Subdesigns in Complementary Path Decompositions and Incomplete Two-fold Designs with Block Size Four

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Abstract. We give a complete solution to the existence problem for subdesigns in complementary P_3-decompositions, where P_3 denotes the path of length three. As a corollary we obtain the spectrum for incomplete designs with block size four and $\lambda = 2$, having one hole.

1. Introduction.

In a recent paper, Rees and Stinson posed, and gave nearly complete solutions for, several problems involving subdesigns in combinatorial designs [7]. Since then, two of these problems have been completely solved (embeddings of Kirkman Triple Systems and embeddings of $(v, 4, 1)$-BIBDs [8, 9, 10]). In this paper, we give a complete solution to a third problem, namely, that of determining the spectrum for complementary path decompositions with subdesigns, where the paths are all isomorphic to P_3, the path with three edges.

A *complementary decomposition* $2K_v \to (P_3, P_3)$ is an edge decomposition of the complete graph K_v into P_3s with the property that upon taking the complement of each path one obtains a second decomposition of K_v into P_3s. (The complement of the path $abcd$ is the path $bdac$.) Note that if D is such a decomposition then the set $\{\{a, b, c, d\}: abcd \in D\}$ is an edge decomposition of $2K_v$ into K_4s, that is, a $(v, 4, 2)$-BIBD. The following result was proven by Granville, Moisiadis, and Rees in [2] (and, with a few small exceptions, also follows from the techniques in this paper):

Theorem 1.1. *There exists a complementary decomposition* $2K_v \to (P_3, P_3)$ *if and only if* $v \equiv 1$ *modulo 3.*

A subdesign (or subsystem) in a complementary decomposition $2K_v \to (P_3, P_3)$ is a complementary decomposition $2K_w \to (P_3, P_3)$ for some complete multisubgraph $2K_w \subseteq 2K_v$. Since this yields a $(v, 4, 2)$-BIBD with a sub-$(w, 4, 2)$-BIBD a necessary condition for existence is that $v \geq 3w + 1$. It was shown in [7] that this is sufficient in all but finitely many cases:

Theorem 1.2. *Let $v \equiv w \equiv 1$ modulo 3, $v \geq 3w + 1$ and $v - w \geq 411$. Then there exists a complementary decomposition $2K_v \to (P_3, P_3)$ containing a subsystem $2K_w \to (P_3, P_3)$.*

We will show that the condition $v - w \geq 411$ can be removed from the hypothesis of Theorem 1.2. Our techniques will be essentially independent of those in [7]; we will use a type of design called an incomplete self-orthogonal latin square for our constructions.

A latin square is called *self-orthogonal* if it is orthogonal to its transpose. An *incomplete self-orthogonal latin square* ISOLS(n, k) is an $n \times n$ array A with entries from an n-set S, such that for some k-subset $S' \subseteq S$:

(i) each cell of A is either empty or contains an element of S;
(ii) the subarray indexed by $S' \times S'$ is empty;
(iii) the elements in row or column s are precisely those of $S \backslash S'$ if $s \in S'$, and those of S if $s \notin S'$; and
(iv) if we superimpose the transpose array A^T onto A we obtain all ordered pairs in $(S \times S) \backslash (S' \times S')$.

Note that an ISOLS(n, 1) is equivalent to a self-orthogonal latin square of order n. The spectrum for ISOLS(n, k) has been almost completely determined (see Heinrich and Zhu [3], and Heinrich, Wu and Zhu [4]):

Theorem 1.3. *Let $k > 0$. There exists an ISOLS(n, k) if and only if $n \geq 3k + 1$, with the exceptions $(n, k) = (6, 1)$ and $(8, 2)$, and possibly $(n, k) \in \{(6m + 2, 2m) : m > 1\}$.*

