Graphs of diameter at most two

Hong-Jian Lai
Department of Mathematics
West Virginia University
Morgantown, WV 26506

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Abstract

A graph H is collapsible if for every even subset $W \subseteq V(H)$, H has a spanning connected subgraph whose set of odd-degree vertices is W. In a graph G, there is a unique collection of maximal collapsible subgraphs, and when all of them are contracted, the resulting contraction of G is a reduced graph. Reduced graphs has been shown to be useful in the study of supereulerian graphs, hamiltonian line graphs, and double cycle covers, (see [2], [3], [4], [6]), among others. It has been noted that subdividing an edge of a collapsible graph may result in a noncollapsible graph. In this note we characterize the reduced graphs of elementary subdivision of collapsible graphs of diameter at most two. We also obtain a converse of a result of Catlin [3] when restricted to graphs of diameter at most two. The main result is used to study some hamiltonian property of line graphs.

INTRODUCTION

We shall use the notation of Bondy and Murty [1], except for contractions, and we allow graphs to have multiple edges but loops are forbidden. We shall use $d_G(u, v)$ to denote the distance between the two vertices u, v in G. When no confusion arises, we use $d(u, v)$ for $d_G(u, v)$. The diameter of G, denoted by diam(G), is defined thus:

$$\text{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

The degree of a vertex v in a graph G will be denoted by $deg_G(v)$, or $deg(v)$. For integer $i \geq 0$, we define

$$D_i(G) = \{v \in V(G) \mid deg_G(v) = i\}.$$

As in [1], $\delta(G)$ denotes the minimum degree of G, and $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of G, respectively.

For a set $X \subseteq E(G)$, we define the contraction G/X to be the graph obtained from G by contracting the edges of X and deleting all resulting loops. When H is a connected subgraph of G, we use G/H for $G/E(H)$.

In [2], Catlin defines the collapsible subgraphs. A graph G is collapsible if for every subset $R \subseteq V(G)$ with $|R|$ even, G has a subgraph Γ such that $G - E(\Gamma)$ is connected and such that the set of odd-degree vertices of Γ is R. The subgraph Γ is called an R-subgraph of G. It is routine to show that G is collapsible if and only if for every subset $W \subseteq V(G)$ with $|W|$ even, G has a connected spanning subgraph whose set of odd-degree vertices is W. In [2], Catlin showed that every vertex of a graph G is in a unique maximal collapsible subgraph of G. The reduction of G is the graph obtained from G by contracting all nontrivial collapsible subgraphs of G. A graph is called reduced if it is the reduction of some graph.

A graph G is eulerian if G is connected and every vertex of G has even degree. Note that the trivial graph K_1 is regarded as both collapsible and having spanning eulerian subgraphs.

Theorem A (Catlin [2]) Let G be a graph.

(a) G is reduced if and only if G has no nontrivial collapsible subgraphs.

(b) If G is reduced, then G is simple with $\delta(G) \leq 3$, and G contains no subgraph isomorphic to K_3.

(c) If each edge of a spanning tree of G is in a collapsible subgraph, then G is collapsible.

(d) If H is a connected subgraph of G and if G is collapsible, then G/H is collapsible; if H is a collapsible subgraph of G and if G/H is collapsible, then G is collapsible.

(e) G has a spanning eulerian subgraph if and only if the reduction of G has a spanning eulerian subgraph.

(f) If G is collapsible, then G has a spanning eulerian subgraph. □.

Reduced graphs of diameter two are characterised in [6]. Let m, l be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,l}$ be two complete bipartite graphs. Let v_1, u_1 be two nonadjacent vertices of degree m in H_1, and v_2, u_2 be two nonadjacent vertices of degree l in H_2. Let $S_{l,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle.

