ALGEBRAIC EQUIVALENCE OF SIGNED GRAPHS WITH ALL EIGENVALUES ≥ -2

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Abstract

We introduce a new concept called algebraic equivalence of sigraphs to study the family of sigraphs with all eigenvalues ≥ -2. First we prove that any sigraph whose least eigenvalue is -2 contains a proper subgraph such that both generate the same lattice in $\mathbb{R}^{\infty}$. Next we present a characterization of the family of sigraphs with all eigenvalues > -2 and obtain Witt's classification of root lattices and the well known theorem which classifies the first mentioned family by using root systems $D_n, n \in \mathbb{N}$ and E_8. Then we prove that any sigraph whose least eigenvalue is less than -2, contains a subgraph whose least eigenvalue is -2. Using this, we characterize the families of sigraphs represented by the above root systems. Finally, we prove that a sigraph generating E_n ($n = 7$ or 8) contains a subgraph generating E_{n-1}. In short, this new concept takes the central role in unifying and explaining various aspects of the theory of sigraphs represented by root systems and in giving simpler and shorter proofs of earlier known results including Witt's theorem and also in proving new results.

1 Introduction

A *sigraph* S is a pair (X, ϕ) where X is a finite set (called the *set of vertices* and denoted by $V(S)$) and $\phi : X \times X \to \{-1, 0, 1\}$ satisfying for all x, y in X, $\phi(x,y) = \phi(y,x)$ and $\phi(x,x) = 0$ (ϕ is called the *edge function.*). If for any x, y in X, $\phi(x,y) = \pm 1$, then we say that x and y are *adjacent* and the set $\{x, y\}$ is an *edge*. The function $\phi^* : X \times X \to \mathbb{Z}$ (the set of integers) defined by

$$\phi^*(x,y) = \begin{cases} \phi(x,y) & \text{if } x \neq y \\ 2 & \text{else} \end{cases}$$

is said to be the *associate edge function* of S. ϕ^* is said to be *linear*, if it has the following property:

for any $A \subseteq X$ and scalars $\alpha_x, x \in A$, if $\sum_{x \in A} \alpha_x \phi^*(x,t) = 0$ holds for all $t \in A$ then it does so for all $t \in X$.

Let $S = (X, \phi)$ be a sigraph and $Y \subseteq X$. Then (Y, ϕ'), where ϕ' is the restriction of ϕ to $Y \times Y$, is said to be the *subgraph* of S on Y and denoted by $S[Y]$. If a sigraph T is a subgraph of S, sometimes we write $T \subseteq S$.

A sigraph S is said to be *minimally forbidden* for a family $\mathcal{F}$ of sigraphs, if S is not in $\mathcal{F}$ but every proper subgraph of S is in $\mathcal{F}$.

Let $\mathbb{R}^\infty$ be the countably infinite dimensional Euclidean space with usual inner-product $< \cdot, \cdot >$ and W, a subset of $\mathbb{R}^\infty$. A sigraph $S = (X, \phi)$ is said to have a *representation* (abbreviated as RPN) ψ in W if $\psi : X \to W$ such that for all x, y in X,

$$\phi^*(x,y) = < \psi(x), \psi(y) > .$$

A sigraph S is said to be *represented* by W, if every component of S has a RPN in W. Let $\mathcal{R}(W)$ denote the family of sigraphs represented by W and $\mathcal{M}(W)$, the class of minimal forbidden sigraphs for $\mathcal{R}(W)$.

Let $\mathcal{B} = \left\{ e_i \middle| i = 1, 2, \cdots \right\}$ be an orthonormal basis for $\mathbb{R}^\infty$. The *root systems* $A_n, D_n, n \in \mathbb{N}$ and E_8 are defined to be

$$\left\{ \pm(e_i - e_j) \middle| 1 \leq i < j \leq n+1 \right\}, \left\{ \pm e_i \pm e_j \middle| 1 \leq i < j \leq n \right\} \text{ and}$$

$$\left\{ \pm\sqrt{2} e_i \middle| 1 \leq i \leq 8 \right\} \bigcup \left\{ \frac{1}{\sqrt{2}}(\pm e_i \pm e_j \pm e_k \pm e_l) \middle| (i,j,k,l) \in \mathcal{D} \right\}$$

respectively ($\mathcal{D}$ is the set of blocks of the unique $3-(8,4,1)$ design with $\{1, \cdots 8\}$ as base set.). Let

$$D_\infty = \bigcup_{n=1}^{\infty} D_n.$$

Let a and b two vectors in E_8 such that $< a, b >= 1$. Then the *root systems* E_7 and E_6 are defined to be

$$\left\{ v \in E_8 \middle| < v, a >= 0 \right\} \text{ and } \left\{ v \in E_7 \middle| < v, b >= 0 \right\}.$$

This definition does not depend upon the choice of a and b (cf, [CGSS] or [VRS].).

