On packings of pairs by quintuples: $v \equiv 3, 9$ or $17 \pmod{20}$

R.C. Mullin
Dept. of Combinatorics and Optimization
University of Waterloo,
Waterloo, Ontario
Canada N2L 3G1, Canada

J. Yin
Dept. of Mathematics of Suzhou University
Suzhou, 215006
P.R. of China

1. Introduction

A (v, k, λ) packing design (briefly packing) is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a v-set, $\mathcal{B}$ is a collection of some k-subsets (called blocks) of $\mathcal{X}$ such that every pair $\{x, y\} \subset \mathcal{X}$ is contained in at most λ blocks of $\mathcal{B}$. The packing number $D(v, k, \lambda)$ is defined to be the maximum number of blocks in a (v, k, λ) packing. A (v, k, λ) packing with $D(v, k, \lambda)$ blocks will be called a maximum packing.

The function $D(v, k, 1)$ is of importance in coding theory since the block incidence vectors of a $(v, k, 1)$ packing form the codewords of a binary code of length v minimum distance $2(k-1)$ and constant weight k. Thus $D(v, k, 1)$ is the maximum number of codewords in such a code.

Schoenheim [20] has shown that

$$D(v, k, \lambda) \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{k-1} \right\rfloor \right\rfloor = B(v, k, \lambda) \tag{1.1}$$

where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x$.

Other upper bounds on the function $D(v, k, 1)$ have been given by Johnson [14] and Best et al. [3]. Lower bounds on the function $D(v, k, \lambda)$ are generally given by construction of (v, k, λ) packings.

The values of $D(v, 3, \lambda)$ for all v and λ have been determined by Shoenheim [20], and Hanani [12]. The values of $D(v, 4, 1)$ have been determined for all v by Brouwer [6] and the values of $D(v, 4, \lambda)$ for all v and $\lambda > 1$ are given by Billington, Stanton and Stinson [4], and Assaf [1], Hartman [13]. Yin [22], [23] has determined the values of $D(v, 5, 2)$ for all v with 11 possible exceptions of v. The values of $D(v, 5, 4)$ for all v are determined by Assaf and Hartman [2]. Recently, an analysis of $D(v, 5, \lambda)$ for all v and $\lambda \equiv 0 \pmod{4}$ was done by Yin [16]. The function of $D(v, 5, 1)$ for $v \equiv 0 \pmod{4}$ has been investigated in [17] by Yin.

In this paper we are concerned about the packing number $D(v, 5, 1)$. The values of $D(v, 5, 1)$ for all $v \equiv 3, 9$ or $17 \pmod{20}$ except $v \in \{29, 49, 243\}$ will be determined. Some infinite families for $D(v, 5, 1)$ with $v \equiv 7, 11$ or $15 \pmod{20}$ are also mentioned. For ease of notation, we write $D(v)$ and $B(v)$ for $D(v, 5, 1)$ and $B(v, 5, 1)$ respectively.

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2. Preliminaries

For definitions of incomplete PBD and incomplete GDD see [11]. By $(v, w; K, \lambda)$-IPBD we mean an incomplete PBD of order v, block sizes from K, hole size ω, and index λ. We say that an incomplete GDD $(\mathcal{X}, Y, \mathcal{G}, \mathcal{A})$ of index λ is a (K, λ)-IGDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The type of the IGDD is defined to be the multi-set of ordered pairs $\{(|G|, |G \cap Y|) : G \in \mathcal{G}\}$. We shall use the 'exponential' notation as in [11]. A $(\{k\}, 1)$-IGDD of type $(n, \omega)^k$ is denoted by $\mathrm{TD}(k, n) - \mathrm{TD}(k, \omega)$. When $Y = \phi$, a (K, λ)-IGDD is essentially a (K, λ)-GDD. A resolvable $(\{k\}, 1)$-GDD of type $(k-1)^s$ is also known as a nearly Kirkman system and denoted NKS $(2, k; s(k-1))$.

We now list some of those results which will be used in this paper.