Theorem B (Lai [7]) Let G be a reduced graph. If $\text{diam}(G) = 2$, then exactly one of the following holds:

(a) $G \cong K_{1,t}, t \geq 2$;

(b) $G \cong K_{2,t}, t \geq 2$;
(c) $G \cong S_{l,m},\ l, m \geq 1$;
(d) G is the Petersen graph. □

In [3], Catlin developed the idea of collapsible graphs. Let H be a graph and let π be a partition of $V(H)$ into two nonempty sets V_1, V_2. We shall denote this by $\pi =< V_1, V_2 >$. Then H is called π-collapsible if for every subset $R \subseteq V(H)$ of even cardinality, the following hold:

(i) if $|R \cap V_1|$ is odd, then H has an R-subgraph;

(ii) if $|R \cap V_2|$ is even, then $H + e$ has an R-subgraph, for any newly added edge $e = v_1v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$.

As examples, the 2-cycle is collapsible, complete graphs of order at least 3 are collapsible; collapsible graphs are π-collapsible, for any partition π. However, the following example, as noted by Catlin in [3], is π-collapsible but not collapsible.

Example 1 Let $C_4 = v_1v_2v_3v_4v_1$ denote the 4-cycle, let $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4\}$, and let $\pi =< V_1, V_2 >$. Then C_4 is π-collapsible.

Suppose that H is a π-collapsible subgraph of G with $\pi =< V_1, V_2 >$. Denote by G/π the graph obtained from G by identifying all vertices of V_1 to form a single vertex v_1, by identifying all vertices of V_2 to form a single vertex v_2, and by joining v_1 and v_2 with exactly one edge. This new edge is denoted by e_π.

Theorem C (Catlin [3]) Let H be a π-collapsible subgraph of G. If G/π is collapsible, then G is collapsible. □

It is easy to construct examples to show the converse of Theorem C is not true in general:

Example 2 Let $H = K_{2,t}$ with $t \geq 2$. Let $e = xy \in E(H)$. Let K be the graph with

$$V(K) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$$

and

$$E(K) = \{x_1x_2, x_2x_3, x_3y_1, y_1y_2, y_2y_3, y_3x_1, x_2y_2, x_3y_3\}.$$

Define $G(t)$ to be the graph obtained from $H - e$ and K by identifying x_1 with x and indentifying y_1 with y. It is shown in [3] that K is collapsible. Note that every edge of $G(t)/K$ is in a 3-cycle and so by (c) and (d) of Theorem A, $G(t)$ is collapsible. Both K and $G(t)$ contain a 4-cycle $x_2x_3y_3y_2x_2$. Let π denote the bipartition of this 4-cycle. Then neither K/π nor $G(t)/\pi$ is collapsible, for their reductions are K_2 and $K_{2,t}$, respectively.

MAIN RESULTS

Theorem 1 Let G be a graph of diameter at most 2 and let C_4, the 4-cycle, be a nonspanning subgraph of G. Let π denote the bipartition of C_4. Then G is collapsible if and only if one of the following holds:

(a) G/π is collapsible.

(b) G is spanned by a subgraph $H \cong K_{2,n-2}$ such that there are two vertices in $D_2(H)$ adjacent in G.

Let $P_4 = x_1x_2x_3x_4$ denote a path of length 3. Let H be a graph disjoint from P_4 that is isomorphic to a $K_{2,t}$ with $t \geq 2$ (respectively, an $S_{l,m}$ with $n \geq 2$ and $m \geq 1$). Let $xy \in E(H)$ be an edge of H that is lying in a 4-cycle of H. Define $K_{2,t}^{+}$ (respectively, $S_{l,m}^{+}$) to be the graph obtained from H and P_4 by identifying x with x_1 and y with x_4.

We say that an edge $e \in E(G)$ is subdivided when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph, denoted by $G(e)$.

Theorem 2 Let G be a collapsible graph of diameter at most 2, and let e be an edge of $E(G)$. If $[G(e)]'$ denotes the reduction of $G(e)$, then exactly one of the following holds:

(a) $[G(e)]' \cong K_1$;

(b) $[G(e)]' \cong K_{2,t}$, $t \geq 2$;

(c) $[G(e)]' \cong S_{l,m}$, $l \geq 2, m \geq 1$;

(d) $[G(e)]' \cong K_{2,t}^{+}$, $t \geq 2$;

(e) $[G(e)]' \cong S_{l,m}^{+}$, $l \geq 2, m \geq 1$.

