph. Another consequence is that chromatic relatedness and distant relatedness between two graphs will be carried over to the same relatedness between the complements of these graphs.

In the proof of the main theorem, a central role will be played by the complete graphs. To show that these graphs are distantly related, we need the following two easy results about chromatic numbers with respect to complete graphs.

**1.4 Proposition.** *For all positive integers $p, n_1, n_2, \ldots, n_p$ and $m \geq 2$,*

$$\chi(K_{n_1,n_2,\ldots,n_p}\colon -K_m) = \left\lceil \frac{p}{m-1} \right\rceil .$$

**1.5 Corollary.** *For any two positive integers $n$ and $m$ with $m \geq 2$,*

$$\chi(K_n\colon -K_m) = \left\lceil \frac{n}{m-1} \right\rceil .$$

The following proposition produces an abundance of graphs that are not chromatically related — we will call such graphs *chromatically alien*. In view of this, our main result, stating that every two graphs are distantly related, is somewhat surprising.

**1.6 Proposition.** *Let $F_1$ and $F_2$ be graphs and let $k$ be any positive integer. If $\chi(F_1\colon -F_2) \geq k+1$, then, for every graph $G$,*

$$\chi(G\colon -F_1) \leq \left\lceil \frac{\chi(G\colon -F_2)}{k} \right\rceil .$$

Proof: Consider an $F_2$-free colouring of $G$ in $\chi(G\colon -F_2)$ colours. A colouring of $G$ in $\left\lceil \frac{\chi(G:-F_2)}{k} \right\rceil$ colours is obtained by grouping these colour classes together $k$ at a time. This colouring is $F_1$-free, since $\chi(F_1\colon -F_2) \geq k+1$, thus the result follows. ∎

**1.7 Corollary.** *For any graph $G$ and any integer $r \geq 2$, we have*

$$\chi(G\colon -K_{2r-1}) \leq \left\lceil \frac{\chi(G\colon -K_r)}{2} \right\rceil .$$

Proof: By Corollary 1.5 we have $\chi(K_{2r-1}\colon -K_r) = 3$; thus, Proposition 1.6 can be applied (with $k = 2$) to get the result. ∎

**1.8 Proposition.** *Let $G_1, G_2$ and $H$ be graphs. Then we have*

$$\chi(G_1 \cup G_2 : -H) \geq \max_{i=1,2} \chi(G_i : -H)$$

*with equality if $H$ is connected.*

**1.9 Theorem.** *For all positive integers $p$ and $n_1 \leq n_2 \leq \ldots \leq n_p$,*

$$\chi(K_{n_1,n_2,\ldots,n_p} : -P_3) = \min_{1 \leq j \leq p} \{p, p-j+n_j\}.$$

Proof: The colour classes of any $P_3$-free colouring of $K_{n_1,n_2,\ldots,n_p}$ induce either null or complete subgraphs and, hence, any such colouring is also a cocolouring of this graph. Thus, Theorem 2 of [4], stating that the cochromatic number of $K_{n_1,n_2,\ldots,n_p}$ is given by $\min_{1 \leq j \leq p}\{p, p-j+n_j\}$, applies. This gives the desired equality. ∎

In [3], Folkman proved the existence of graphs with any specified $F$-free chromatic number. In particular, he was able to keep the clique number of such graphs down to that of $F$, a fact that is crucial to the proof of our main result.

**1.10 Theorem (Folkman).** *For each integer $k \geq 2$ and each graph $F$ there exists a graph $H_k$ with $\chi(H_k : -F) = k$ and $\omega(H_k) = \omega(F)$.*

## 2. Main results.

We first establish distant relatedness between any two *complete* graphs, thence extending to the general case by proving that any connected graph is related to some complete graph. By using Proposition 1.3, we then prove that the corresponding relatedness holds between any two null graphs and between any disconnected graph and some null graph. A link between the null graphs and the complete graphs completes the chain of relatednesses.

From the inequality in Corollary 1.7 it follows that $K_r$ and $K_{2r-1}$ are chromatically alien. The following theorem of Broere and Frick [1] asserts that all the complete graphs in between are in fact chromatically related.

**2.1 Theorem.** *Let $n, r, k$ be integers with $n \geq r \geq 2$ and $k \geq 2$. Then there exists a graph $G$ such that*

$$\chi(G : -K_r) = \chi(G : -K_{r+1}) = \ldots = \chi(G : -K_n) = k$$

*if and only if $n \leq 2r-2$.*

**2.2 Corollary.** *Let $p, q$ and $k$ be integers with $k \geq 2$. Then we have $K_p \asymp_k K_q$ if and only if there exists an integer $r$ with $r \leq p, q \leq 2r-2$.*

Since any two graphs are 1-chromatically related, this result implies that $K_p \asymp K_q$ if and only if there exists some integer $r$ such that $r \leq p, q \leq 2r-2$. In particular, this means that the result of Corollary 1.7 is best possible. Also, using this result, it is easy to see that any two complete graphs on more than two vertices are distantly related. We cannot deduce distant relatedness between $K_2$ and $K_3$ from this result; the next two lemmas take care of this case.

