On the Minimum Graphs Which Contain all Small Trees

Yang Yuansheng

Dalian University of Technology
People's Republic of China

Abstract. Let E_n denote the minimum number of edges in a graph that contains every tree with n edges. This article provides two sets of data concerning $(n+1)$-vertex graphs with E_n edges for each $n \leq 11$: first, a minimum set of trees with n edges such that all trees with n edges are contained in such a graph whenever it contains the trees in the minimum set; second, all mutually nonisomorphic graphs that contain all trees with n edges.

1. Introduction.

Let E_n denote the minimum number of edges in a graph that contains every tree in $\mathcal{T}_n$, the set of all trees with n edges. Let S_n and P_n denote the star and path, respectively, in $\mathcal{T}_n$. Following [1] we refer to a graph that contains S_n and P_n as a star-path containment graph, and let $(SP)_n$ denote a star-path containment graph on $n+1$ vertices. For $n \geq 2$, an edge-minimum $(SP)_n$ has $2n-2$ edges when $\star$, the central vertex of S_n, is an interior vertex of P_n, and has $2n-1$ edges when $\star$ is a terminal vertex of P_n: see Figure 1.1. We always use $\star$ to denote the center of S_n and will often omit nonpath edges between $\star$ and other vertices.

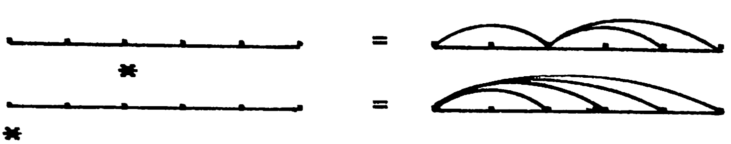

Figure 1.1

In accordance with [1], we let

$e_n = E_n -$ minimum number of edges in an $(SP)_n$;

$\sigma_n =$ number of unlabeled, mutually nonisomorphic graphs with $n+1$ vertices and E_n edges that contain $\mathcal{T}_n$;

$\mu_n =$ minimum cardinality of $\mathcal{S}_n \subseteq \mathcal{T}_n$ such that every $(n+1)$-vertex E_n-edge graph that contains $\mathcal{S}_n$ also contains $\mathcal{T}_n$.

We refer to such an $\mathcal{S}_n$ with $|\mathcal{S}_n| = \mu_n$ as a minimum sufficient set.

Table I summarizes the main enumeration results for $n \leq 11$ taken from Fishburn [1].

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Table I

n	E_n	e_n	σ_n	$\|\mathcal{T}_n\|$	μ_n
1	1	0	1	1	1
2	2	0	1	1	1
3	4	0	1	2	2*
4	6	0	2	3	2*
5	8	0	2	6	2
6	11	1	7	11	3
7	13	1	13	23	3
8	16	2	26*	47	5
9	18	2	2*	106	5
10	22	4		235	
11	24	4		551	

We have checked these independently for $n \leq 11$ and found that the numbers for μ_3, μ_4, σ_8, and σ_9, in Table I are incorrect. Table II summarizes our main enumeration results.

Table II

n	E_n	e_n	σ_n	$\|\mathcal{T}_n\|$	μ_n
1	1	0	1	1	1
2	2	0	1	1	1
3	4	0	1	2	1
4	6	0	2	3	1
5	8	0	2	6	2
6	11	1	7	11	3
7	13	1	13	23	3
8	16	2	25	47	5
9	18	2	17	106	5
10	22	4	776	235	14
11	24	4	2307	551	38

The values for E_n, e_n, and $\mathcal{T}_n$ $(1 \leq n \leq 11)$ are easy to verify. In this article we discuss only σ_n and μ_n.

2. Minimum sufficient sets.

It is easily seen for $n = 1$ and $n = 2$ that S_n, which is identical to P_n, forms a minimum sufficient set.

There are 2 trees in $\mathcal{T}_3$. Graph $GC3.1$ with $n+1 = 4$ vertices and $E_3 = 4$ edges contains all trees in $\mathcal{T}_3$ except S_3. Hence, we have $S_3 \in \mathcal{S}_3$, and so $\mu_3 \geq 1$. There is only one graph (G3.1) with 4 vertices and 4 edges that contains S_3. Graph G3.1 also contains P_3, and, hence, G3.1 contains all trees in $\mathcal{T}_3$. Therefore, $\mathcal{S}_3 = \{S_3\}$, $\mu_3 = 1, \sigma_3 = 1$. (See Figure 2.1.)

