On the Game Chromatic Number of some Classes of Graphs

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Abstract. Consider the following two-person game on the graph G. Player I and II move alternatingly. Each move consists in coloring a yet uncolored vertex of G properly using a prespecified set of colors. The game ends when some player can no longer move. Player I wins if all of G is colored. Otherwise Player II wins. What is the minimal number $\gamma(G)$ of colors such that Player I has a winning strategy? Improving a result of Bodlaender [1990] we show $\gamma(T) \leq 4$ for each tree T. We, furthermore, prove $\gamma(G) = 0(\log |G|)$ for graphs G that are unions of k trees. Thus, in particular, $\gamma(G) = 0(\log |G|)$ for the class of planar graphs. Finally we bound $\gamma(G)$ by $3\omega(G) - 2$ for interval graphs G. The order of magnitude of $\gamma(G)$ can generally not be improved for k-fold trees. The problem remains open for planar graphs.

1. Introduction.

Consider the following two-person game on a graph G. Players I and II move alternatingly with Player I moving first, say. Each move consists in choosing a vertex, say v, which is not yet colored and assigning one color from a prespecified set of colors to it so that the resulting partial coloring of G has no two adjacent vertices bearing the same color. The game ends as soon as one of the two players can no longer execute a feasible move. Player I wins if all vertices of G are colored; otherwise Player II wins.

The *game chromatic number* $\gamma(G)$ of G is the smallest number colors such that there is a winning strategy for Player I. Bodlaender [1990] introduces the game chromatic number and studies its computational complexity. He shows, for example, that $\gamma(T) \leq 5$ holds for trees T and exhibits trees satisfying $\gamma(T) \geq 4$. (Most of his results, however, deal with a variation of the above game, in which the vertices of G have to be chosen in a prespecified order.)

We study $\gamma(G)$ for several classes of graphs. In Section 2, we improve Bodlaender's bound to $\gamma(T) \leq 4$ for trees and introduce a modified coloring game on trees, which is useful for analyzing other classes of graphs. In Section 3, we look at graphs G whose edge sets are unions of edge sets of k trees and prove $\gamma(G) = O(\log |G|)$ for fixed k. We, furthermore, exhibit an infinite number of graphs G that are unions of two trees and satisfy $\gamma(G) \geq c \cdot \log |G|$ for some constant $c > 0$. A direct application of these results yields $\gamma(G) = O(\log |G|)$ for planar graphs G. (The problem of determining nontrivial lower bounds remains open for planar graphs.) Section 4 is devoted to interval graphs G, which turn out to satisfy $\gamma(G) \leq 3\omega(G) - 2$. Interval graphs G with $\gamma(G) \geq 2\omega(G)$ can be constructed.

It may be interesting to observe that many of our results for upper bounds on $\gamma(G)$ actually refer to a generalization of the coloring game in the following way. Instead of "coloring" vertices, the players just "mark" vertices of the graph G alternatingly. Player I loses as soon as some unmarked vertex of G is adjacent to more than K marked vertices. What is the minimum number K such that Player I has a winning strategy?

2. Trees.

In this section, we will consider graphs that do not contain cycles. There is no loss in generality when we assume that these graphs are connected, that is, are trees. Bodlaender [1990] has shown that the game chromatic number $\gamma(T)$ of a tree T satisfies $\gamma(T) \leq 5$ and that there are trees T with $\gamma(T) \geq 4$.

Theorem 1. *If T is a tree, then $\gamma(T) \leq 4$.*

Proof: We will give a winning strategy for the coloring game described in the Introduction using only 4 colors.

Initially, Player I chooses an arbitrary vertex r of T, which will, henceforth, be called the *root*, and assigns some color to it. During the whole game, Player I maintains a subtree T_0 of T that contains all the vertices colored so far. Player I initializes $T_0 = \{r\}$.

Suppose now that Player II has just moved by coloring vertex v. Let P be the (unique) directed path from r to v in T and let u be the last vertex P has in common with T_0. Then Player I does the following:

(1) Update $T_0 := T_0 \cup P$.
(2) If u is uncolored, assign a feasible color to u.
(3) If u is colored and T_0 contains an uncolored vertex $v \in T_0$, assign a feasible color to v.
(4) If all vertices in T_0 are colored, color any vertex v adjacent to T_0 and update $T_0 := T_0 \cup \{v\}$.

It is clear that this strategy of Player I guarantees each player the existence of an uncolored vertex with at most 3 colored neighbors until the whole tree is colored. ∎

Let us now consider a modification of the coloring game in which Player II is allowed to color 3 vertices in one move. We denote by $\overline{\gamma}$ the modified game coloring number.

Theorem 2. *There is a constant* c *such that for every tree* T *with* n *vertices the modified game coloring number* $\overline{\gamma}(T)$ *satisfies* $\overline{\gamma}(T) \leq c \cdot \log n$.