Lemma 2.1. ([12]) *If* $v \equiv 1$ *or* $5 \pmod{20}$ *and* $v \geq 5$, *then there is a* $(v, 5, 1)$*-BIBD.*

Lemma 2.2. ([11] *If* $v \equiv 9$ *or* $17 \pmod{20}$ *and* $v \geq 37, v \neq 49$, *then there is a* $(v, 9; \{5\}, 1)$*-IPBD.*

Lemma 2.3. ([18], [19]) *If* $v \geq 24$, $v \notin E$ *and* $v \equiv 0 \pmod{12}$, *then there exists an NKS* $(2, 4; v)$, *where* $E = \{84, 132, 264, 372, 456, 552, 660, 804, 852, 6312\}$.

Lemma 2.4. ([11]) *There exists a* $\mathrm{TD}(5, n)$ *if* $n \geq 4$ *and* $n \neq 6, 10$. *There exists a* $\mathrm{TD}(6, n)$ *if* $n \geq 5$ *and* $n \neq 6, 10, 14, 18, 22, 26, 30, 34, 42$.

Lemma 2.5. ([8]) *If* $\mathrm{TD}(6, t)$ *and* $\mathrm{TD}(5, m + m_j) - \mathrm{TD}(5, m_j)$ *(for* $j = 1, 2, \ldots, t$*) all exist, then also a* $TD(5, mt + \sum_{1 \leq j \leq t} m_j) - \mathrm{TD}(5, \sum_{1 \leq j \leq t} m_j)$ *exists.*

Lemma 2.6. *([10]) There exists a* $TD(5, 10) - \mathrm{TD}(5, 2)$.

As a consequence of Lemmas 2.4–2.6, we have

Lemma 2.7. *There exists a* $(\{5\}), 1)$*-IGDD of type* $(72 + 15, 15)^5$ *or* $(72 + 5, 5)^5$.

Proof: Use Lemma 2.5 with $t = 9$, noting that there exists a $\mathrm{TD}(5, 8) - \mathrm{TD}(5, 0)$, a $\mathrm{TD}(5, 9) - \mathrm{TD}(5, 1)$, and a $\mathrm{TD}(5, 10) - \mathrm{TD}(5, 2)$. ∎

Lemma 2.8. ([5]) *Let* q *be a prime power. Then there exists a* $(q^3 + 1, q + 1, 1)$*-RBIBD.*

Lemma 2.9. ([15]) *Let* q *be a prime power. Then there exists a* $(q^3 + q^2 + q + 1, q + 1, 1)$*-RBIBD.*

Now we give some families of GDD or IGDD.

Lemma 2.10. *Let n be a positive integer and $n \neq 1, 7, 9$ or 10. Then there exists a* $(\{5\}), 1)$*-GDD of type* $(24)^{5n}(4u)^1$, *where* $0 \leq u \leq 6$.

Proof: For these values of n except $n = 4$ or 5, an RTD$(6, 5n+1)$ exists by [7]. Taking a parallel class of blocks in an RTD$(6, 5n+1)$ as groups we obtain a $(\{5n+1, 6\}, 1)$-GDD of type 6^{5n+1}. When $n = 4$ or 5, we have a $(30n+6, 6, 1)$-RBIBD from taking $q = 5$ in Lemmas 2.8 and 2.9. Therefore a $(\{5n+1, 6\}, 1)$-GDD of type 6^{5n+1} also exists for $n = 4$ or 5. Delete $6 - u$ points from one group in a $(\{5n+1, 6\}, 1)$-GDD of type 6^{5n+1}. Give each point of the resulting design a weight of 4. Apply the Fundamental Construction (see [21]). This produces the required result. All GDDs required as ingredients come from Lemma 2.1. ∎

Lemma 2.11. *Suppose there exists a* TD$(6, n)$, *and* $0 \leq u \leq n$. *Then the following designs exist:*

(1) *a* $(\{5\}, 1)$*-IGDD of type* $(4n, 4)^5(4u, 0)^1$ *or* $(4n, 4)^5(4u, 4)^1$;
(2) *a* $(\{5\}), 1)$*-IGDD of type* $(8n, 8)^5(8u, 0)^1$ *or* $(8n, 8)^5(8u, 8)^1$.