A.J. Hoffman who initiated the study of graphs with all eigenvalues ≥ -2 has proved (see [H].) that if a connected graph's least eigenvalue ≥ -2 then either it is a generalized line graph or it has at most 36 vertices (Generalized line graphs are precisely the graphs in $\mathcal{R}(D_\infty)$; but Hoffman defined them differently; the equivalence of these two definitions has been shown in Theorem 4.2 of [CGSS].).

An improvement of the above result is the following one:

Theorem 1.1 Let S be a connected sigraph and $\lambda(S)$, its least eigenvalue. Then the following are equivalent:
(1) $\lambda(S) \geq -2$.
(2) S is represented by $\mathbb{R}^\infty$.
(3) S is represented by D_∞ or E_8.

The equivalence of (1) and (2) is easy to demonstrate (see the paragraph preceding Remark 2.6.). (2) $\Rightarrow$ (3) has been proved in [W] which has classified all the root lattices of $\mathbb{R}^n, n \in \mathbb{N}$; for a combinatorial proof, the reader is referred to [CGSS].

As a consequence of this theorem, root systems have gained much attention to investigate the properties of $\mathcal{R}(\mathbb{R}^\infty)$. The graphs in $\mathcal{M}(D_\infty)$ have been computed in [CDS], [RSV] and [V1]. A description of the graphs in $\mathcal{M}(\mathbb{R}^\infty)$ has been given in [VRS]. In [BN], all the graphs in $\mathcal{M}(\mathbb{R}^\infty)$ have been computed. They are 1812 in number. In [V2], a characterization for the family $\mathcal{R}(D_\infty)$ has been found and 10 has been shown to be the best possible upper bound for the order of any sigraph in $\mathcal{M}(\mathbb{R}^\infty)$.

In this paper, we introduce a new concept called *algebraic equivalence* of sigraphs to study the families $\mathcal{R}(\mathbb{R}^\infty)$ and $\mathcal{M}(\mathbb{R}^\infty)$. Roughly speaking, an algebraic equivalent of a sigraph S is obtained by replacing a subset A of $V(S)$ by a set of vertices which are "integral combinations of elements of A" and defining the edge function accordingly. Using this, we give a new proof of Theorem 1.1. The crux of this proof is in showing that a connected sigraph with least eigenvalue > -2 is an algebraic equivalent of a Dynkin graph. This concept is used quite extensively in the proof of the following

Theorem 1.2 A sigraph whose least eigenvalue < -2 contains a subgraph of order ≤ 9 whose least eigenvalue is -2.

This result for the particular case of graphs has been verified, by using computer calculations, in [D] and later in [BN]. Using this theorem we characterize algebraically the family $\mathcal{R}(\mathbb{R}^\infty)$:

Theorem 1.3 A sigraph is represented by $\mathbb{R}^\infty$ if and only if its associate edge function is linear.

An outline of the paper is as follows: In the next section we introduce necessary terminology and prove some preliminary results. In the third section first we prove the following

Theorem 1.4 If the least eigenvalue of a sigraph S is -2, then S has a representation ξ in $\mathbb{R}^\infty$ such that for some $a \in V(S), \xi(a)$ is an integral combination of elements of $\{\xi(x) \mid x \in V(S) - \{a\}\}$ (while classifying the root systems, a similar result has been proved – cf. the second chapter of [C] – Any root system is generated by a linearly independent subset.).

Next we characterize the family of sigraphs with all eigenvalues > -2:

Theorem 1.5 A connected sigraph whose least eigenvalue > -2 is an algebraic equivalent of one of the graphs $P_n, Q_n, n \in \mathbb{N}$ or $R_k, k = 6, 7, 8$ (these graphs generate $A_n, D_n, n \in \mathbb{N}$ and $E_k, k = 6, 7, 8$ respectively; in the literature, they are known as Dynkin graphs.).