Proof: The winning strategy for Player I is as follows. Before his r-th move, the set V^r of uncolored vertices of T partitions into non-empty connected components $S_1^r, \ldots, S_\ell^r$ $(\ell \leq n)$. Each such component S_i^r is weighted with the number $m(S_i^r)$ of colored vertices adjacent to S_i^r.

Player I now chooses a component S_i^r of maximal weight $m(S_i^r)$ and colors a vertex $v \in S_i^r$ so that $S_i^r \backslash \{v\}$ decomposes into connected components each having weight at most $1 + \lceil m(s_i^r)/2 \rceil$. It is clear that Player I can indeed find such a vertex v. The Theorem will follow if we can show that after Player I's r-th move each component of $V^r \backslash \{v\}$ has weight $O(\log n)$.

It is convenient to consider the *reduced weights* $s(S) = m(S) - 1$ of connected components S of uncolored vertices. The next property is obvious.

Claim A: Assume that the subset $C \subseteq S$ of the connected component S is colored and denote by $S_1, \ldots, S_t$ the connected components induced on $S \backslash C$. Then the reduced weights satisfy

$$s(S_1) + \cdots + s(S_t) \leq s(S) + |C|.$$

We need another technical fact.

Claim B: Assume that $s, s_1, \ldots, s_t$ and k are nonnegative integers such that $s \geq k + 6$, $s_i \geq k_1$ $(i = 1, \ldots, t)$, and $s_1 + \cdots + s_t \leq s + k$. Then either $t = 1$ or

$$(4/3)^s \geq (4/3)^{s_1} + \cdots + (4/3)^{s_t}.$$

We will use claim A and claim B with $k = |C| \leq 3$ in order to analyse the move of player II. Informally, the two properties imply that player II cannot create many large connected components and increase their "potential" at the same time. To be more definite, let us say that a component S of uncolored vertices is *large* if its reduced weight satisfies $s(S) \geq 18$; otherwise it is *small*. It follows from claim A that Player II cannot induce large components when coloring at most 3 vertices of S unless claim B becomes applicable in the analysis.

Let $S_1, \ldots, S_u$ be the nonempty connected components of uncolored vertices after move $r - 1$ of player I. We associate with this collection of components the *potential*

$$\phi_{r-1} = (4/3)^{s_1} + (4/3)^{s_2} + \cdots + (4/3)^{s_u},$$

where $s_i = m(S_i) - 1$. Note that the contribution of small components to the potential ϕ is always bounded by

$$n(4/3)^{17} < 134\,n$$

Claim C: $\phi_r - \phi_{r-1} < 134\,n$.

To prove claim C, consider the situation after Player I's $(r-1)$st move. Player II colors 3 vertices and thus induces a partition of the remaining uncolored vertices into connected components $S_1^r, \ldots, S_\ell^r$. Let s be the maximal reduced weight occurring in this partition and denote by ϕ'_{r-1} the associated potential. If $s \leq 17$, then Player I will also keep all components small in his r-th move. Thus $\phi_r - \phi_{r-1} \leq \phi_r < 134\,n$.

Assume, therfore, that $s \geq 18$. Suppose Player I colors k vertices, $1 \leq k \leq 3$, of S_1, say, so that S_1 induces the new components $S_1^r, \ldots, s_\ell^r$. If $s_1 \leq 8$, then the new components are all small. If $s_1 \geq 9$, then claim B says that either exactly 1 large component is created or the net contribution to ϕ'_{r-1} arises at most from small components. Moreover, if exactly 1 large component arises from S_1, then the net contribution to ϕ'_{r-1} comes from small components plus possibly a value bounded by

$$\begin{cases} (1/3) \cdot (4/3)^{s-1} & \text{if } k = 1 \\ (2/3) \cdot (4/3)^{s-1} & \text{if } k = 2 \\ (4/3)^{s-1} & \text{if } k = 3 \end{cases}$$

In other words, we obtain the bound

$$\phi'_{r-1} - \phi_{r-1} < (4/3)^{s-1} + 134\,n.$$

On the other hand, Player I's strategy for carrying out move r yields a decrease of ϕ'_{r-1} of at least

$$(4/3)^s - 2(4/3)^{1+s/2}.$$

Because $s \geq 18$, we observe

$$(4/3)^{s-1} \leq (4/3)^s - 2(4/3)^{1+s/2}.$$

Hence

$$\phi_r - \phi_{r-1} = (\phi_r - \phi'_{r-1}) + (\phi'_{r-1} - \phi_{r-1}) < 134\,n.$$

The relation $\phi_r < 134\,n^2$ now is a direct consequence of claim C. It implies that $s(S) = O(\log n)$ holds for all connected components S of uncolored vertices occurring after any move of Player I, which proves the theorem. ∎

It will follow from the proof of Theorem 3 together with Theorem 4 below that Theorem 1 cannot substantially be improved. If we define

$$\overline{g}(n) = \max\{\overline{\gamma}(T) \mid T \text{ tree on } n \text{ vertices}\},$$

then there is a constant $\overline{c} > 0$ such that $\overline{g}(n) > \overline{c} \log n$ for infinitely many n's.