Proof: Delete $n - u$ points from one group of a TD$(6, n)$ to yield a $(\{5, 6\}, 1)$-GDD of type n^5u^1. Remove one block of size 5 or 6 from the above GDD. It is shown in [9] that a $(\{5\}, 1)$-GDD of type 8^6 exists. Hence we can use the Fundamental Construction to get (1) by giving points of the resulting design a weight of 4 and to get (2) by giving the points a weight of 8. ∎

In analogy with Lemma 2.11, we have

Lemma 2.12. *Suppose that there exists a* TD$(6, n)$, *and* $0 \leq u \leq n$. *Then the following designs exist:*

(1) *a* $(\{5\}, 1)$*-GDD of type* $(4n)^5(4u)^1$*; and*
(2) *a* $(\{5\}, 1)$*-GDD of type* $(8n)^5(8u)^1$.

3. Maximum incomplete packing designs and their construction

The concept of a maximum incomplete packing design (MIPD) has been used by Yin in [22] to determine packing numbers $D(v, 5, 2)$. For simplicity, we shall not state the most general form, but only the special case required to meet the paper.

Let v and w be non-negative integers. A maximum incomplete packing design, denoted by (v, w)-MIPD, is defined to be a triple $(\mathcal{X}, Y, \mathcal{B})$ where $\mathcal{X}$ is a v-set, $Y \subset \mathcal{X}$ is a w-set, $\mathcal{B}$ is a collection of $B(v) \backslash B(w)$ 5-subsets (called blocks) of $\mathcal{X}$ which has the following properties:

(1) each pair of distinct points x and y from $\mathcal{X}$, where at least one of x and y does not lie in Y, occurs in at most one block of $\mathcal{B}$;
(2) no block contains any pair of Y;
(3) there are exactly $v - w$ pairs of $(X \backslash Y) \times (X \backslash Y)$ blocks of B;
(4) $w \equiv v \equiv 3 (\mathrm{mod}\, 4)$.

We adopt the convention that $B(w) = 0$ for $w < 5$, and we admit $Y = \phi$. The set Y is referred to as the hole of the design.

The following two lemmas are straightforward.

Lemma 3.1. *If $D(w) = B(w)$ and a (v, w)-MIPD exists, then $D(v) = B(v)$.*

Lemma 3.2. *If (v, w)-MIPD and (w, u)-MIPD both exist, then a (v, u)-MIPD also exists.*

The significance of MIPDs defined as above is that the known techniques used in construction of IPBD work also for them. Especially, we have the following constructions.

Construction 3.3. *Let $q \geq 0$. Suppose that the following designs exist:*

(1) *a $(\{5\}, 1)$-IGDD of type $\{(t_1, u_1)(t_2, u_2), \dots, (t_n, u_n)\}$, and*
(2) *a $(t_i + q, u_i + q)$-MIPD for $1 \leq i \leq n$.*

Then there exists a $(t + q, u + q)$-MIPD where $t = \sum t_i$ and $u = \sum u_i$.

Construction 3.4. *Suppose that the following designs exist:*

(1) *a $(\{5\}, 1)$-GDD of type $\{t_1, t_2, \dots, t_n\}$; and*
(2) *a $(t_i + q, q)$-MIPD for $1 \leq i \leq n - 1$.*

Then there exists a $(t + q, t_n + q)$-MIPD where $t = \sum t_i$.

As an immediate corollary of Construction 3.4 and Lemma 2.12, we have

Lemma 3.5. *Suppose that there exists a* TD$(6, t)$, *and $0 \leq u \leq t$. Then*

(1) *a $(20t + 4u + q, 4u + q)$-MIPD exists if a $(4t + q, q)$-MIPD exists; and*
(2) *a $(40t + 8u + q, 8u + q)$-MIPD exists if a $(8t + q, q)$-MIPD exists.*

Finally, we note the following results for MIPDs.