A consequence of Theorems 1.4 and 1.5 is Theorem 1.1.

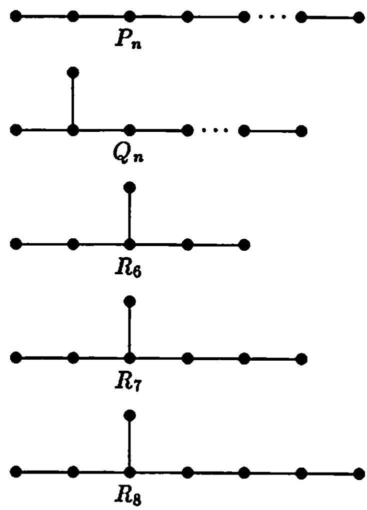

Figure 1.6

A *root lattice* is an additive subgroup $\mathcal{L}$ of $\mathbb{R}^n$, generated by a set X of vectors such that for all x, y in $X, < x, x >= 2$ and $< x, y >\in \mathbb{Z}$. The vectors in $\mathcal{L}$ of norm $\sqrt{2}$ are called the *roots* of $\mathcal{L}$. A root lattice is said to be *irreducible* if it is not a direct sum of proper sublattices. In [W], the root lattices have been classified:

Theorem 1.7 If R is the set of roots of an irreducible root lattice then there exists an *automorphism* θ of $\mathbb{R}^\infty$ (a bijective linear inner-product preserving map from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$) such that $\theta(R)$ is one of the root systems $A_n, D_n, n \in \mathbb{N}$ or $E_k, k = 6, 7, 8$.

We derive this result from Theorems 1.4 and 1.5.

In the fourth section, first we derive some properties of $\mathcal{R}(D_\infty)$ and $\mathcal{M}(D_\infty)$. Next we prove Theorems 1.2 and 1.3. Then we present characterizations of $\mathcal{R}(D_\infty)$ and $\mathcal{R}(E_8)$, as the following theorems:

Theorem 1.8 A sigraph S is represented by D_∞ if and only if the following hold:

(1) No subgraph is an algebraic equivalent of R_6 (in Figure 1.6).
(2) If any subgraph $S[A]$ is switching equivalent to T_i for some i, $i \leq 3$ (in Figure 1.11), then for all x, y in $V(S)$, the associate edge function of $S[A \cup \{x, y\}]$ is linear.

Theorem 1.9 A sigraph is represented by E_8 if and only if the following hold:
(1) No subgraph is an algebraic equivalent of P_9 or Q_9 (in Figure 1.6).
(2) The associate edge function is linear.

A sigraph S is said to *generate* a root system W, if S has a RPN ξ such that the lattices generated by W and $\xi(V(S))$ are same (Note that S has to be connected.). A consequence of Theorem 1.8 is the following

Theorem 1.10 A sigraph generating E_n ($n = 7$ or 8) contains a subgraph generating E_{n-1}.

Here is a list of sigraphs which occur often in this paper:

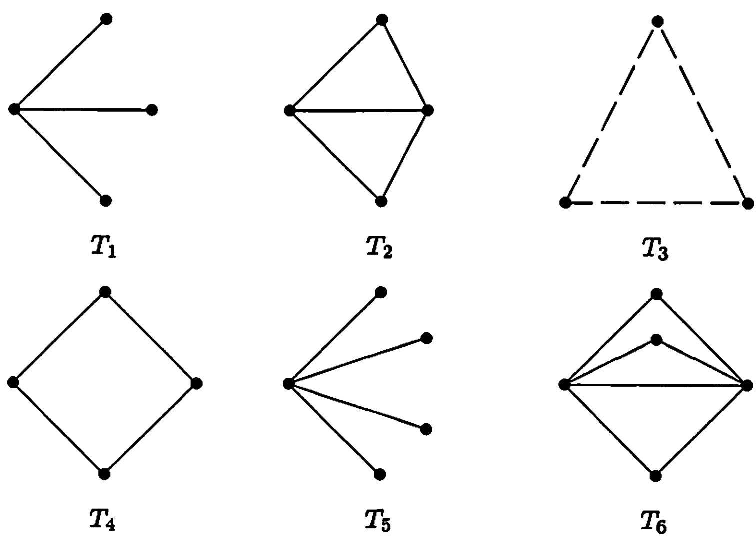

Figure 1.11 Solid lines are positive edges and broken lines are negative.