3. Unions of trees.

We now turn our attention to *k-fold trees*, that is, to graphs that can be obtained as a union of k trees. If G is a union of trees $T_1, \ldots, T_k$, there is no loss in generality when we assume that each tree T_i is a spanning tree of G. To keep our discussion simple, we will only consider 2-fold trees, that is, the case $k = 2$. Note that the usual chromatic number of a 2-fold tree G satisfies $\chi(G) \leq 4$. The situation turns out to be quite different for the game chromatic number $\gamma(G)$.

Theorem 3. *There is a constant* c *such that each 2-fold tree* G *on* n *vertices satisfies* $\gamma(G) \leq c \log n$.

Proof: Assume G is the union of the trees T_1 and T_2. We will bound the game chromatic number of G by the modified game chromatic numbers of T_1 and T_2:

$$\gamma(G) \leq \overline{\gamma}(T_1) + \overline{\gamma}(T_2).$$

To see that this relation holds, compare the situation for Player I at move $r + 2$ with the situation at move r: some "opponent" has colored 3 vertices in the meantime. A winning strategy for Player I can thus consist in playing according to the modified coloring game relative to T_1 if r is even and relative to T_2 if r is odd. ∎

Theorem 4. *There is an infinite class of 2-fold trees* G *satisfying* $\gamma(G) \geq \frac{1}{3} \log_2 n$, *where* n *is the number of vertices of* G.

Proof: We construct a graph G from the complete graph K_t on $t = 2^k$ vertices as follows: we replace each edge of K_t by $2t$ parallel edges and subdivide each edge. It is easy to see that G is a 2-tree. We claim that the game chromatic number satisfies $\gamma(G) \geq k + 1$.

Let $\overline{X}_0$ be the vertex set of K_t and $xy(s)$ be the vertex introduced on the sth edge between x and y by the subdivision. We describe a coloring strategy for Player II which will eventually force one of the players to use a $(k + 1)$st color. This strategy is divided into k rounds; the ith round consists of 2^{k-i} plays.

At the start of the $(i + 1)$st round, $i = 0, \ldots, k - 1$, there will be a subset $\overline{X}_i \subseteq \overline{X}_0$ of 2^{k-i} uncolored vertices, each of which is adjacent to a vertex already colored with color α, for $\alpha = 1, \ldots, i$. Let M_i be a matching of $\overline{X}_i$ in K_t. On his jth play of the ith round, Player II colors an uncolored vertex of the form $xy(s)$ with color $i + 1$, where xy is the jth edge of M_i. Note that such a vertex will always be present, because there have been less than $2t$ plays so far.

At the end of the ith round each of the vertices in $\overline{X}_i$ will be adjacent to a vertex colored $i + 1$ and at least half will be uncolored. Thus, the uncolored vertices of $\overline{X}_i$ will be sufficient to form $\overline{X}_{i+1}$. Clearly, after k rounds, one of the vertices in $\overline{X}_k$ will require the $(k + 1)$st color. ∎

It is straightforward to extend the modified game on a tree to the case where the opponent may color k vertices. With the potential function

$$\phi = \left(\frac{k+1}{k}\right)^{s_1} + \cdots + \left(\frac{k+1}{k}\right)^{s_n}$$

then the analogue of Theorem 2 can be proved. Hence, also the statement of Theorem 3 holds for k-fold trees (k fixed). As an application we are lead to

Corollary 5. *There is a constant* c *such that each planar graph* G *on* n *vertices satisfies*

$$\gamma(G) \le c \log n.$$

Proof: Because each planar graph contains some vertex of degree at most 5, each planar graph is a 5-fold tree. ∎

We do not know whether Corollary 5 is "best possible" in any sense. In fact, we know nothing about the game chromatic number of series-parallel graphs. (Series-parallel graphs are, in particular, planar 2-fold trees.)

4. Interval graphs.

Recall that the graph G is an *interval graph* if G is isomorphic to some graph $G(I)$ where the vertices of $G(I)$ are a set I of intervals of the real line and two distinct intervals $i, k \in I$ are considered adjacent in $G(I)$ if $i \cap k \neq \phi$. It is convenient to think of an interval graph $G = G(I)$ in terms of its interval representation I. There is no loss in generality when we assume that all intervals $i \in I$ have mutually distinct left endpoints $\ell(i)$ and mutally distinct right endpoints $r(i)$. It is well-known that the interval graph G allows a feasible coloring with $\omega(G)$ colors, where $\omega(G)$ denotes the size of the largest clique in G.