THE PROOF OF THEOREM 1

Lemma 1 Let $n \geq 5$ be an integer and let H be a graph isomorphic to a $K_{2,n-2}$. Let G be a spanning supergraph of H such that there is an edge incident with two vertices in $D_2(H)$, then for any edge $e \in E(G)$, $G(e)$ is collapsible.

Proof: Let $x, y \in D_2(H)$ be two vertices of G such that $xy \in E(G)$. Thus $H_1 = H + xy$ is a spanning subgraph of G. By (c) of Theorem A, it suffices to show that for any $e \in E(H_1)$, $H_1(e)$ is collapsible.

If $e \neq xy$, then $H_1(e)$ contains a 3-cycle C_3. Note that either every edge in $H_1(e)/C_3$ is in a 3-cycle or $G(e)/C_3$ has a 2-cycle C_2 such that every edge in $[G(e)/C_3]/C_2$ is in a k-cycle with $k \in \{2, 3\}$. It follows from (d) of Theorem A that $G(e)$ is collapsible.

If $e = xy$, then since $n \geq 5$, $n - 2 \geq 3$. It follows from Theorem 11 of [3] that $G(e)$ is collapsible. □

Lemma 2 Let H be a graph that contains a nonspanning π-collapsible subgraph K, where $\pi = < V_1, V_2 >$ is a bipartition of $V(H)$. If e_π is contained in a collapsible subgraph of H/π, then the reduction of H/π is the same as the reduction of H.

Proof: Let H' and $(H/\pi)'$ denote the reductions of H and H/π, respectively. Since e_π is in a collapsible subgraph L_1 of H/π, the subgraph

$$L = H[(E(L_1) - \{e_\pi\}) \cup E(C_4)]$$

is a collapsible subgraph of H, by Theorem C. By the definition of contractions, we have

$$H/L \cong (H/\pi)/L_1. \tag{1}$$

By (d) of Theorem A, (1) implies that $H' \cong (H/\pi)'$. □

By Theorem C and Lemma 1, it suffices to show that if G is collapsible, then either (a) or (b) of Theorem 1 holds.

Note that the hypothesis of Theorem 1 implies that $n \geq 5$. Let G' denote the reduction of G/π. If $G' = K_1$, then by (d) of Theorem A and by Theorem C, G/π is collapsible and so (a) of Theorem 1 holds. Thus we assume that

$$G' \text{ is not collapsible.} \tag{2}$$

Since contracting the edges does not increase the diameter and since the operation to get G/π from G does not increase the diameter either, we have $\text{diam}(G') \leq 2$. Thus by Theorem B, one of the conclusions of Theorem B must hold or $G' \cong K_2$.

Since G is collapsible, if e_π is in a collapsible subgraph of G/π, then by Lemma 2, G' is collapsible, a contradiction. Hence we assume that

$$e_\pi \in E(G'). \tag{3}$$

If G' does not have a cut edge, then by Theorem B, G' is either a $K_{2,t}$, or an $S_{n,m}$, or the Petersen graph. In any case, by (3), one of the edges of G' is e_π. It follows that the diameter of G would be at least 3, contrary to the hypothesis of $\text{diam}(G) \leq 2$.

Since G is collapsible, G is 2-edge-connected. If G' has a cut-edge, then this cut-edge must be created in the process of getting e_π. This, in conjunction with (3), implies that e_π is the only cut-edge of G'. By Theorem B, we have

$$G' \cong K_2. \tag{4}$$

Let the C_4 subgraph be $C_4 = x_1x_2x_3x_4x_1$ and let $\pi = < V_1, V_2 >$ with $V_1 = \{x_1, x_3\}, V_2 = \{x_2, x_4\}$. Then by (4), $E(C_4)$ is an edge-cut of G. Let G_1 and G_2 denote the two sides of $G - E(C_4)$ such that $V_1 \subseteq V(G_1)$ and $V_2 \subseteq V(G_2)$. Since $\mathrm{diam}(G) \leq 2$, at most one of the $V(G_i)$'s contains more than two vertices. Since $n \geq 5$, we may assume that

$$|V(G_1)| = 2, \ |V(G_2)| = n - 2 \geq 3.$$

Since $\mathrm{diam}(G) \leq 2$, for every vertex $w \in V(G_2) - \{x_2, x_4\}$, we must have $wx_2, wx_4 \in E(G)$. Thus G has a spanning subgraph $H \cong K_{2,n-2}$ with $D_2(H) = \{x_2, x_4\}$. Since H is not collapsible but G is collapsible, and since $E(C_4)$ is an edge-cut of G, there must be vertices in $D_2(H)$ that are adjacent. Hence (b) of Theorem 1 holds. □

