The Number of Rooted Maps with a Fixed Number of Vertices

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Abstract. Let $T_g(m,n)$ (respectively, $P_g(m,n)$) be the number of rooted maps, on an orientable (respectively, non-orientable) surface of type g, which have m vertices and n faces. Bender, Canfield and Richmond [3] obtained asymptotic formulas for $T_g(m,n)$ and $P_g(m,n)$ when $\epsilon \le m/n \le 1/\epsilon$ and $m,n \to \infty$. Their formulas can not be extended to the extreme case when m or n is fixed. In this paper, we shall derive asymptotic formulas for $T_g(m,n)$ and $P_g(m,n)$ when m is fixed and derive the distribution for the root face valency. We also show that their generating functions are algebraic functions of a certain form. By the duality, the above results also hold for maps with a fixed number of faces.

1. Introduction.

A *map* is a connected graph G embedded in a surface S in such a way that every component of $S - G$ (called a *face*) is a topological disk. A map is *rooted* by distinguishing an edge, a direction along the edge and a side of the edge. Throughout we use $g = 1 - \chi/2$ to denote the *type* of a surface with Euler characteristic χ. For an orientable surface, g is the same as the genus. (See [2] for more details about type.)

Consider m-vertex rooted maps which have some distinguished faces indexed by a finite set I. Let $\vec{\mathrm{M}}_{g,m}(x,y,z_I)$ be the generating function for such maps on an orientable surface of type g, where x marks the number of faces which are neither the root face nor the distinguished faces, y marks the root face valency and $z_I = \{z_I : i \in I\}$ marks the valencies of the distinguished faces. We similarly define $\tilde{M}_{g,m}(x,y,z_I)$ for non-orientable surfaces, and define $M_{g,m}(x,y,z_I) = \vec{\mathrm{M}}_{g,m}(x,y,z_I) + \tilde{M}_{g,m}(x,y,z_I)$. Let

$$T_g(m,n) = [x^{n-1}]\vec{\mathrm{M}}_{g,m}(x,1,z_\emptyset),\ P_g(m,n) = [x^{n-1}]\tilde{M}_{g,m}(x,1,z_\emptyset);$$
$$T_g^k(m,n) = [x^{n-1}y^k]\vec{\mathrm{M}}_{g,m}(x,y,z_\emptyset),\ P_g^k(m,n) = [x^{n-1}y^k]\tilde{M}_{g,m}(x,y,z_\emptyset).$$

Then $T_g(m,n)$ (respectively, $P_g(m,n)$) is the number of rooted maps, on an orientable (respectively, non-orientable) surface of type g, which have m vertices and n faces. $T_g^k(m,n)$ and $P_g^k(m,n)$ are the number of such maps with root face valency k. Bender, Canfield, and Richmond [3] obtained asymptotic formulas for $T_g(m,n)$ and $P_g(m,n)$ when $\epsilon \le m/n \le 1/\epsilon$ and $m,n \to \infty$. In this paper, we study the extreme case when one of m and n is fixed while the other goes to infinity. By duality, we only need to study the case when m is fixed. The one-vertex maps have been studied in [4, 7, 8, 9], its generating functions (with respect to the number of edges) have been calculated by Canfield [6] for $g \leq 3$. Some results on two-vertex maps have been obtained in [10].

To state our results, we first introduce some notations. Throughout this paper, R denotes $\sqrt{1-4x}$. If $\mathbf{u}_I$ is a set of indeterminates indexed by a finite set I, then $P(R;\mathbf{u}_I)$ denote a polynomial of R and $\mathbf{u}_I$ with rational coefficients and $\mathcal{Q}(R;\mathbf{u}_I)$ denote the ring whose elements are of the form

$$R^{-a}(1+R)^{-b}\prod_{i\in I}\left(1-(1-R)u_i/2\right)^{-c_i}P(R;\mathbf{u}_I),$$

where a, b, and c_i, are non-negative integers. When $I=\emptyset$, we simply denote them, respectively, by $P(R)$ and $\mathcal{Q}(R)$. We shall prove the following results.