2. Constructing complementary P_3-decompositions from self-orthogonal latin squares.

Let A be a self-orthogonal latin square of order n and A^T be its transpose. We may assume that A is written on the symbols $1, 2, \ldots, n$ and, furthermore, that the (i, i)-entry in A is i, for each $i = 1, 2, \ldots, n$. For each i and j with $1 \leq i$, $j \leq n$ let us denote the (i, j)-entry in A by $i * j$ and the (i, j) entry in A^T by $i \cdot j$. Let $v = 3n + 1$ and label the vertices of K_v with $(\{1, 2, \ldots, n\} \times Z_3) \cup \{\infty\}$. Consider the following collection of paths in K_v (note that since $i \cdot j$ is the symbol $j * i$, $i * j$ and $i \cdot j$ are distinct when $i \neq j$): $(i * j, x + 1)\ (i, x)(j, x)(i \cdot j, x + 1)$ where $1 \leq i < j \leq n$ and $x = 0, 1, 2$ (addition is modulo 3 in the second coordinate), together with

$$\begin{array}{cccc} \infty & (i,0) & (i,1) & (i,2) \\ (i,0) & (i,2) & \infty & (i,1) \end{array}$$

where $1 \leq i \leq n$. Since A is idempotent and $i \cdot j = j * i$ it can be readily verified that the above forms an edge-decomposition of K_v. Moreover, since A and A^T are orthogonal, we obtain a second edge-decomposition upon taking the complement of each of the above paths:

(i,x)	$(i\cdot j,x+1)$	$(i*j,x+1)$	(j,x)
$(i,0)$	$(i,2)$	∞	$(i,1)$
∞	$(i,0)$	$(i,1)$	$(i,2)$

where $1 \leq i < j \leq n$ and $x = 0,1,2$.

Now consider an ISOLS(n,k) A, with symbol sets $S = \{1,2,\dots,n\}$ and $S' = \{n-k+1, n-k+2, \dots, n\}$. Again we may assume that the (i,i)-entry in A is i for $1 \leq i \leq n-k$. Let $v = 3n+1$ and $w = 3k+1$. Then the foregoing construction, with i restricted to $1 \leq i \leq n-k$, will yield and edge-decomposition of the graph $K_{v-w} \vee \overline{K}_w$ (where $\vee$ denotes the usual join function and $\overline{K}_w$ is the empty graph with w vertices) into P_3s with the property that upon taking the complement of each path one obtains a new decomposition of $K_{v-w} \vee \overline{K}_w$ into P_3s. By constructing a complementary decomposition $2K_w \rightarrow (P_3, P_3)$ (the existence of which is guaranteed by Theorem 1.1) on the 'hole' in the above design, we have now established the following:

Theorem 2.1. *If there is an ISOLS(n,k) then there is a complementary decomposition $2K_{3n+1} \rightarrow (P_3,P_3)$ containing a subsystem $2K_{3k+1} \rightarrow (P_3,P_3)$.*

3. The results.

It will be assumed throughout this section that the reader is familiar with the definitions and notation for group-divisible designs (GDDs) and pairwise balanced designs (PBDs).

We will need the following preliminary result.

Lemma 3.1. *For each $u \geq 4$ there is a 4-GDD of type $6^u(3u-3)^1$. Also, there is a 4-GDD of type $3^4 6^2$.*

Proof: A 4-GDD of type $3^4 6^2$ appears in the appendix of [7]. A 4-GDD of type $6^u(3u-3)^1$ is obtained by adjoining a group 'at infinity' to a resolvable 3-GDD of type 6^u. These latter designs exist for all $u \geq 4$ (see [1,6]). ∎

Theorem 3.2. *Let $v \equiv w \equiv 1$ modulo 3, $v \geq 3w+1$, $v \not\equiv 3w+4$ and $(v,w) \neq (19,4)$. Then there exists a complementary decomposition $2K_v \rightarrow (P_3,P_3)$ containing a subsystem $2K_w \rightarrow (P_3,P_3)$.*

Proof: If $w = 1$ apply Theorem 1.1. Now suppose that $w \geq 4$ and let $n = \frac{v-1}{3}$ and $k = \frac{w-1}{3}$. From Theorem 1.3 there is an ISOLS(n,k). Apply Theorem 2.1.

