Another Theorem on Families of Sets

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The following result can be proved with the aid of the famous Hall marriage theorem:

Given a family $(A_i : i \epsilon I)$ of $|I| = n+1$ non-zero subsets of a set S with $|S| = n$. Then there are disjoint non-zero subsets I_1 and I_2 of I with $\bigcup_{i \epsilon I_1} A_i = \bigcup_{i \epsilon I_2} A_i$.

In [1] I proved a generalization of this result using Rado's extension of Hall's theorem. Tverberg gave another proof in [2], which depends on his generalization of Rado's theorem for point sets in R^n.

I do not know any combinatorial proof of the following.

Theorem. *Given a family $(A_i : i \epsilon I)$ of $|I| = n+2$ subsets of a finite set S with $|S| = n$. Then there are disjoint non-zero subsets I_1 and I_2 of I such that*

$$\bigcup_{i \epsilon I_1} A_i = \bigcup_{i \epsilon I_2} A_i$$

and

$$\bigcap_{i \epsilon I_1} A_i = \bigcap_{i \epsilon I_2} A_i \quad .$$

Proof. Let $S = \{1,2,\dots,n\}$. To any subset $A \subset S$ associate the incidence vector of the pair of subsets $(A, S-A)$, $v \epsilon R^{2n}$, $v = (x_1,\dots,x_n,y_1,\dots,y_n)$ with $x_i = 1$ if $i \epsilon A$, $x_i = 0$ if $i \notin A$ and $y_i = 1 - x_i$ for $i = 1,2,\dots,n$. Observe that v belongs to the subspace V of all vectors v for which $x_1 + y_1 = \cdots = x_n + y_n$. The dimension of V is $n+1$.

Now let $v_i \epsilon V$ correspond to A_i, $i \epsilon I$. Since $|I| = n+2$ and $dim V = n+1$, there is a non-trivial linear relation between these vectors v_i, $i \epsilon I$, which we can write

$$\sum_{i \in I_1} a_i v_i = \sum_{i \in I_2} b_i v_i \quad ,$$

where I_1 and I_2 are non-void subsets of I and a_i, $b_j > 0$ for $i \in I_1$ and $j \in I_2$, $I_1 \cap I_2 = \emptyset$. Then it follows

$$\bigcup_{i \in I_1} A_i = \bigcup_{i \in I_2} A_i$$

and

$$\bigcup_{i \in I_1} (S - A_i) = \bigcup_{i \in I_2} (S - A_i) \quad ,$$

and the theorem follows by the de Morgan laws for sets.

References.

[1] B. Lindstrom, *A theorem on families of sets*, J. Comb. Theory 13 (1972), 274-277.

[2] H. Tverberg, *On equal unions of sets*, In: Studies in Pure Mathematics (Frestschrift in honor of R. Rado), ed. by L. Mirsky, Academic Press, London 1971.