Theorem 1. $\vec{M}_{g,m}(x,y,\mathbf{z}_I), \tilde{M}_{g,m}(x,y,\mathbf{z}_I) \in \mathcal{Q}(R;y,\mathbf{z}_I)$. *Therefore, they are algebraic functions.*

Corollary 1. $\vec{M}_{g,m}(x,1,\mathbf{z}_\emptyset)$ *and* $\tilde{M}_{g,m}(x,1,\mathbf{z}_\emptyset)$ *are of the form* $R^{-a}(1+R)^{-b}P(R)$ *for some non-negative integers* a *and* b *and polynomial* P.

Theorem 2. *For any* $\epsilon>0$, *let* k *depend on* n *such that* $k/n\in[\epsilon,1-\epsilon]$ *and* $y=2+k/(n-k)$. *Then there are positive numbers* $t(g,m;y)$ *and* $p(g,m;y)$ *such that*

$$T_g^k(m,n) \sim t(g,m;y)\,n^{4g+2m-5/2}e^{y^{-k}}\left(\frac{y^2}{y-1}\right)^n,$$
$$P_g^k(m,n) \sim p(g,m;y)\,n^{4g+2m-5/2}e^{y^{-k}}\left(\frac{y^2}{y-1}\right)^n,$$

uniformly for all such k *as* $n\to\infty$.

Theorem 3. *There are positive contants* $t(g,m)$ *and* $p(g,m)$ *such that*

$$T_g(m,n) \sim t(g,m)\,n^{3(2g+m-2)/2}4^n,$$
$$P_g(m,n) \sim p(g,m)\,n^{3(2g+m-2)/2}4^n,$$

as $n\to\infty$.

To avoid considerable repetitions, we shall rely heavily on [2]. Refer to [2] for those notations and terminologies not defined here.

2. Functional equations.

For convenience, we shall use $M_{g,m}(x,y,I)$ to denote $M_{g,m}(x,y,z_I)$, etc., throughout this section. Using the argument similar to that used in [2], we obtain that for $w \notin I$,

$$
\begin{aligned}
&M_{g,m}(x,y,I)\\
&= y^2 \sum_{\ell=1}^{m-1} \sum_{(j,S)=(0/2,\emptyset)}^{(g,m)} M_{j,\ell}(x,y,S)\, M_{g-j,m-\ell}(x,y,I-S)\\
&+ 2y^3 \frac{\partial}{\partial z_w} M_{g-1,m}(x,y,I+\{w\}) \,|_{z_w=y}\\
&+ y^2 \frac{\partial}{\partial y}\left(y M_{g-1/2,m}(x,y,I)\right)\\
&+ \sum_{i\in I} \frac{y z_I}{z_I - y}\left[z_I M_{g,m}(x,z_I,I-\{i\}) - y M_{g,m}(x,y,I-\{i\})\right]\\
&+ \frac{xy}{y-1}\left(M_{g,m}(x,y,I) - M_{g,m}(x,1,I)\right) + \delta_{m,1}\delta_{g,0}\delta_{I,\emptyset},
\end{aligned}
$$

and

$$
\begin{aligned}
&\vec{\mathrm{M}}_{g,m}(x,y,I)\\
&= y^2 \sum_{\ell=1}^{m-1} \sum_{(j,S)=(0/2,\emptyset)}^{(g,m)} \vec{\mathrm{M}}_{j,\ell}(x,y,S)\, \vec{\mathrm{M}}_{g-j,m-\ell}(x,y,I-S)\\
&+ y^3 \frac{\partial}{\partial z_w} \vec{\mathrm{M}}_{g-1,m}(x,y,I+\{w\}) \,|_{z_w=y}\\
&+ \sum_{i\in I} \frac{y z_I}{z_I - y}\left[z_I \vec{\mathrm{M}}_{g,m}(x,z_I,I-\{i\}) - y \vec{\mathrm{M}}_{g,m}(x,y,I-\{i\})\right]\\
&+ \frac{xy}{y-1}\left(\vec{\mathrm{M}}_{g,m}(x,y,I) - \vec{\mathrm{M}}_{g,m}(x,1,I)\right) + \delta_{m,1}\delta_{g,0}\delta_{I,\emptyset},
\end{aligned}
$$