THE PROOF OF THEOREM 2

In this section, for $v \in V(G), N_G(v)$ denotes the set of vertices in G that are adjacent to v.

Let $n = |V(G)|$. When $n \leq 4$, Theorem 2 is obvious. Thus we assume that $n \geq 5$. Let G and e satisfy the hypothesis of Theorem 2 and suppose that $G(e)$ is not collapsible. Denote $e = x_1y_1, \ x_1, y_1 \in V(G)$ and let z_1 denote $v(e)$. By cotradiction, we assume that the conclusions of Theorem 2 is false. Let G be a counterexample of Theorem 2 with as few vertices as possible. Thus, by the minimality of G, we have

$$G(e) \text{ is reduced.} \tag{5}$$

By (5) and by (b) of Theorem A,

$$G(e) \text{ is } K_3\text{-free.} \tag{6}$$

Lemma 3 $G(e)$ contains no 4-cycle C with $E(C) \cap \{x_1z_1, z_1y_1\} = \emptyset$ (such a 4-cycle C is called a forbidden 4-cycle).

Proof: Suppose that $G(e)$ has a forbidden 4-cycle C. Let π denote the bipartition of $V(C)$. Since $x_1z_1, z_1y_1 \notin E(C)$, we have

$$(G/\pi)(e) = G(e)/\pi. \tag{7}$$

Let $[(G/\pi)(e)]'$ and $[G(e)/\pi]'$ denote the reductions of $(G/\pi)(e)$ and $G(e)/\pi$, respectively. Then by (7)

$$[(G/\pi)(e)]' = [G(e)/\pi]'. \tag{8}$$

By Theorem 1, either G/π is collapsible, or G has a spanning subgraph $H \cong K_{2,n-2}$, with two verices in $D_2(H)$ adjacent in G. If the latter case holds, then by Lemma 1, and by (c) of Theorem A, we have $[G(e)]' \cong K_1$ and so (a) of Theorem 2 holds, contrary to the assumption that G is a counterexample. Hence G/π must be collapsible.

Note that $\text{diam}(G/\pi) \leq \text{diam}(G) \leq 2$. By the minimality of G, $[G(e)/\pi]'$ must satisfy one of the conclusions of Theorem 2. If $e_\pi \notin E([G(e)/\pi]')$, then e_π is in a collapsible subgraph of $[G(e)/\pi]'$. It follows by Lemma 2 that

$$[G(e)]' = [G(e)/\pi]', \tag{9}$$

and so by (8) and (9), one of the conclusions of Theorem 2 must hold for $[G(e)]'$, contrary to the assumption that G is a counterexample. Hence we may assume that

$$e_\pi \in E([G(e)/\pi]'). \tag{10}$$

But then (10) and any one of (b), (c), (d), and (e) of Theorem 2 would imply that the diameter of G exceeds 2, a contradiction. This proves the lemma. □

If $\text{diam}(G(e)) \leq 2$, then by (5) and Theorem B, (a) or (b) or (c) of Theroem 2 must hold, contrary to the assumption that G is a counterexmaple. Hence by the hypothesis of $\text{diam}(G) \leq 2$, we may assume that

$$\text{diam} G(e) = 3. \tag{11}$$

By $\text{diam}(G) \leq 2$ again, for any distinct vertices $u, v \in V(G(e)) - \{z_1\}$, either

$$d_{G(e)}(u, v) \leq 2, \tag{12}$$

or in $G(e)$,

all shortest (u, v)-paths are of length 3 and contain x_1z_1, y_1z_1. (13)

By (11), there are distinct vertices $x, y \in v(G)$ such that $d_{G(e)}(x, y) = 3$. By (13), we may assume that $y = y_1$ and that xx_1z_1y is a shortest (x, y)-path.