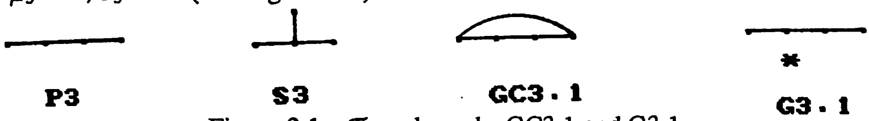


Figure 2.1 $\mathcal{T}_3$ and graphs GC3.1 and G3.1

Similarly, there are 3 trees in $\mathcal{T}_4$. Graph GC4.1 with $n+1 = 5$ vertices and $E_4 = 6$ edges contains all trees in $\mathcal{T}_4$ except S_4. Hence, we have $S_4 \in \mathcal{S}_4$, $\mu_4 \geq 1$. There are only two graphs (G4.1 and G4.2) with 5 vertices and 6 edges that contain S_4. Both G4.1 and G4.2 contain P_4 and T_4, hence, contain all trees in $\mathcal{T}_4$. Therefore, we have $\mathcal{S}_4 = \{S_4\}$, $\mu_4 = 1$, $\sigma_4 = 2$. (See Figure 2.2.)

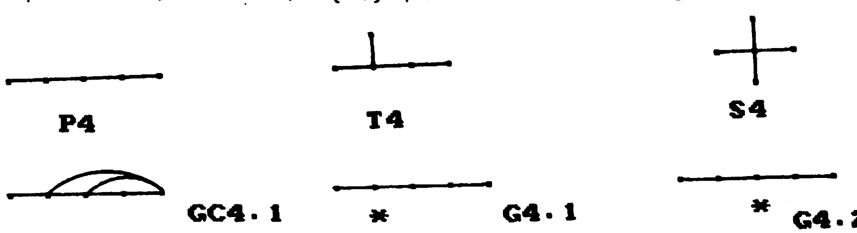


Figure 2.2 $\mathcal{T}_4$ and graphs GC4.1, G4.1, G4.2

From the above, we have

Theorem 2.1. *For n from 1 to 4, the values of μ_n and σ_n are :* $\mu_1 = \sigma_1 = 1$; $\mu_2 = \sigma_2 = 1$; $\mu_3 = \sigma_3 = 1$; $\mu_4 = 1$; $\sigma_4 = 2$.

In fact, S_n must be a member of every minimum sufficient set. If G is a graph on $n+1$ vertices with E_n edges that contains $\mathcal{T}_n$, and if one edge incident to $\star$ is deleted and replaced by an edge elsewhere, then the modified graph can contain virtually all trees in $\mathcal{T}_n$ except S_n.

The trees in the minimum sufficient set need not be unique. Let us take $n = 5$, for example. There are 6 trees in $\mathcal{T}_5$. Graph GC5.1 with $n+1 = 6$ vertices and $E_5 = 8$ edges contains all trees in $\mathcal{T}_5$ except $T5.1$ and $T5.2$. Hence, either $T5.1 \in \mathcal{S}_5$ or $T5.2 \in \mathcal{S}_5$, and so $\mu_5 \geq 2$. (See Figure 2.3.)

Figure 2.3 T5.1, T5.2, and GC5.1

There are 13 trees besides S_{10} in a minimum sufficient set for $n = 10$. (They are noted as T10.1 – T10.13 in Figure 2.4.) (For n from 5 to 9, see [1].)

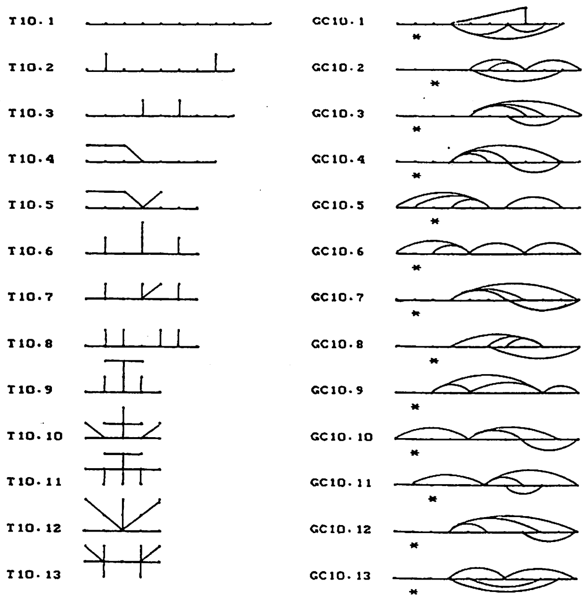


Figure 2.4 T10.1 – T10.13 and GC10.1 – GC10.13

There are 37 trees besides S_{11} in a minimum sufficient set for $n = 11$; they are available from the author on request.

Two things need to be verified for each alleged minimum sufficient set. First, we must demonstrate that each set is sufficient, that is, every graph with $n + 1$ vertices and E_n edges that contains the trees in the set contains all trees in $\mathcal{T}_n$. This is discussed further in the next section. Secondly, we must show that the set is minimum, and to that end we take $n = 10$, for example, to see how to establish appropriate lower bounds for μ_n.

There are 235 trees in $\mathcal{T}_{10}$. Graph GC10.i ($1 \leq i \leq 13$) with $n + 1 = 11$ vertices and $E_{10} = 22$ edges contains all trees in $\mathcal{T}_{10}$ except T10.i ($1 \leq i \leq 13$), ana so $\mu_{10} \geq 14$. (See Figure 2.4.)