Multiplying by $1-y$ and rearranging terms, we can rewrite the above recursions as

$$
\begin{aligned}
&A(x,y)\, M_{g,m}(x,y,I) \qquad (1)\\
&= y^2(1-y) \sum_{\ell=1}^{m-1} \sum_{(j,S)=(0/2,\emptyset)}^{(g,m)} M_{j,\ell}(x,y,S)\, M_{g-j,m-\ell}(x,y,I-S)\\
&+ 2y^3(1-y) \frac{\partial}{\partial z_w} M_{g-1,m}(x,y,I+\{w\}) \,|_{z_w=y}\\
&+ y^2(1-y) \frac{\partial}{\partial y}\left(y M_{g-1/2,m}(x,y,I)\right)
\end{aligned}
$$

$$+ (1-y)\sum_{i\in I}\frac{yz_I}{z_I-y}\left[z_I M_{g,m}(x,z_I,I-\{i\}) - yM_{g,m}(x,y,I-\{i\})\right]$$
$$+ xy\left(M_{g,m}(x,1,I)\right) + (1-y)\delta_{m,1}\delta_{g,0}\delta_{I,\emptyset},$$

and

$$A(x,y)\vec{\mathrm{M}}_{g,m}(x,y,I) \tag{2}$$
$$= y^2(1-y)\sum_{\ell=1}^{m-1}\sum_{(j,S)=(0/2,\emptyset)}^{(g,m)}\vec{\mathrm{M}}_{j,\ell}(x,y,S)\vec{\mathrm{M}}_{g-j,m-\ell}(x,y,I-S)$$
$$+ y^3(1-y)\frac{\partial}{\partial z_w}\vec{\mathrm{M}}_{g-1,m}(x,y,I+\{w\})\,|_{z_w=y}$$
$$+ (1-y)\sum_{i\in I}\frac{yz_I}{z_I-y}\left[z_I\vec{\mathrm{M}}_{g,m}(x,z_I,I-\{i\}) - y\vec{\mathrm{M}}_{g,m}(x,y,I-\{i\})\right]$$
$$+ xy\left(\vec{\mathrm{M}}_{g,m}(x,y,I)\right) + (1-y)\delta_{m,1}\delta_{g,0}\delta_{I,\emptyset},$$

where

$$A(x,y) = xy^2 + 1 - y. \tag{3}$$

3.

Proof of Theorem 1: Since $A(x,y) = 0$ has a unique power series solution

$$y = f(x) = \frac{2}{1+R},$$

Equations (1) and (2) determine $M_{g,m}(x,y,I)$ and $\vec{\mathrm{M}}_{g,m}(x,y,I)$ recursively in lexicographic order of $(g,m,|I|)$, where $|I|$ is the cardinality of I. Setting $g=0$, $m=1$, $I=\emptyset$ and $y=f$ in (2), we obtain

$$\vec{\mathrm{M}}_{0,1}(x,1,\emptyset) = \frac{2}{1+R}.$$

Substituting it into (2), we obtain

$$\vec{\mathrm{M}}_{0,1}(x,y,\emptyset) = \frac{1}{1-(1-R)y/2}. \tag{4}$$

Thus, Theorem 1 holds for $(g,m,|I|) = (0,1,0)$. Suppose that Theorem 1 holds for $(j,\ell,|S|) < (g,m,|I|)$ with respect to lexicographic order. Then

$$\vec{\mathrm{M}}_{j,\ell}(x,y,S) \in \mathcal{Q}(R;y,z_S) \text{ for all } (j,\ell|S|) < (g,m,|I|).$$

Setting $y = f$ in (2), we have $\vec{M}_{g,m}(x, 1, I) \in \mathcal{Q}(R; \mathbf{Z}_I)$; substituting it into (2) and cancelling out the factor $1 - (1 + R)y/2$, we have

$$\vec{M}_{g,m}(x, y, I) \in \mathcal{Q}(R; y, z_I).$$

Similarly, we can show that $M_{g,m}(x, y, I) \in \mathcal{Q}(R; y, z_I)$. Therefore,

$$\vec{M}_{g,m}(x, y, I) = M_{g,m}(x, y, I) - \vec{M}_{g,m}(x, y, I) \in \mathcal{Q}(R; y, z_I)$$

and, thereby, establishes Theorem 1.