Notation When a sigraph is denoted by S, its set of vertices is X. For any sigraph, we denote its edge function by ϕ, unless otherwise stated.

The least eigenvalue of a sigraph S is denoted by $\lambda(S)$; for any a in X, its *neighbourhood* $N(a)$ is the set of its adjacent vertices; its degree is denoted by *deg* a; we write $S - a$ for $S[X - a]$. The sigraph whose set of vertices contains $X - a$ and one more vertex denoted by $-a$ such that its subgraph on $X - a$ is $S - a$ and $\phi(-a, x) = -\phi(a, x)$ for all x in $X - a$, is denoted by $S(-a)$. $S_u = (X, \phi_u)$ is the *underlying graph* of S where

$\phi_u(x,y) = |\phi(x,y)|$, for all $x, y \in X$. For any $x, y \in X$ and $T \subseteq S$, $d(x,y)$ is the length of a shortest path joining x and y and $d(x,T) = \min_{t \in V(T)} d(x,t)$.

Λ denotes D_∞ or E_8 and Ω is Λ or $\mathbb{R}^\infty$. For any $W \subseteq \mathbb{R}^\infty$, $\mathcal{L}(W)$ and $\mathcal{I}(W)$ are the set of linear combinations and the set of integral combinations respectively of elements of W; $dimW$ is the dimension of $\mathcal{L}(W)$;

$$\mathcal{R}^*(W) = \left\{ S \in \mathcal{R}(W) \middle| \lambda(S) > -2 \right\} \text{ and } \mathcal{R}_0(W) = \mathcal{R}(W) - \mathcal{R}^*(W).$$

2 Algebraic Equivalence of Sigraphs

The following proposition is useful to find the inter-relations of the RPNs of any sigraph in $\mathcal{R}(\mathbb{R}^\infty)$.

Proposition 2.1 If a sigraph S has two RPNs ξ_1 and ξ_2 in $\mathbb{R}^\infty$ then there is an automorphism θ of $\mathbb{R}^\infty$ such that $\theta \circ \xi_1 = \xi_2$.

Proof: Since

$$\sum_{x \in X} \alpha_x \xi_1(x) = 0 \quad (\alpha_x, x \in X \text{ are scalars}) \implies \sum_{x \in X} \alpha_x \xi_2(x) = 0$$

for when $i = 1, 2$

$$\begin{aligned} < \sum_x \alpha_x \xi_i(x), \sum_x \alpha_x \xi_i(x) > &= \sum_{x,y} \alpha_x \alpha_y < \xi_i(x), \xi_i(y) > \\ &= \sum_{x,y} \alpha_x \alpha_y \phi^*(x,y) \end{aligned}$$

it can be verified easily that the map θ^* from $\mathcal{L}(\xi_1(X))$ to $\mathcal{L}(\xi_2(X))$ defined by

$$\theta^* \Big(\sum_{x \in X} \alpha_x \xi_1(x) \Big) = \sum_{x \in X} \alpha_x \xi_2(x) \quad (\alpha_x, x \in X \text{ are scalars})$$

is a well defined isomorphism. Since $\mathcal{L}(\xi_i(X)), i = 1, 2$ are finite dimensional, θ^* can be extended to an automorphism θ of $\mathbb{R}^\infty$. Clearly $\theta \circ \xi_1 = \xi_2$. This completes the proof.

Proposition 2.2 If two sigraphs S and T have RPNs ψ and ξ respectively in $\mathbb{R}^\infty$ and $T \subseteq S$, then ξ can be extended to a RPN of S.

Proof: By the previous proposition, there is an automorphism θ of $\mathbb{R}^\infty$ such that $\theta \circ \psi_T = \xi$, where ψ_T is the restriction of ψ to $V(T)$. Clearly $\theta \circ \psi$ has the required property.

Let $\mathcal{D} = (\mathbf{E}, \beta)$ be a $3-(8,4,1)$ design with $\mathbf{E} = \{1, \cdots 8\}$ as the set of points; β is a set of 14 blocks. We list a few well known properties of $\mathcal{D}$ (cf. [VRS] for details.):

Remark 2.3
(1) For any three distinct $x, y, z \in \mathbf{E}$, there exists a unique block in β, containing x, y, z.
(2) $B \in \beta \Rightarrow (E - B) \in \beta$.
(3) $B_1, B_2 \in \beta, B_1 \neq B_2 \Rightarrow |B_1 \cap B_2| = 0$ or 2; $B_1 \Delta B_2 \in \beta$, if $|B_1 \cap B_2| = 2$
((2) and (3) follow from (1).).