In a similar way we have $\mu_6 \geq 3$, $\mu_7 \geq 3$, $\mu_8 \geq 5$, $\mu_9 \geq 5$, and $\mu_{11} \geq 38$. From above we have

Lemma 2.2. *For* $5 \leq n \leq 11$, *we have* $\mu_n \geq \beta_n$, *where* $\beta_5 = 2$, $\beta_6 = \beta_7 = 3$, $\beta_9 = \beta_9 = 5$, $\beta_{10} = 14$, *and* $\beta_{11} = 38$.

3. Nonisomorphic containment graphs.

In view of Lemma 2.2 we now carry out the following procedure by computer:

(1) for $1 \leq i \leq \beta_n$, construct the set $SGn.i$ of all graphs with $n + 1$ vertices and E_n edges that contain $Tn.i$ and S_n;

(2) determine $SGn = SGn.1 \cap SGn.2 \cap \ldots \cap SGn.\beta_n$;
(3) verify that every graph in SGn contains every tree in $\mathcal{T}_n$.

We conclude the following, where M_n denotes the set of trees Tn.i and S_n $(1 \leq i \leq \beta_n)$.

Theorem 3.1. *For* $5 \leq n \leq 11$, $\mu_n = \beta_n$ *and* $M_n = S_n$.

We illustrate our procedure in the case $n = 6$. There are two trees in M_6 besides S_6. (See Figure 3.1.) There are 13 mutually nonisomorphic graphs with $n+1 = 7$ vertices and $E_6 = 11$ edges that contain T6.1 and S_6. (See Figure 3.2.) There are 9 mutually nonisomorphic graphs with 7 vertices and 11 edges that contain T6.2 and S_6. (See Figure 3.3.) There are 7 graphs that belong to both SG6.1 and SG6.2. (See Figure 3.4.) Every graph in SG6 contains all trees in $\mathcal{T}_6$. Hence, we have $\mu_6 = 3$, $\sigma_6 = |SG6| = 7$. We find similarly that $\sigma_5 = 2$, $\sigma_7 = 13$, $\sigma_8 = 25$, $\sigma_9 = 17$, $\sigma_{10} = 776$, and $\sigma_{11} = 2307$.

Figure 3.1 The trees in M_6 besides S_6

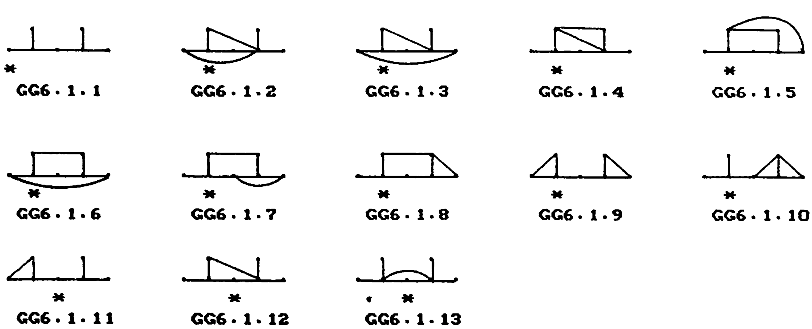


Figure 3.2 The set SG6.1

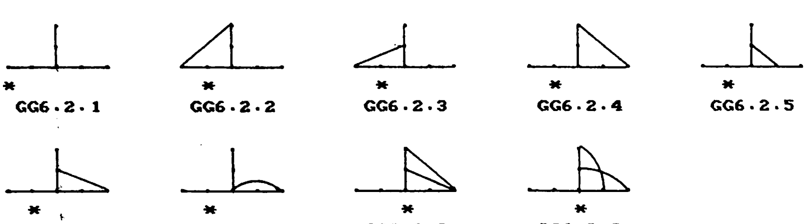
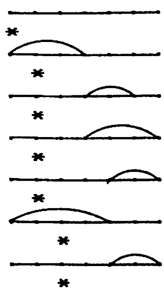


Figure 3.3 The set SG6.2



G6.1 $\cong$ GG6.1.12 $\cong$ GG6.2.2

G6.2 $\cong$ GG6.1.1 $\cong$ GG6.2.3

G6.3 $\cong$ GG6.1.8 $\cong$ GG6.2.5

G6.4 $\cong$ GG6.1.5 $\cong$ GG6.2.6

G6.5 $\cong$ GG6.1.7 $\cong$ GG6.2.7

G6.6 $\cong$ GG6.1.6 $\cong$ GG6.2.1

G6.7 $\cong$ GG6.1.9 $\cong$ GG6.2.8

Figure 3.4 The set SG6

The graphs in SGn are called minimum containment graphs in [1]. Figure 3.5 shows all the graphs in SGn for $n = 8$ and $n = 9$. For n from 5 to 7, see [1]. There are 776 graphs in SG_{10} and 2307 graphs in SG_{11}: they are available from the author on request.

Acknowledgement

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References

1. P.C. Fishburn, *Minimum graphs that contain all small trees*, Ars Combinatoria 25 (1988), 133–165.