By carrying out the first few calculations, (with the assistance of the symbolic manipulation system Maple) we obtain

$$\begin{aligned}
\vec{M}_{0,2}(x, 1, \emptyset) &= \frac{2}{R^2(1+R)}, \\
\vec{M}_{0,3}(x, 1, \emptyset) &= \frac{4}{R^5(1+R)}, \\
\vec{M}_{0,4}(x, 1, \emptyset) &= \frac{2(8 - 2R - R^2)}{R^8(1+R)}, \\
\vec{M}_{0,5}(x, 1, \emptyset) &= \frac{4(21 - 11R - 4R^2 + R^3)}{R^{11}(1+R)}, \\
\bar{M}_{1/2,1}(x, 1, \emptyset) &= \frac{2}{R^2(1+R)}, \\
\bar{M}_{1/2,2}(x, 1, \emptyset) &= \frac{9+R}{R^5(1+R)}, \\
\bar{M}_{1/2,3}(x, 1, \emptyset) &= \frac{59 - 6R - 9R^2}{R^8(1+R)}, \\
\bar{M}_{1/2,4}(x, 1, \emptyset) &= \frac{1773 - 627R - 469R^2 + 67R^3}{4R^{11}(1+R)}, \\
\bar{M}_{1/2,5}(x, 1, \emptyset) &= \frac{14325 - 8874R - 4436R^2 + 1954R^3 + 119R^4}{4R^{14}(1+R)}, \\
\vec{M}_{1,1}(x, 1, \emptyset) &= \frac{1}{R^5},
\end{aligned}$$

$$\tilde{M}_{1,1}(x,1,\emptyset) = \frac{2(3+R)}{R^5(1+R)},$$

$$\vec{\mathrm{M}}_{1,2}(x,1,\emptyset) = \frac{14+9R-3R^2}{R^8(1+R)},$$

$$\tilde{M}_{1,2}(x,1,\emptyset) = \frac{3(29+4R-5R^2)}{R^8(1+R)},$$

$$\vec{\mathrm{M}}_{1,3}(x,1,\emptyset) = \frac{2(83+27R-38R^2-2R^3)}{R^{11}(1+R)},$$

$$\tilde{M}_{1,3}(x,1,\emptyset) = \frac{1059-109R-357R^2+15R^3}{R^{11}(1+R)},$$

$$\vec{\mathrm{M}}_{1,4}(x,1,\emptyset) = \frac{1864+38R-1200R^2+75R^3+63R^4}{R^{14}(1+R)},$$

$$\tilde{M}_{1,4}(x,1,\emptyset) = \frac{48567-17532R-21706R^2+4700R^3+931R^4}{4R^{14}(1+R)},$$

$$\vec{\mathrm{M}}_{1,5}(x,1,\emptyset) = \frac{2(10203-2845R-7715R^2+1953R^3+807R^4-93R^5)}{R^{17}(1+R)},$$

$$\tilde{M}_{1,5}(x,1,\emptyset) = \frac{3(180303-113193R-88092R^2+43872R^3+6589R^4-1611R^5)}{4R^{17}(1+R)},$$

$$\vec{\mathrm{M}}_{0,2}(x,y,\emptyset) = \frac{2y(y(R^2+4R-1)-2R+2)}{R^2(2-y+yR)^3},$$

$$\vec{\mathrm{M}}_{0,3}(x,y,\emptyset) = \frac{4y\vec{\mathrm{P}}_{0,3}(R,y)}{R^5(2-y+yR)^5},$$

$$\tilde{M}_{1/2,1}(x,y,\emptyset) = \frac{2y(y(R^2+4R-1)-2R+2)}{R^2(2-y+yR)^3},$$