By the hypothesis that G is collapsible, we have $\kappa'(G) \geq 2$ and so

$$\kappa'(G(e)) \geq 2. \tag{14}$$

This, in conjunction with (5) and (b) of Theorem A, implies

$$2 \leq \delta(G(e)) \leq 3.$$

Thus we have

$$2 \leq \delta(G) \leq 3. \tag{15}$$

Lemma 4 $\kappa(G(e)) \geq 2$.

Proof: By contradiction, we assume that $G(e)$ has a cut-vertex v and so it has two nontrivial connected subgraphs G_1 and G_2 such that

$$E(G(e)) = E(G_1) \cup E(G_2) \text{ and } V(G_1) \cap V(G_2) = \{v\}.$$

By (11), we may assume that in G_2, there is a vertex u such that

$$d_{G(e)}(v, u) \geq 2. \tag{16}$$

Thus every vertex in $V(G_1) - \{v\}$ is adjacent to v, by (11). By (5) and by (a) of Theorem A, both G_1 and G_2 are reduced. It follows that G_1 must have a cut-edge, contrary to (14). □

By Menger's Theorem ([1], page 16), by Lemma 4 and (11), we have

$$\text{every two edges of } G(e) \text{ is in a cycle of length at most 6.} \tag{17}$$

We shall divide the rest of the proof into several cases.

Case 1 Either $deg(x_1)$ or $deg(y)$ is equal to $\delta(G(e))$.

(1A) $deg(y) = 2$.
Let z, z_1 be the two vertices adjacent to y in $G(e)$. By (11) and (17), we may assume that $G(e)$ has a 6-cycle

$$H_1 = yz_1x_1xx_2zy.$$

Since diam(G) $\leq$ 2 and by $deg(y) = 2$, for every vertex $w \in V(G(e)) - \{y, z, z_1, x_1\}$

$$\text{either } zw \text{ or } x_1w \text{ is in } E(G(e)). \tag{18}$$

By (6) and (18), and by Lemma 3,

$$deg(x_2) = deg(x) = 2. \tag{19}$$

Since G is collapsible, $G \neq H_1$. But by (14), (18) and (19), for every $w \in V(G) - V(H_1)$, we must have

$$wz, wx_1 \in E(G(e)). \tag{20}$$

By (6) and Lemma 3, $E(G(e))$ consists of edges in $E(H_1)$ and edges described in (20) only. It follows that G is a subdivision of K_t, for some $t \geq 2$, and so G

is not collapsible, a contradiction.

(1B) $deg(y) = \delta(G) = 3$.

Let z_1, z_2, z_3 be the vertices adjacent to y in $G(e)$. By (5), (6) and (17), we may assume that there are two more vertices z_4, z_5 such that in $G(e)$,

$$yz_2, z_2z_4, z_4x, yz_3, z_3z_5, z_5x \in E(G(e)). \tag{21}$$

By $\delta(G) = 3$, and by (6), (12) and (13), we can find z_6, z_7 so that

$$z_2z_7, z_7z_5, z_3z_6, z_6z_4 \in E(G(e)). \tag{22}$$

Let H_2 denote the induced subgraph of $G(e)$ with

$$V(H_2) = \{y, x_1, x, z_1, z_2, z_3, z_4, z_5, z_6, z_7\}.$$

By (5), (6) and Lemma 3, $E(H_2)$ consists of the edges described in (21) and (22), together with $\{yz_1, z_1x_1, x_1x\}$. Since $\delta(G) = deg(y) = 3$, for every $w \in V(G(e)) - V(H_2)$,

$$\text{one of } wx_1, wz_2, wz_3 \text{ is in } E(G(e)). \tag{23}$$

Since H_2/x_1y is a subgraph of the Petersen graph, it is not collapsible, and so there must be a vertex $w_1 \in V(G(e)) - V(H_2)$.

Claim 1 If $w_1x_1 \in E(G(e))$, then $w_1z_6, w_1z_7 \in E(G(e))$.

