More on Order 10 Turn-Squares

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Abstract. We verify that 6 more of the turn squares of order 10 cannot be completed to a triple of mutually orthogonal latin squares of order 10. We find a pair of orthogonal latin squares of order 10 with 6 common transversals, 5 of which have only a single intersection, and a pair with 7 common transversals.

We start with the upper left block

0	4	3	2	1
4	3	2	1	0
3	2	1	0	4
2	1	0	4	3
1	0	4	3	2

This block is repeated in the lower right corner and 5 is added to all digits to produce the other two blocks.

The cells (i,j), $(i,j+5)$, $(i+5,j)$, $(i+5,j+5)$ $0 \leq i,j \leq 4$ contain a latin 2-subsquare. If we turn the 2-subsquare by 90 degrees, then a new latin square is obtained. The turn is completely determined by marking the cell in the upper left hand block. We give the name "turn-square" to the initial square and all squares obtained by turning some 2-subsquares. An order 10 latin square usually has about 850 transversals whereas the turn-squares, with the exception of the trivial one have 2816-5504 transversals.

We have previously classified the order 10 turn-squares [3]. The square with no cell marked has no transversals. There is one square with 1 cell marked, and 2 squares with 2 cells marked. These have no mates by Mann's Theorem [4,5].

The square with 00, 01, 10 marked has 5504 transversals. It has a rich automorphism group, and Parker [6] showed in 1963 that it did not complete to a triple of MOLS. The square with 24, 31, 32, 44 marked has 2816 transversals. The authors [1] showed in 1982 that it did not complete to a triple of MOLS. The squares with 00, 11, 22 marked and with 11, 22, 33, 44 marked both have 3584 transversals. The authors [2] showed in 1984 that these squares do not complete to a triple of MOLS.

We have now shown that 6 more of the turn-squares do not complete to a triple of MOLS . They are;

Cells Marked	Transversals
00, 01, 20	4352
00, 01, 40	4096
00, 01, 22	4224
00, 11, 24	4096
24, 32, 33, 43	4224
14, 22, 33, 41	4224

Our procedure is as follows. We call any cell of the 4 cells in a turning, a special cell. Then Parker [7] has shown that any transversal must contain an odd number of special cells. In all of the above cases there must be a transversal which contains 3 special cells if there are to be the 10 parallel transversals which form an orthogonal mate. We start with such a transversal and use the automorphisms of the square to reduce the number of starting possibilities.

Two of the squares did produce some interesting output.

The square 00, 01, 20 with 4352 transversals produced a pair of orthogonal squares with 7 common transversals. See Figure 1. The leftmost square is the base square. It was rearranged to aid the automorphism search, and 10 has replaced 0 as a digit. The digit j in the i-th position of the transversal indicates cell (i, j).

9	4	1	6	3	8	2	7	5	10	1	2	3	4	5	6	7	8	9	10
3	8	10	5	2	7	1	6	4	9	9	3	4	8	6	2	5	10	1	7
2	7	4	9	1	6	10	5	8	3	10	1	7	6	4	3	9	5	2	8
1	6	3	8	10	5	4	9	2	7	2	9	6	10	1	7	8	3	4	5
10	5	2	7	4	9	3	8	1	6	5	6	2	7	10	4	3	9	8	1
4	9	6	1	8	3	7	2	10	5	6	5	8	9	7	10	4	1	3	2
8	3	5	10	7	2	6	1	9	4	8	4	1	2	9	5	6	7	10	3
7	2	9	4	6	1	5	10	3	8	3	8	9	5	2	1	10	6	7	4
6	1	8	3	5	10	9	4	7	2	7	10	5	1	3	8	2	4	6	9
5	10	7	2	9	4	8	3	6	1	4	7	10	3	8	9	1	2	5	6

1.	4	10	1	7	2	9	5	6	3	8
2.	6	4	5	2	1	8	10	9	7	3
3.	6	7	2	8	5	3	4	9	10	1
4.	7	8	4	5	9	10	2	1	3	6
5.	8	10	7	4	3	1	2	6	5	9
6.	9	3	4	7	10	5	6	1	2	8
7.	9	3	10	8	4	1	6	5	2	7

Figure 1. Pair with 7 Common Transversals

The square 00, 01, 40 with 4096 transversals produced a pair of orthogonal squares with 6 common transversals, 4 of which are parallel and a fifth has only a single intersection with the four. See Figure 2. The rearranged base square is on the left with 10 replacing 0.

9	4	2	7	3	8	1	6	5	10	1	2	3	4	5	6	7	8	9	10
3	8	1	6	2	7	10	5	4	9	2	3	8	7	10	1	9	6	4	5
2	7	10	5	1	6	4	9	3	8	6	8	5	3	1	9	10	4	7	2
1	6	4	9	10	5	3	8	2	7	4	5	1	8	3	10	6	7	2	9
10	5	3	8	4	9	2	7	6	1	7	1	9	5	8	2	4	3	10	6
4	9	7	2	8	3	6	1	10	5	5	10	6	9	4	3	1	2	8	7
8	3	6	1	7	2	5	10	9	4	9	4	2	10	7	8	5	1	6	3
7	2	5	10	6	1	9	4	8	3	10	7	4	2	6	5	3	9	1	8
6	1	9	4	5	10	8	3	7	2	3	9	7	6	2	4	8	10	5	1
5	10	8	3	9	4	7	2	1	6	8	6	10	1	9	7	2	5	3	4

1.	1	<u>5</u>	<u>2</u>	<u>7</u>	4	<u>10</u>	<u>3</u>	8	<u>6</u>	<u>9</u>
2.	2	6	3	4	7	<u>10</u>	1	5	8	<u>9</u>
3.	4	3	6	<u>7</u>	1	2	10	9	5	8
4.	9	<u>5</u>	<u>2</u>	8	10	1	<u>3</u>	7	<u>6</u>	4
5.	<u>10</u>	1	5	2	8	4	9	3	7	6
6.	<u>10</u>	2	8	9	3	7	5	6	4	1

Figure 2. Pair with Nearly 5 Parallel Transversals.

There was only one pair with 7 common transversals. Pairs with 6 common transversals occured from time to time, but they contained 4 parallel transversals very infrequently.

References

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