Theorem 3.3. *For each $w \equiv 1$ modulo 3 there is a complementary decomposition $2K_{3w+4} \rightarrow (P_3,P_3)$ containing a subsystem $2K_w \rightarrow (P_3,P_3)$.*

Proof: If $w = 1$ apply Theorem 1.1, and if $w = 4$ apply Theorem 2.1 to an ISOLS$(5,1)$. If $w = 7$ adjoin a point to each group in a 4-GDD of type $3^4\ 6^2$

to obtain a PBD($\{4,7\}$; 25) and construct a complementary path decomposition on each block. If $w \geq 10$ adjoin a point to each group in a 4-GDD of type $6^{\frac{w+2}{3}}(w-1)^1$ to obtain a PBD($\{4,7,w*\}$; $3w+4$) and construct a complementary path decomposition on each block. ∎

Theorem 3.4. *There is a complementary decomposition $2K_{19} \rightarrow (P_3, P_3)$ containing a subsystem $2K_4 \rightarrow (P_3, P_3)$.*

Proof: Vertex set $(Z_5 \times \{1,2,3\}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.
Develop the following paths modulo 5:

$\infty_1(0,1)(3,1)(3,2)$	$\infty_3(0,3)(3,3)(3,1)$	$(4,1)(3,2)(2,1)(4,2)$
$(4,2)\infty_1(0,3)(1,3)$	$(4,1)\infty_3(0,2)(1,2)$	$(2,2)(1,3)(0,2)(2,3)$
$\infty_2(0,2)(3,2)(3,3)$	$\infty_4(0,1)(3,2)(1,3)$	$(0,3)(1,1)(2,3)(0,1)$
$(4,3)\infty_2(0,1)(1,1)$	$(1,2)\infty_4(0,3)(2,1)$	$\infty_1\infty_2\infty_3\infty_4$
		$\infty_2\infty_4\infty_1\infty_3$

∎

Collecting Theorem 3.2, 3.3, and 3.4, we now have established

Theorem 3.5. *There is a complementary decomposition $2K_v \rightarrow (P_3, P_3)$ containing a (proper) subsystem $2K_w \rightarrow (P_3, P_3)$ if and only if $v \equiv w \equiv 1$ modulo 3 and $v \geq 3w+1$.*

Recalling that a complementary decomposition $2K_v \rightarrow (P_3, P_3)$ gives rise to a $(v,4,2)$-BIBD we get the following as a by-product of Theorem 3.5:

Corollary 3.6. *There is a $(v,4,2)$-BIBD containing a $(w,4,2)$-BIBD as a (proper) subdesign if and only if $v \equiv w \equiv 1$ modulo 3 and $v \geq 3w+1$.*

An *incomplete* PBD (of index λ) is a triple (X, Y, B) where X is a set of points, Y is a subset of X (called the *hole*) and B is a collection of subsets of X (blocks), satisfying:

(i) each unordered pair of points from X occurs either in Y or in exactly λ blocks; and
(ii) for each block $B_i \in B$, $|Y \cap B_i| \leq 1$.

A $(v, w; K)$-IPBD of index is an incomplete PBD with $|X| = v$, $|Y| = w$, and $|B_i| \in K$ for each block $B_i \in B$.

The spectrum for incomplete PBDs of index 1 with block size 4 has been determined by Rees and Stinson [9] and Mills [5]:

Theorem 3.7. *Let $w > 0$. There exists a $(v, w; \{4\})$-IPBD of index 1 if and only if $v \geq 3w+1$ and either*

(i) *$v \equiv 1$ or 4 modulo 12 and $w \equiv 1$ or 4 modulo 12; or*
(ii) *$v \equiv 7$ or 10 modulo 12 and $w \equiv 7$ or 10 modulo 12.*

Note that this yields a broader spectrum of pairs (v, w) than that which occurs by considering only embeddings of $(w, 4, 1)$-BIBDs. The same phenomena does not occur when $\lambda = 2$, however; it is not difficult to verify that if a $(v, w; \{4\})$-IPBD of index 2 exists, then $v \equiv w \equiv 1$ modulo 3 and $v \geq 3w + 1$. Hence, the spectrum for these designs is an immediate consequence of Corollary 3.6:

Theorem 3.8. *There exists a $(v, w; \{4\})$-IPBD of index 2 if and only if $v \equiv w \equiv 1$ modulo 3 and $v \geq 3w + 1$.*

Proof: Remove the blocks from the sub-$(w, 4, 2)$-BIBD to create a hole of size w. ∎

Conclusion.

In concluding, we would like to thank the referee for pointing out that R. Wei has determined the spectrum for incomplete BIBDs with block size four and $\lambda = 3$, while G. Kong and L. Zhu have settled the case $\lambda = 6$. These results, together with our Theorem 3.8, and the results of [9], will determine the spectrum for $(v, w; \{4\})$-IPBD of index for any $\lambda \geq 1$.

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