Lemma 3.6. *If $s \equiv 0 \pmod 4$, $s \geq 8$ and $v = 3s \notin E$, then there exists a $(4s - 1, s - 1)$-MIPD where E is the same as in Lemma 2.3.*

Proof: It was pointed out in Lemma 2.3 that an NKS$(2,4;v)$ exists for each $v = 3s$. Adjoin new points to $(\frac{v}{3} - 1)$ parallel classes of a NKS$(2,4;v)$. This produces a $(\{5\}, 1)$-GDD of type $3^s(s-1)^1$. The collection of blocks this GDD form a $(4s - 1, s - 1)$-MIPD. ∎

Lemma 3.7. *If $s \equiv 0 \pmod 4$, $s \geq 12$ and $3s \notin E$, then an*

(1) *$(15gs + (s - 1), s - 1)$-MIPD,*
(2) *$(15gs + (s - 1), 4s - 1)$-MIPD,*
(3) *$(15gs + (4s - 1), s - 1)$-MIPD and*
(4) *$(15gs + (4s - 1), 4s - 1)$-MIPD*

all exist where E is the same as above and g is a positive integer.

Proof: For these values of s, a TD$(5, \frac{3s}{4})$ exists from Lemma 2.4. By Lemma 2.1 we have also a $(\{5\}, 1)$-GDD of type 4^{5g} or 4^{5g+1} for each positive integer g. Give points of such a GDD weight $\frac{3s}{4}$. The Fundamental Construction guarantees that a $(\{5\}, 1)$-GDD of type $(3s)^{5g}$ or $(3s)^{5g+1}$ exists. Apply Construction 3.4 with $n = 5g$ and $5g + 1$ respectively $t_1 = t_2 = \ldots = t_n = 3s$ and $q = s - 1$ and Lemma 3.6 to obtain a $(15gs + (s - 1), 4s - 1)$-MIPD and a $(15gs+(4s-1), 4s-1)$-MIPD respectively. And hence a $(15gs+(s-1)$-MIPD and a $(15gs+(4s-1), s-1)$-MIPD all exist by Lemma 3.2 and Lemma 3.6. ∎

4. Packing numbers $D(v)$ for $v \equiv 3 \pmod{20}$

Let MIPD$(w) = \{v : \text{a } (v, w)\text{-MIPD exists}\}$.

Lemma 4.1. *If $v \in \{3, 23, 43, 63, 83, 103, 123, 143, 163, 183\}$, then $D(v) = B(v)$.*

Proof: For $v = 3$, there is nothing to do. For the other values of v, we construct directly a $(v, 5, 1)$-packing with $B(v)$ blocks as follows, and then the conclusion follows from (1.1).

$v = 23$	0	1	4	6	13	(mod 23)
$v = 43$	0	1	5	13	15	(mod 43)
	0	3	9	20	27	(mod 43)
$v = 63$	0	1	7	36	55	(mod 63)
	0	2	12	23	26	(mod 63)
	0	4	17	22	47	(mod 63)
$v = 83$	0	1	9	29	69	(mod 83)
	0	2	12	18	39	(mod 83)
	0	3	22	48	53	(mod 83)
	0	4	11	36	70	(mod 83)
$v = 103$	0	1	17	64	74	(mod 103)
	0	2	23	45	78	(mod 103)
	0	3	34	62	71	(mod 103)
	0	4	18	42	54	(mod 103)
	0	5	11	88	95	(mod 103)

$v = 103$	0	1	17	64	74	(mod 103)
	0	2	23	45	78	(mod 103)
	0	3	34	62	71	(mod 103)
	0	4	18	42	54	(mod 103)
	0	5	11	88	95	(mod 103)

$v = 123$	0	1	8	76	99	(mod 123)
	0	2	11	60	95	(mod 123)
	0	3	15	44	81	(mod 123)
	0	4	20	71	89	(mod 123)
	0	5	26	36	106	(mod 123)
	0	6	33	19	83	(mod 123)
$v = 143$	0	1	9	20	43	(mod 143)
	0	2	12	26	90	(mod 143)
	0	3	16	57	96	(mod 143)
	0	4	21	49	87	(mod 143)
	0	5	27	75	112	(mod 143)
	0	6	35	67	97	(mod 143)
	0	7	25	40	99	(mod 143)
$v = 163$	0	1	10	25	59	(mod 163)
	0	2	13	87	103	(mod 163)
	0	3	17	70	124	(mod 163)
	0	4	22	69	137	(mod 163)
	0	5	28	55	130	(mod 163)
	0	6	35	72	118	(mod 163)
	0	7	43	64	84	(mod 163)
	0	8	52	40	71	(mod 163)
$v = 183$	0	1	11	27	125	(mod 183)
	0	2	14	42	64	(mod 183)
	0	3	18	104	152	(mod 183)
	0	4	23	55	116	(mod 183)
	0	5	29	46	123	(mod 183)
	0	6	36	74	117	(mod 183)
	0	7	44	83	170	(mod 183)
	0	8	53	88	158	(mod 183)
	0	9	63	84	136	(mod 183)