$$\tilde{M}_{1/2,2}(x,y,\emptyset) = \frac{y\tilde{P}_{1/2,2}(R,y)}{R^5(2-y+yR)^5},$$

$$\vec{\mathrm{M}}_{1,1}(x,y,\emptyset) = \frac{(1+R)y\vec{\mathrm{P}}_{1,1}(R,y)}{R^5(2-y+yR)^5},$$

$$\tilde{M}_{1,1}(x,y,\emptyset) = \frac{2y\tilde{P}_{1,1}(R,y)}{R^5(2-y+yR)^5},$$

Where

$$\begin{aligned}\vec{\mathrm{P}}_{0,3}(R,y) = {} & y^3(9R^4+18R^3-16R^2+6R-1)\\ & + y^2(-8R^4-14R^3+42R^2-26R+6)\\ & + y(-20R^2+32R-12)+(8-8R),\end{aligned}$$

$$
\begin{aligned}
\tilde{P}_{1/2,2}(R,y) &= y^3(R^5+99R^4+154R^3-138R^2+53R-9)\\
&\quad + y^2(-86R^4-92R^3+352R^2-228R+54)\\
&\quad + y(-20R^3-148R^2+276R-108)-8R^2-64R+72,\\
\vec{\mathrm{P}}_{1,1}(R,y) &= y^3(R^4+26R^3-16R^2+6R-1)\\
&\quad + y^2(-22R^3+42R^2-26R+6)\\
&\quad + y(-20R^2+32R-12)-8R+8,\\
\tilde{P}_{1,1}(R,y) &= y^3(R^5+45R^4+46R^3-42R^2+17R-3)\\
&\quad + y^2(-38R^4-8R^3+100R^2-72R+18)\\
&\quad + y(-20R^3-28R^2+84R-36)\\
&\quad + (-8R^2-16R+24).
\end{aligned}
$$

Our results on one-vertex maps are independent verifications of some of the results given in [6]. (Note that the generating functions given in [6] are by the number of edges, thus, differ from ours by a factor $x^{2g} = (1-R^2)^{2g}/4^{2g}$.) The above results also suggest that $\vec{\mathrm{M}}_{g,m}(x,y,\emptyset)$ and $\tilde{M}_{g,m}(x,y,\emptyset)$ have no factor $(1+R)$ in their denominators and that $\vec{\mathrm{M}}_{g,m}(x,1,\emptyset)$ and $\tilde{M}_{g,m}(x,1,\emptyset)$ only have $(1+R)$ to the first power in their denominators. This could probably be proved by using a more delicate inductive argument similar to the one used above. Another interesting fact is that $\vec{\mathrm{M}}_{0,2}(x,y,\emptyset)$ equals $\tilde{M}_{1/2,1}(x,y,\emptyset)$ which can be proved directly by the following combinatorial argument: For any rooted two-vertex planar map, add a cross-cap in a face which is incident to both vertices (say, the first such face in the cyclic order around the root vertex) and identify the two vertices through the cross-cap, the resulting map is a rooted one-vertex map on the projective plane. Clearly, this process is reversible.

4.

Proof of Theorem 2 and Theorem 3: Let $\boldsymbol{\alpha}$ be a vector of positive integers indexed by I and let $|\boldsymbol{\alpha}| = \sum \alpha_i$. As in [2], we define

$$
\begin{aligned}
\vec{\mathrm{M}}^{(n)}_{g,m}(x,y,I,\boldsymbol{\alpha}) &= \frac{\partial^{n+|\boldsymbol{\alpha}|}}{\partial y^n \prod \partial z_I^{\alpha_i}} \vec{\mathrm{M}}_{g,m}(x,y,z_I)\,|_{z_I=y},\\
\vec{\mathrm{M}}^{(n)}_{g,m}(x,I,\boldsymbol{\alpha}) &= \frac{\partial^{n+|\boldsymbol{\alpha}|}}{\partial y^n \prod \partial z_I^{\alpha_i}} \vec{\mathrm{M}}_{g,m}(x,y,z_I)\,|_{z_I=y=f}.
\end{aligned}
$$

Similarly define $\tilde{M}^{(n)}_{g,m}(x,y,I,\boldsymbol{\alpha})$, $\tilde{M}^{(n)}_{g,m}(x,I,\boldsymbol{\alpha})$, $M^{(n)}_{g,m}(x,y,I,\boldsymbol{\alpha})$ and $M^{(n)}_{g,m}(x,I,\boldsymbol{\alpha})$. Let $\approx$ be as defined as in [2]. We first establish the following results.