Suppose that $w_1z_6 \notin E(G(e))$. By (12) and (13), $d_{G(e)}(w_1, z_6) = 2$ and so there must be some vertex $w_2 \in V(G(e))$ such that

$$w_1w_2, w_2z_6 \in E(G(e)).$$

Then by (23), either w_2x_1, or w_2z_2, or w_2z_3 is an edge of $G(e)$. By (6), it must be $w_2z_2 \in E(G)$. By $\delta(G) = 3$, there is some $w_3 \in V(G) - [V(H_2) \cup \{w_1, w_2\}]$ such that $w_3w_1 \in E(G)$. By (6), $w_3 \notin V(H_2) \cup \{w_1, w_2\}$. By (23), $G(e)$ has either a k-cycle, $2 \leq k \leq 3$, or a forbidden 4-cycle, contrary to (5) or Lemma 3.

Similarly, w_1z_7 is in $E(G(e))$. □

Claim 2 The degrees of z_4, z_5, x, x_1 in $G(e)$ are at most 3.

It follows from Claim 1 and Lemma 3 that $deg_{G(e)}(x_1)$ is at most 3.

Suppose that there is a vertex $w \notin V(H_2)$ such that $wz_4 \in E(G(e))$. By (23), one of wx_1, wz_2, wz_3 is in $E(G(e))$. But then by Claim 1 again, $G(e)$ contains either a K_3 or a forbidden 4-cycle, contrary to either (6) or Lemma 3.

Similarly, z_5, x must have degree 3 also. □

By Claim 2, we have $G \cong G(e)/z_1x_1$ is isomorphic to the Petersen graph and so G is not collapsible, a contradiction.

(1C) $deg_{G(e)}(x_1) = \delta(G)$.

If y is adjacent to a vertex w such that

$$d_{G(e)}(w, x_1) = 3,$$

then we are back to Case 1A or Case 1B, by renaming the vertices w, y, z_1, x_1 by x, x_1, z_1, y, respectively. Hence we may assume that

$$\text{for any } w \notin \{y, z_1, x_1, x\},\ wy \in E(G(e)) \Longrightarrow wx_1 \in E(G(e)). \qquad (24)$$

By (24), we have $deg_{G(e)}(y) = deg_{G(e)}(x_1) = \delta(G)$, and so we are back to Case 1A or Case 1B again.

Case 2 $deg_{G(e)}(x) = \delta(G)$ and $deg_{G(e)}(x_1) > \delta(G)$, $deg_{G(e)}(y) > \delta(G)$.

(2A) $\delta(G) = 2$.

Let x_1, x_2 be the vertices in $G(e)$ adjacent to x. By (17) and by the assumption of $d_{G(e)}(x, y) = 3$, there is some $z \in V(G(e))$ such that

$$H_3 = yzx_2xx_1z_1y$$

is a 6-cycle of $G(e)$. Since $deg_{G(e)}(x_1) \geq 3$, there is some vertex $w_1 \in V(G(e)) - V(H_3)$ such that $w_1x_1 \in E(G(e))$. By (12) and (13), we must have

$$d_{G(e)}(w_1, x_2) \leq 2.$$

By Lemma 3, there must be a vertex $w_2 \notin V(H_3) \cup \{w_1\}$ with $w_1w_2, w_2x_2 \in E(G(e))$. By (12), (13) and Lemma 3, there are must be a vertex $w_3 \notin V(H_3) \cup \{w_1, w_2\}$ with $w_2w_3, w_3y \in E(G(e))$. Then by $N_{G(e)}(x) = \{x_1, x_2\}$, and by (12) and (13), either w_3x_1 or w_3x_2 is in $E(G(e))$. It follows that $G(e)$ contains either a forbidden 4-cycle or a 3-cycle, contrary to Lemma 3 or to (6).

(2B) $\delta(G) = 3$.