Let E_8 be defined by this design $\mathcal{D}$, as in the introduction. Using the above remark, it is easy to verify that E_8 shares the following properties with D_∞:

Remark 2.4 For all x, y in Λ,
(1) $-x \in \Lambda$,
(2) $< x, x >= 2$,
(3) $x \neq \pm y \Rightarrow < x, y >= \pm 1$ or 0,
(4) $< x, y >= 1 \Longleftrightarrow x - y \in \Lambda$ and
(5) $\{x \in \mathcal{I}(\Lambda) | < x, x >= 2\} = \Lambda$.

Proposition 2.5 If a connected sigraph S has a RPN ψ in $\mathbb{R}^\infty$ such that for some $a \in X, \psi(a) \in \mathcal{I}(\psi(X - a))$ then $S - a$ is connected and $S - a \in \mathcal{R}(\Lambda) \Rightarrow S \in \mathcal{R}(\Lambda)$.

Proof: By hypothesis, $\mathcal{I}(\psi(X - a)) = \mathcal{I}(\psi(X))$ and therefore irreducible, since S is connected. Hence $S - a$ is also connected.

Suppose $S - a \in \mathcal{R}(\Lambda)$. By Proposition 2.2, S has a RPN ξ such that $\xi(X - a) \subseteq \Lambda$. Then by Proposition 2.1 and hypothesis $\xi(a) \in \mathcal{I}(\xi(X - a))$. Hence the conclusion follows from (2.4.5).

Now let us give a series of definitions which play important roles in proving the main theorems and corresponding remarks.

Let ξ be a RPN of a sigraph S; ξ is said to be *linearly independent* (*dependent*) *representation* (abbreviated as *LIR* (*LDR*)) if $\xi(X)$ is a linearly independent (dependent) subset of $\mathbb{R}^\infty$.

Let A be the adjacency matrix of a sigraph S with $\lambda(S) \geq -2$; then $A + 2I$ is positive semi-definite and therefore $A + 2I = MM^T$, for some matrix M. Hence we have a RPN ξ of S in $\mathbb{R}^\infty$ where $\xi(x)$, $x \in X$ are row vectors of M. Conversely it is easy to see that if S has a RPN then $\lambda(S) \geq -2$. Further one can note the following

Remark 2.6 For a sigraph S
(1) $\lambda(S) > -2 \Longleftrightarrow S$ has an LIR and
(2) $\lambda(S) = -2 \Longleftrightarrow S$ has an LDR.

Two sigraphs S and T are said to be *switching equivalent* (abbreviated as *SE*) if there are functions $f : X \to V(T)$ and $\sigma : X \to \{-1, 1\}$ such that f is a bijection and for all x, y in X

$$\sigma(x)\phi(x, y)\sigma(y) = \phi(f(x), f(y)).$$

Note that for any $a \in X, S$ is SE to $S(-a)$.

Remark 2.7 If two sigraphs S and T are SE, then
(1) $S \in \mathcal{R}(\Omega) \Rightarrow T \in \mathcal{R}(\Omega)$ and
(2) $S \in \mathcal{M}(\Omega) \Rightarrow T \in \mathcal{M}(\Omega)$.

Remark 2.8 If a subgraph T of a sigraph $S \in \mathcal{R}(D_\infty)$ is SE to T_1 then two vertices of T have the same neighbourhood in S.

Let S be a sigraph and $a, b \in X$ such that $\phi(a,b) = 1$ and

$$(**) \qquad \text{for all } x \in X,\ S[a,b,x] \text{ is not SE to } T_3.$$

Clearly for all $x \in X$,

$$\phi^*(a,x) - \phi^*(b,x) = 0 \text{ or } \pm 1.$$

The sigraph whose set of vertices contains $X - b$ and one more vertex denoted by $a - b$ such that its subgraph on $X - b$ is $S - b$ and for all $x \in X - b$,

$$\phi(a-b,x) = \phi^*(a,x) - \phi^*(b,x)$$

is said to be an *algebraic transform* (abbreviated as AT) of S and denoted by $S(a:b)$.