Lemma 1. *For $a = 4g + 2m + 2|I| + |\alpha| + n - 1$ and $y > 2$, there are positive numbers $\vec{M}^{(n)}_{g,m}(I, \alpha; y)$ and $\tilde{M}^{(n)}_{g,m}(I, \alpha; y)$ such that*

$$\vec{M}^{(n)}_{g,m}(x, y, I, \alpha) \approx \vec{M}^{(n)}_{g,m}(I, \alpha; y)(1 - x/r(y))^{-a},$$
$$\tilde{M}^{(n)}_{g,m}(x, y, I, \alpha) \approx \tilde{M}^{(n)}_{g,m}(I, \alpha; y)(1 - x/r(y))^{-a},$$

as $x \to r(y) = (y-1)/y^2$.

Lemma 2. *For $b = (6g + 3m + 3|I| + |\alpha| + n - 2)/2$, there are positive numbers $\vec{M}^{(n)}_{g,m}(I, \alpha)$ and $\tilde{M}^{(n)}_{g,m}(I, \alpha)$ such that*

$$\vec{M}^{(n)}_{g,m}(x, I, \alpha) \approx \vec{M}^{(n)}_{g,m}(I, \alpha)(1 - 4x)^{-b},$$
$$\tilde{M}^{(n)}_{g,m}(x, I, \alpha) \approx \tilde{M}^{(n)}_{g,m}(I, \alpha)(1 - 4x)^{-b},$$

as $x \to 1/4$.

Proof: (Lemma 1) The proof is very similar to that of [2, Theorem 3], by using induction on the lexicographic order of $(g, m, |I|, n)$. By Theorem 1, for any $y > 2$, the smallest positive singularity of $\vec{M}_{g,m}(x, y, I, \alpha)$ is the solution to $1 - (1-R)y/2 = 0$, that is, $r(y) = (y-1)/y^2$. (Clearly, $0 < r(y) < 1/4$ for $y > 2$.) From (4), we obtain

$$\vec{M}^{(n)}_{0,1}(x, y, \emptyset, 0) = n! \left(\frac{1-R}{2}\right)^n \left(1 - \frac{1-R}{2}y\right)^{-(n+1)}. \tag{5}$$

Thus, Lemma 1 holds for $g = 0$, $m = 1$ and $I = \emptyset$. The rest of the proof is essentially the same as that of [2, Theorem 3]. ∎

Thc proof of Lemma 2 is essentially the same as that of [2, Theorem 3], while the initial case can be verified from (5). Theorem 3 now follows immediately from Lemma 2 and [1, Theorem 4] by setting $y = f$ in (1) and (2). (*c.f.* [2, Section 6])

We now use a local limit theorem to prove Theorem 2. We have

$$-\frac{d}{ds}\log r(e^s) = \frac{y-2}{y-1}, -\frac{d^2}{ds^2}\log r(e^s) = \frac{y}{(y-1)^2},$$

and

$$|r(ye^{i\theta})| = \frac{1}{y}\left|1 - \frac{1}{y}e^{-i\theta}\right| = r(y)\sqrt{1 + \frac{2y}{(y-1)^2}(1 - \cos\theta)}.$$

Therefore, for any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|r(ye^{i\theta})| \ge r(y)(1 + \delta_1) \text{ for } 2 + \epsilon \le y \le 1/\epsilon \text{ and } \delta_2 \le |\theta| \le \pi.$$

Theorem 2 now follows from Lemma 1, [1, Theorem 4] and [5, Corollary 2].

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