Let x_1, x_2, x_3 be the vertices in $G(e)$ adjacent to x. By (17), by the assumption of $d_{G(e)}(x, y) = 3$ and by Lemma 3, there are vertices x_4, x_5 such that

$$x_2x_4, x_4y, x_3x_5, x_5y \in E(G(e)).$$

By (12), (13), we have $deg_{G(e)}(x_3, x_4) = 2$, and so by Lemma 3, there is a vertex $x_6 \notin \{y, x, z_1, x_1, x_2, x_3, x_4, x_5\}$ with

$$x_6x_4, x_3x_6 \in E(G(e)).$$

By (6) and by $deg_{G(e)}(y) \geq 4$, there is a vertex $x_7 \notin \{x_1, x_2, x_3, x_4, x_5, x_6, x, y, z_1\}$ with $x_7y \in E(G(e))$. By (12) and (13), we have $d_{G(e)}(x_7, x) = 2$ and so by $N_{G(e)}(x) = \{x_1, x_2, x_3\}$, either x_1x_7, or x_2x_7 or x_3x_7 is in $E(G(e))$. It follows by Lemma 3 that $x_1x_7 \in E(G(e))$. Similarly, $d_{G(e)}(x_6, x_7) = 2$ and so there is some vertex w with $wx_6, wx_7 \in E(G(e))$. Then by $N_{G(e)} = \{x_1, x_2, x_3\}$, and by (12) and (13), one of x_1w, x_2w, x_3w is in $E(G(e))$. It follows that $G(e)$ has either a K_3 or a forbidden 4-cycle, contrary to either (6) or Lemma 3.

Case 3 All $deg_{G(e)}(x_1), deg_{G(e)}(y), deg_{G(e)}(x)$ are greater than $\delta(G)$.
Let $z \in V(G(e)) - \{z_1\}$ be a vertex with $deg_{G(e)}(z) = \delta(G)$.

(3A) $zy \in E(G(e))$.
If $d_{G(e)}(z, x_1) = 3$, then by (13), $zy \in E(G(e))$, and so we are back to Case 2 with the path zyz_1x_1 replacing xx_1z_1y. Hence $d_{G(e)}(z, x_1) \leq 2$. If $zx_1 \in E(G(e))$, then by replacing xx_1z_1y by zx_1z_1y, we are back to Case 2 again. Thus we assume that there is some $x_2 \in V(G(e))$ with $x_2z, x_2x_1 \in E(G(e))$. If $deg_{G(e)}(z) = 2$, then by (12), $d_{G(e)}(x, z) = 2$ and so by (6) and $N_{G(e)}(z) = \{y, x_2\}$, we must have $xy \in E(G(e))$, contrary to the assumption of $d_{G(e)}(x, y) = 3$. Thus we assume that

$$deg_{G(e)}(z) = 3.$$

Let y, x_2, x_3 be the vertices in $G(e)$ that are adjacent to z and that zx_2, x_2x_1 are in $E(G(e))$. By (12) and (6), and by $xy \notin E(G(e))$, we have $xx_3 \in E(G(e))$. By $deg_{G(e)}(x) \geq 4$, we assume that x_4, x_5 are in $V(G(e))$ with $x_4x, x_5x \in E(G(e))$. By (12) with $\{u, v\} = \{z, x_i\}, (i = 1, 2)$, and by (6), we must have $x_4y, x_5y \in E(G(e))$. It follows that $G(e)$ contains a forbidden 4-cycle, contrary to Lemma 3.

(3B) $zx_1 \in E(G(e))$.
Since $deg_{G(e)}(x) \geq deg_{G(e)}(z) + 1$, by (12) with $\{u, v\}$ being z and one vert-dex adjacent to $x, G(e)$ contains either a 3-cycle or a forbidden 4-cycle, contrary to (6) or to Lemma 3.

(3C) $zx \in E(G(e))$.
Since $deg_{G(e)}(y) \geq deg_{G(e)}(z) + 1$, by (12) and (13) with u, v being nd one vertex adjacent to y, $G(e)$ must contain a 3-cycle or a forbidden 4-cycle, contrary to (6) or Lemma 3.