Similarly for any $a, b \in X$, when $\phi(a,b) = -1$ and $(**)$ holds, $S(a:b)$ can be defined and in this case the new vertex is denoted by $a + b$.

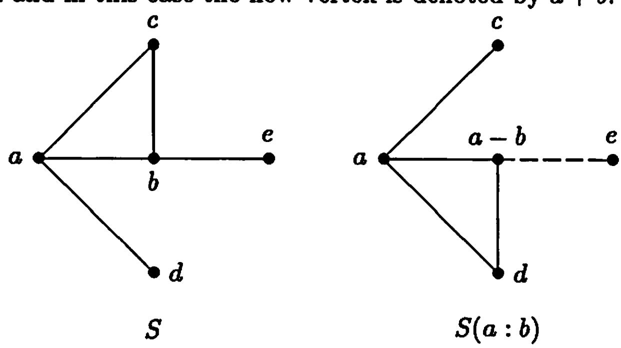

Figure 2.9

Remark 2.10
(1) It is easy to verify that S is also an AT of $S(a:b)$.
(2) If S has a RPN ξ in Ω then by (2.4.5), $\xi(a) - \alpha\xi(b) \in \Omega$ where $\alpha = \phi(a,b)$ and $S(a:b)$ also has a RPN η in Ω defined by

$$\eta(x) = \begin{cases} \xi(x) & \text{if } x \in X - b \\ \xi(a) - \alpha\xi(b) & \text{else.} \end{cases}$$

(3) If a sigraph $T \in \mathcal{R}^*(I\!R^\infty)$, then for any adjacent $x, y \in V(T)$, $T(x:y)$ exists.

Now let us define the concept of algebraic equivalence which plays the central role in the remaining portion of this paper.

A sigraph S is said to be an *algebraic equivalent* (abbreviated as AE) of another sigraph T if there is a sequence $\{S_i\}_{i=1}^n$ of sigraphs such that $S = S_1, T = S_n$ and for $i = 1, \cdots n-1, S_i$ is an SE or AT of S_{i+1}.

Remark 2.11 If two sigraphs S and T are AE, then the following hold:
(1) S is connected $\Rightarrow T$ is connected
(2) $S \in \mathcal{R}(\Omega) \Rightarrow T \in \mathcal{R}(\Omega)$
(3) S has an LIR (LDR) ξ in $\Omega \Rightarrow T$ has an LIR (LDR) η in Ω such that $\mathcal{I}(\xi(X)) = \mathcal{I}(\eta(V(T)))$.

3 Properties of $\mathcal{R}(\Omega)$

First we give the proof of Theorem 1.4: Suppose the theorem does not hold for S. We can assume that

(a) the theorem holds for any sigraph S' with $|V(S')| < |X|$.

By (a), it follows that

(b) any proper subgraph of S is in $\mathcal{R}^*(I\!R^\infty)$.

By hypothesis and (b), S has a RPN ξ such that

$$(**) \qquad \sum_{x \in X} \alpha_x \xi(x) = 0$$

where $\alpha_x, x \in X$ are nonzero scalars. Define

$$\sigma(S) = \left(\sum_{x \in X} |\alpha_x| \right) \Big/ \left(\min_{x \in X} |\alpha_x| \right).$$

Note that by Proposition 2.1 and (b), this definition depends upon neither the RPN ξ nor the equation (**). Let us assume that among the sigraphs with $|X|$ vertices for which the theorem does not hold, S is chosen such that $\sigma(S)$ is as maximum as possible.

Since any SE of S also satisfies our requirement, we can have $\alpha_x > 0$, for all $x \in X$, by replacing x by $-x$, if necessary. Then for any a, b in X, $\phi(a, b) \neq 1$ for otherwise assuming $\alpha_a \geq \alpha_b$, it can be seen by (**) and Proposition 2.1, the theorem does not hold for $S(a : b)$, contradicting our choice of S since $\sigma(S(a : b)) > \sigma(S)$.

Since for polygons, the theorem holds and S is connected by (b), S_u is a tree. Therefore for all $x \in X, \deg x \leq 3$, for otherwise by (b), S_u would be isomorphic to T_5 and the conclusion would hold for S. Let

$$A = \left\{ x \in X \,\middle|\, \deg x = 3 \right\}.$$

Obviously $A \neq \phi$. If $|A| > 1$, S is isomorphic to the first graph in Figure 3.1. If $|A| = 1$, $S_u[X-A]$ is a disjoint union of three paths and by hypothesis and (b), S_u is isomorphic to one of the other graphs in Figure 3.1. However it can be verified that the conclusion holds for the graphs in Figure 3.1 (The required vertex is starred; note that by deleting that vertex, we get a Dynkin graph.) and consequently for S also. A contradiction. Therefore the theorem is proved.