Lemma 4.2. *Suppose that m is an non-negative integers and $q = 7, 23$ or 31. Then $120m + q \in MIPD(q)$.*

Proof: Taking $s = 8$ in Lemma 3.6 yields $31 \in$ MIPD(7). So, when $m \neq 1, 7, 9$ or 10, the conclusion follows from Construction 3.4 and Lemma 2.10. For $m = 1$ and $q = 7$, note that by Lemma 2.4 a TD(5, 24) exists. This may be viewed as a $(\{5\}, 1)$-GDD of type $24^5 0^1$. Since there exists an (31,7)-MIPD as shown above, then there exists a (127,7)-MIPD by Construction 3.4. For $m = 1$ and $q = 23$, note that since there exists a TD(6, 7), then by Lemma 2.11(1), there exists a $(\{5\}, 1)$-IGDD of type $(28, 4)^5$. By applying Construction 3.3 with $q = 3$, a

(143,23)-MIPD is obtained. For $m = 1$ and $q = 31$, note that by Lemma 2.4 there exists a TD$(6,7)$. By deleting a block and the points on it from a TD$(6,7)$, a $(\{5,6\},1)$-GDD of type 6^6 is obtained. If each point is assigned a weight of 4 and the fundamental construction [21] is applied, a $(\{5\},1)$-GDD of type 24^6 is obtained. Since there exists (31,7)-MIPD, then there exists a (151,31)-MIPD by Construction 3.4.

Before treating the cases $m = 7$, 9, and 10, we require a (191,47)-MIPD, a (255,63)-MIPD and a (255,15)-MIPD. These may be obtained by applying Lemma 3.6 to $s = 48$, 64, and 16 respectively, applying Lemma 3.2 to obtain the last case from that proceeding it. The cases $m = 7, 9$, and 10 are now treated in the following table, applying Lemmas 3.5(1) and 3.2. (The required TD's come from Lemma 2.4).

m	q	$120m+q$	$4t$	$4u$	q	auxiliary$MIPD$
7	7	847	144	80	47	(127,7)
7	23	863	144	96	47	(143,23)
7	31	871	144	104	47	(151,31)
9	7	1087	192	64	63	(127,7)
9	23	1103	192	80	63	(143,23)
9	31	1111	192	88	63	(151,31)
10	7	1207	192	184	63	(247,7)*
10	23	1223	240	8	15	(23,23)
10	31	1231	240	16	15	(31,31)

* This is the case $m = 2, q = 7$ covered above.

The auxilliary MIPDs for $m = 7$ and 9 come from the case $m = 1$.

This covers all cases for m and q, and completes the proof. ∎

Lemma 4.3. *If* $v \in \{383,403,423,443,703,723\}$, *then* $D(v) = B(v)$.

Proof: It has been shown in Lemma 3.6 and Lemma 4.2 that $\{95,143\} \subset$ MIPD (23). Apply Lemma 3.5(2) with $(8t,8u,q) = (72,0,23)$, $(72,40,23)$ and $(120,80,23)$. This works for $v \in \{383,423,703\}$ by Lemma 3.1 and Lemma 4.1. Since a $(\{5\},1)$-IGDD of type $(77,5)^5$ and $(87,15)^5$ exists by Lemma 2.7 we can take $q = 18$ and 8 respectively in Construction 3.3 to get $403 \in$ MIPD(43) and $443 \in$ MIPD(83) respectively. The result for $v \in \{403,443\}$ then follows from Lemmas 3.1 and 4.1. In view of Lemma 2.11, we have a $(\{5\},1)$-IGDD of type $(28,4)^5(4,4)^1$. Give points of such a IGDD weight 5 to yield a $(\{5\},1)$-IGDD of type $(140,20)^5(20,20)^1$. Thus the result for $v = 723$ can be taken care of by Construction 3.3 with $q = 3$. ∎

We now give our main results of this section.