**2.3 Lemma.** $K_2 \asymp P_3$.

Proof: Let $m \geq 1$. The graph establishing $m$-chromatic relatedness is an $m$-partite complete graph $K_{n_1,n_2,\ldots,n_m}$ : By Theorem 1.9 and Proposition 1.4 it is sufficient to choose the numbers $n_1, n_2, \ldots, n_m$ in such a way that $m - j + n_j = m$. This is achieved by letting $n_j = j$ for $j = 1, 2, \ldots, m$. Then, for all $m \geq 1$, $\chi(K_{n_1,n_2,\ldots,n_m} : -P_3) = \chi(K_{n_1,n_2,\ldots,n_m} : -K_2) = m$. ∎

**2.4 Lemma.** $K_3 \asymp P_3$.

Proof: Let $p$ be even. Again, $\chi(K_{n_1,n_2,\ldots,n_p} : -K_3) = \lceil \frac{p}{3-1} \rceil = \frac{p}{2}$ by Proposition 1.4 and $\chi(K_{n_1,n_2,\ldots,n_p} : -P_3) = \min_{1 \leq j \leq p} \{p, p-j+n_j\}$ by Theorem 1.9. Equality of these two chromatic numbers will follow if $p - j + n_j \geq \frac{p}{2}$, with equality for some $j$. Let

$$n_j = j \qquad \text{if } 1 \leq j \leq \frac{p}{2}$$

$$n_j = j - \frac{p}{2} \quad \text{if } \frac{p}{2} < j \leq p.$$

Then $p - j + n_j = \frac{p}{2}$ for all $j > \frac{p}{2}$. Thus we obtain the value $\frac{p}{2}$ for the $P_3$-free chromatic number of $K_{n_1,n_2,\ldots,n_p}$ for every even $p \geq 4$. By choosing $m = \frac{p}{2}$ for all even $p \geq 4$, we get

$$\chi(K_{n_1,n_2,\ldots,n_p} : -P_3) = \chi(K_{n_1,n_2,\ldots,n_p} : -K_3) = m$$

for all $m \geq 2$. Therefore, $P_3$ and $K_3$ are chromatically related. ∎

We now summarize the preceding three results as Theorem 2.5. By using Proposition 1.3, we also obtain the corresponding result about the null graphs.

**2.5 Theorem.** $K_m \sim K_n$ *for all* $m, n \geq 2$.

**2.6 Corollary.** $N_m \sim N_n$ *for all* $m, n \geq 2$.

**2.7 Theorem.** $G \asymp K_n$ *for every connected non-complete graph $G$ and every integer* $n > \omega(G)$.

Proof: For any $m \geq 2$, we know from Folkman's theorem that there is a graph $G_m$ with $\chi(G_m : -G) = m$ and $\omega(G_m) = \omega(G)$. Let $H_m = G_m \cup K_r$, where $r = (m-1)(n-1) + 1$. From Proposition 1.8 then follows that $\chi(H_m : -G) = \max\{\chi(G_m : -G), \chi(K_r : -G)\} = m$, as $\chi(K_r : -G) = 1$ and $G$ is connected. Also,

$$\begin{aligned}
\chi(H_m : -K_n) &= \max\{\chi(G_m : -K_n),\ \chi(K_r : -K_n)\} \\
&= \max\{1, \chi(K_r : -K_n)\} \\
&= \max\left\{1, \left\lceil \frac{r}{n-1} \right\rceil\right\} \\
&= \max\{1, m\} \\
&= m.
\end{aligned}$$

Here, the second equality follows from $n > \omega(G) = \omega(G_m)$; the third from Corollary 1.5 and the fourth from the choice of $r$. Thus, we have $m$-chromatic relatedness between $G$ and $K_n$ for all $m \geq 2$, proving chromatic relatedness. ∎

**2.8 Corollary.** *$G \asymp N_n$ for every disconnected non-trivial graph $G$ and every integer $n > \beta(G)$.*

Proof: Suppose that $G$ is disconnected and non-trivial, and $n > \beta(G)$. Then $\overline{G}$ is connected and non-complete and $\omega(\overline{G}) = \beta(G) < n$. By Theorem 2.7 we have $\overline{G} \asymp K_n$ and, thus, $G \asymp N_n$. ∎

Using the results proved up till now, we see that any two connected graphs are distantly related through the complete graphs and any two disconnected graphs are distantly related using the null graphs. The following result now establishes distant relationship between any two graphs by linking the complete graphs with the null graphs.

**2.9 Proposition.** *Every complete graph is distantly related to every null graph.*

Proof: Let $S$ be any self-complementary graph of order at least two. Since $S$ is necessarily connected, Theorem 2.7 may be applied to find a complete graph $K_n$ with $S \asymp K_n$. From this also follows that $S \asymp N_n$. Thus, we have distant relatedness holding between every complete graph and every null graph by applying Theorem 2.5 and Corollary 2.6. ∎

We are now in a position to state our final result.

**2.10 Theorem.** *Every two non-trivial graphs are distantly related.*

## References


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