(3D) $zx, zx_1, zy \notin E(G(e))$.
By (12) and the assumption of Case 3D, there are x_2, x_3, x_4 such that

$$zx_2, x_2x, z_3, x_3x_1, zx_4, x_4y \in E(G(e)).$$

Hence $deg_{G(e)}(z) = 3$ and so $deg_{G(e)}(x) \geq 4$. By (12), (6) and by $N_{G(e)}(z) = \{x_2, x_3, x_4\}$, there is a vertex adjacent to x and one of x_3, x_4. By $deg_{G(e)}(x) \geq 4$, by (12) and by Lemma 3, we may assume that x_5 is adjacent to both x and x_3. By (12) and Lemma 3 again, there is a vertex x_6 adjacent to both y and x_2. By the same reason once more, there is some vertex w adjacent to x_5 and x_6.

Since $N_{G(e)}(z) = \{x_2, x_3, x_4\}$, and by (12), one of wx_2, wx_3, wx_4 is an edge of $G(e)$. It follows that $G(e)$ has either a 3-cycle or a forbidde 4-cycle, contrary to (6) or to Lemma 3.

Since all cases lead to contradictions, the proof of Theorem 2 is complete.

AN APPLICATION

We conclude this note with an application of Theorem 2. The line graph of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. A trail T of G is called a dominating trail of G if $G - V(T)$ is edgeless.

Theorem D (Harary and Nash-Williams [5]) Let G be a graph with at least 3 edges. $L(G)$ has a hamilton cycle if and only if G has a dominating eulerian subgraph. □

Imitating the proof of Theorem D, one has:

Lemma 4 Let e be an edge of G. Then $L(G)$ has a hamilton path starting with e if and only if G has a dominating trail starting with e. □

A dominating trail of G starting with an edge $e \in E(G)$ is called a dominating e-trail of G.

Lemma 5 ([7], Corollary 9) If G is collapsible, then for any $v, u \in V(G)$, (possibly $u = v$), there is a spanning (v, u)-trail in G. □

Corollary 1 If G is a graph of diameter at most two, then for any edge $e \in E(G)$, in $L(G)$, the line graph of G, has a hamilton path starting with e.

Proof: Let e be an edge of G. To avoid trivial cases, we assume that G has at least 3 edges. By Lemma 4, it suffices to show that G has a dominating e-trail.

Case 1 G is collapsible.

Note that G has an e-trail if and only if $G(e)$, the graph obtained from G by subdividing the edge e once, has a dominating trail with $v(e)$ at an end of the trail, (call such trails dominating $v(e)$-trails). Note that any spanning $v(e)$-trail (spanning trail starting with $v(e)$) is a dominating $v(e)$-trail.

If $G(e)$ is collapsible, then by Lemma 5, $G(e)$ has a $v(e)$-trail and we are done. So suppose that $G(e)$ is not collapsible. Let $[G(e)]'$ denote the reduction of $G(e)$. By Theorem 2, $[G(e)]' \in \{K_{2,t},\ S_{l,m},\ K^+_{2,t},\ S^+_{l,m}\}$, $(t \geq 2, l \geq 2, m \geq 1)$. Since G is collapsible, $v(e)$ must be a trivial vertex of $[G(e)]'$. It is then easy to check that $[G(e)]'$ has a spanning $v(e)$-trail. Thus G has a spanning trail starting with $v(e)$ and so G has a dominating e-trail.

Case 2 G is not collapsible.
Let G' denote the reduction of G. By Theorem B, $G' \in \{K_2, K_{1,t}, K_{2,t}, S_{l,m}, P\}$, where $t \geq 2$, $l, m \geq 1$ and P is the Petersen graph.

If $G' = P$, then by Corollary 7 of [7], $G = G' = P$ and so G has a spanning e-trail. By $\text{diam}(G) \leq 2$, if $G' = K_{1,t}$, then $G = G' = K_{1,t}$ again and so G has a dominating e-trail. Hence we assume $G' \in \{K_2, K_{2,t}, S_{l,m}\}$. Since $\text{diam}(G) \leq 2$, at most one vertex of G' is nontrivial, and if H denotes the only nontrivial collapsible subgraph of G, then H is spanned by $K_{1,t'}$, $(t' \geq 1)$. Thus G has a spanning e-trail again.

This proves Corollary 1. □

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