Theorem 4.4. *If* $v \equiv 3 \pmod{20}$ *and* $v \neq 243$, *then* $D(v) = B(v)$.

Proof: From the above lemmas, we need only to consider the case $v \geq 203$ and $v \neq \{243, 383, 403, 423, 443, 703, 723\}$. It is sufficient to show $v \in \text{MIPD}(w)$ such that $D(w) = B(w)$. We apply recursively Lemma 3.5(1)in Table 1 to give this proof. All of the required TDs have been shown to exist in Lemma 2.4. ∎

Table 1

$v = 5 \cdot (4t) + 4u + q$	$4t$	q	$4u + q$	$4t + q \in \text{MIPD}(q)$
203–223	36	11	23–43	Lemma 3.6
263–303	48	15	23–63	Lemma 3.6
323–363	60	19	23–63	Lemma 3.6
463–543	92	3	3–83	NKS(2,4; 72)
523–603	96	31	43–123	Lemma 3.6
583–683	108	35	43–143	Lemma 3.6
743–863	144	7	23–143	Lemma 4.2, 31 $\in$ MIPD(7)
883–923	156	51	103–143	Lemma 3.6
943–1043	180	11	43–143	Lemma 3.7 $s = 12, g = 1$ (1)
1063–1103	192	63	103–143	Lemma 3.6
1123–1223	216	11	43–143	Lemma 3.7 $s = 12, g = 1$ (3)
1223–1343	240	15	23–143	Lemma 3.7 $s = 16, g = 1$ (1)
1343–1463	264	7	23–143	Lemma 4.2, 31 $\in$ MIPD(7)
1463–1583	288	15	23–143	Lemma 3.7 $s = 16, g = 1$ (3)
1603 -1643	300	99	103–143	Lemma 3.6
1643–1743	276	91	263–363	Lemma 3.6
1763–1883	300	99	263–383	Lemma 3.6
1903–1943	360	23	103–143	Lemma 3.7 $s = 24, g = 1$ (1)
1943–2123	336	111	263–443	Lemma 3.6
2143–2263	360	119	343–463	Lemma 3.6
2283–2503	396	131	303–523	Lemma 3.6
2523–2783	444	147	303–563	Lemma 3.6
2783–3123	504	167	263–603	Lemma 3.6
3143–3643	576	191	263–763	Lemma 3.6
3623–4243	672	223	263–883	Lemma 3.6
4263–4463	720	239	663–863	Lemma 3.6
≥ 4463	$120m$ ($m \geq 7$)	23	263–863	Lemma 4.2

5. Packing numbers $D(v)$ for $v \equiv 9$ or $17 \pmod{20}$

Lemma 5.1 *If* $v \equiv 9, 13$ *or* $17 \pmod{20}$, *then* $D(v) \leq B(v) - 1$.

Proof: Let $(\mathcal{X}, \mathcal{B})$ be a $(v, 5, 1)$ packing such that v satisfies the given congruence. Define Y_x to be the number of blocks in $\mathcal{B}$ which contain x for any $x \in \mathcal{X}$. Then $Y_x \leq \frac{v-1}{4}$ by the definition of a packing. From (1.1) we have also $|\mathcal{B}| \leq \frac{v(v-1)-12}{20} = B(v)$, which implies that there is at least one point of $\mathcal{X}$ such that $Y_x < \frac{v-1}{4}$. Therefore, there must be 4 pairs of $\mathcal{X}$ involving x, say $\{x_i, x\}(1 \le i \le 4)$, which do not appear in any block of $\mathcal{B}$. This implies that $Y_{x_i} < \frac{v-1}{4}$ for each $1 \le i \le 4$. It follows that $|\mathcal{B}| = (\sum_{x\in\mathcal{X}} Y_x)/5 \le \frac{1}{5}\left(\frac{v(v-1)}{4} - 5\right) = \frac{v(v-1)-20}{20}$, and hence $|\mathcal{B}| \le B(v) - 1$. ∎