Theorem 6. *The game chromatic number* $\gamma(G)$ *of the interval graph* $G = G(I)$ *satisfies*

$$\gamma(G) \le 3\omega(G) - 2.$$

Proof: We give a winning strategy for Player I using $3\omega(G) - 2$ colors. At each turn Player I assigns a feasible color to the unique interval $i \in I$ such that,

(a) if possible, i contains the last interval colored by Player II and
(b) subect to (a) i has the largest right endpoint $r(i)$.

It remains to prove that this strategy works. First we introduce some notation. At a given stage of the game, let C be the set of colored intervals. Define the colored left, right, and middle degree $c\ell d(i)$, $crd(i)$, and $cmd(i)$ by

$$\begin{aligned} c\ell d(i) &= |\{j \in C\backslash\{i\}: \ell(i) \in j\}| \\ crd(i) &= |\{j \in C\backslash\{i\}: r(i) \in j\}| \\ cmd(i) &= |\{j \in C\backslash\{i\}: j \subset i\}|. \end{aligned}$$

Clearly, $c\ell d(i)$ and $crd(i)$ never exceed $\omega(G) - 1$. If $cmd(i) \neq 0$, let

$$w(i) = \max\{\ell(k): k \in I \text{ and } k \subset i\}.$$

The Theorem now follows from the Lemma 7.

Lemma 7. *At the end of every play by Player I, for every uncolored interval i, there are at least $cmd(i)$ colored intervals k such that both $r(i) \in k$ and $w(i) \in k$; hence, $cmd(i) \leq \omega(G) - 2$.*

Proof: We argue by induction on the number plays. After the first play the result holds trivially. So assume the result is true for the first s plays by Player I and consider the $(s + 1)$st play. First note that Player I always colors a maximal interval; thus, $cmd(i)$ does not increase during Player I's turn, for any interval i.

Suppose that $cmd(i)$ increased during the previous play by Player II, for some uncolored interval i. Then Player II colored an interval $j \subset i$. Thus i satisfies condition (a) of Player I's strategy. If Player I colors i, then we are no longer concerned about i; otherwise Player I colors an interval, which contains j, but has $r(j) > r(i)$. This increases the number of colored intervals k such that both $r(i) \in k$ and $w(i) \in k$. ∎

To complete the proof of Theorem 6, note that at the end of any play by Player I, any interval i is adjacent of $d = c\ell d(i) + crd(i) + cmd(i)$ colored intervals. So $d \leq 3\omega(G) - 4$, and at the start of Player I's next turn, $d \leq 3\omega(G) - 3$ and $3\omega(G) - 2$ colors suffice. ∎

We do not know whether the upper bound in Theorem 6 can be improved in general. When proving lower bounds on the game chromatic number of the class of interval graphs, we may assume that Player II plays first on a graph G, by taking two disjoint copies of G. It is easy to show that for each ω, there is an interval graph G with $\omega(G) = \omega$ and

$$\gamma G \geq 2\omega(G) - 2.$$

Indeed, let Player II play first on the graph $K_{\omega-1} + I_{2(\omega-1)}$, where $G + H$ denotes disjoint copies of G and H with all possible edges between the vertices of G and H, and I_n is an independent set on n vertices. On each play Player II colors a vertex of the independent set with an unused color. He can use at least $\omega - 1$ colors before Player I colors all the vertices of $K_{\omega-1}$ with $\omega - 1$ different colors.

Similarly, $I_{2(\omega-1)} + K_{\omega-1} + I_1 + K_{\omega-1} + I_{2(\omega-1)}$ has game chromatic number $2\omega - 1$ when Player II goes first. We mention without going into details that interval graphs can be constructed with game chromatic number 2ω.

Finally, we observe that the following strategy, which we call the *greedy strategy*, is not effective for Player I. When following the greedy strategy, Player I always colors a vertex whose neighborhood has been colored with the maximum number of colors.

Theorem 8. *For every k, there exists an interval graph G such that $\omega(G) = 3$, but Player II can force k colors if Player I uses the greedy strategy.*

Proof: Again, assume that Player II goes first. Let G consist of 2^{k-2} disjoint copies of $K_2 + I_k$ Player II's strategy consists of $k-2$ rounds. At the start of the ith round there are 2^{k-1-i} copies of $K_2 + I_k$ such that none of the points in the K_2 have been colored and exactly $i-1$ of the points in the I_k have been colored, using the colors $1, \ldots, i-1$. Player II completes the round in 2^{k-1-i} plays by coloring one point from each of these I_k with color $i+1$. Player I must respond by completely coloring 2^{k-2-i} of the cliques K_2. Thus, after $k-2$ rounds k colors will have been used. ∎

References

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