Lemma 5.2. *If a $(v,9;\{5\},1)$-IPBD exists, then $D(v) = B(v) - 1$.*

Proof: It is easy to show that $v \equiv 9$ or 17 (mod 20) whenever a $(v,9;\{5\},1)$-IPBD exists. Let $\mathcal{X}, Y, \mathcal{A}$ be a $(v,9;\{5\},1)$-IPBD. Since $D(9) = 2$ from the Table in [3], we can construct a $(v,5,1)$ packing on Y with two blocks. Use $\mathcal{B}$ for its block set. Then it is readily checked that $(\mathcal{X}, \mathcal{A}\cup\mathcal{B})$ is a $(v,5,1)$ packing with $B(v) - 1$ blocks. The conclusion then follows from Lemma 5.1. ∎

Combining Lemma 2.2 and Lemma 5.2 with the Table in [3] we are able to give our main result of this section.

Theorem 5.3. *For all positive integers $v \equiv 9$ or 17 $\pmod{20}$, we have $D(v) = B(v) - 1$ with exception $v = 17$ and possible exceptions $v = 29, 49$.*

Unfortunately we do not have an analogous result for the case $v \equiv 13 \pmod{20}$, since this would constitute the case $v \equiv 1 \pmod{4}$.

6. Packing number for $v \equiv 7, 11$ or 15 (mod 20)

Lemma 6.1 *Let n be a positive integer, and suppose that $v = 100n + 7$. Then $D(v) = B(v)$.*

Proof: For $n \ne 12, D(20n+3) = B(20n+3)$ and a TD$(5, 20n+1)$ exists from Theorem 4.4 and Lemma 2.4. Let $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ be a TD$(5, 20n+1)$. Add two new points ∞_1, ∞_2 to each group of a TD$(5, 20n+1)$ and then construct a $(v,5,1)$ packing with $B(20n+3)$ blocks on $G\bigcup\{\infty_1,\infty_2\}$ such that pair $\{\infty_1,\infty_2\}$ does not occur in any block. Write $\mathcal{A}_G$ for its block set for each $G \in \mathcal{G}$. Then $(\mathcal{X}\cup\{\infty_1\infty_2\}, \mathcal{A}\cup(\bigcup \mathcal{A}_G))$ is a packing with $B(100n+7)$ blocks. The conclusion follows from (1.1).

For $n = 12$, see Lemma 6.4. ∎

Lemma 6.2. *Let n be a positive integer $\ne 12$, and suppose that $v = 100n + 11$. Then $D(v) = B(v)$.*

Proof: Add one new point to a TD$(5, 20n+2)$. The proof is similar to the above. ∎

Lemma 6.3. *Let n be a positive integer, and suppose that $v = 100n + 15$. Then $D(v) = B(v)$.*

Proof: For $n \ne 12$, the conclusion follows from the fact that a TD$(5, 20n+3)$ exists and $D(20n+3) = B(20n+3)$ for these values of n.

For $n = 12$, proceed as follows. Note that by Lemma 2.12, a $(\{5\}, 1)$-GDD of type 192^6 exists, and by Lemma 3.6, a $(255, 63)$-MIPD exists. Apply Const 3.4 with $n = 6, t_1 = t_2 = \ldots = t_6 = 192$ and $q = 63$ to for $(1215, 63)$-MIPD, Since by Lemma 4.1 we have $D(63) = B(63)$, it follows that $D(1215) = B(1215)$ as required. ∎

Note that $D(7) = B(7)$ and $31 \in$ MIPD(7). Combining Lemma 3.1 with Lemma 4.2, we have also the following.

Lemma 6.4. *If* $v \equiv 7$ *or* $31 \pmod{120}$, *then* $D(v) = B(v)$.

7. Conclusion

We have determined the packing numbers $D(v)$ for $v \equiv 3, 9$ or $17 \pmod{20}$ with possible exceptions of $v \in \{29, 49, 243\}$. The results shown in section 3 can be used to investigate the case $v \equiv 7, 11, 15$ or $19 \pmod{20}$ which is currently under consideration. Further results will be reported in a subsequent paper.

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