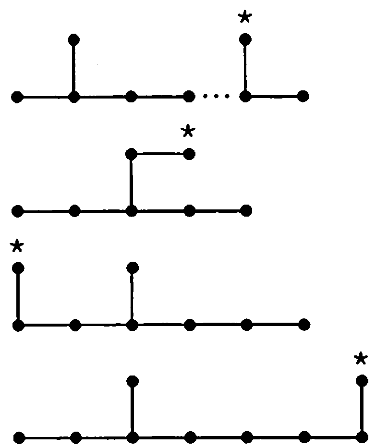

Figure 3.1

An interesting consequence of this theorem, Proposition 2.5 and (2.6.2) is the following

Corollary 3.2 For any sigraph $S \in \mathcal{M}(\Lambda)$, $\lambda(S) \neq -2$.

Now let us prove Theorem 1.5: Let S be a connected sigraph in $\mathcal{R}^*(\mathbb{R}^\infty)$ and ψ, a RPN of S. Let

$$w = \sum_{x \in X} \psi(x).$$

If η is a RPN of an AE T of S in $\mathcal{I}(\psi(X))$ such that for some nonzero scalars $\alpha_x, x \in V(T)$,

$$\sum_{x \in X} \alpha_x \eta(x) = w$$

then define

$$\sigma(\eta) = \sum_{x \in X} |\alpha_x|$$

(Note that by (2.11.3), $T \in \mathcal{R}^*(\mathbb{R}^\infty)$ and therefore α_x, $x \in V(T)$ are uniquely determined by w.). Among such RPNs, choose a RPN ξ of a sigraph R such that $\sigma(\xi)$ is as maximum as possible (Note that the number of RPNs in $\mathcal{I}(\psi(X))$ is finite.). Now by the same argument of the above proof, it follows that R_u is one of the graphs in Figure 1.6.

Remark 3.3 By Theorems 1.4, 1.5 and Proposition 2.5, any irreducible root lattice is generated by a connected sigraph AE to one of the graphs in Figure 1.6. Since $P_n, Q_n, n \in \mathbb{N}$ and $R_k, k = 6, 7, 8$ generate A_n, D_n and E_k respectively, by (2.11.3), Witt's theorem and consequently by Remark 2.6, Theorem 1.1 follow.

The main tool for proving Theorem 1.2 is the following

Proposition 3.4. If G is a graph having a vertex p adjacent to all other vertices, then either a subgraph $H \in \mathcal{R}_0(\mathbb{R}^\infty)$ such that $p \in V(H)$ or $G \in \mathcal{R}(\mathbb{R}^\infty)$.

Proof: First note that

(a) any $x \in X - p$ can be replaced by $p - x$ (for if the theorem holds for $G(p : x)$, it does so for G also).

If the components of $G - p$ are at most two and they are complete graphs then it is easy to verify that $G \in \mathcal{R}(D_\infty)$. Otherwise there exists $A \subseteq V(G)$ such that $p \in A$ and $G[A]$ is isomorphic to T_1 or T_2. We can assume the first possibility by (a) and also

(b) for all $a, b \in A$ and $x, y \in X$, $G[p, a, b, x, y] \notin \mathcal{R}_0(\mathbb{R}^\infty)$.

Then it follows that any $x \in X - A$, replaced by $p - x$ if necessary, is adjacent to exactly one of $\{x_1, x_2, x_3\} = A - p$. By (b), $G[N(x_i)], i \leq 3$, are complete graphs; two of them can be assumed to have more than one vertex for otherwise it can be easily verified that G is in $\mathcal{R}(D_\infty)$.

Now define a sigraph T with $V(T) = V(G) - A$ and edge function ρ given as follows:

$$\text{(c)} \qquad \text{for all } x, y \in V(T), \rho(x,y) = \begin{cases} -1 & \text{if } \phi(x,y) = 0 \\ 0 & \text{if } x, y \in N(x_i) \text{ for some } i \leq 3 \\ 1 & \text{else.} \end{cases}$$

It can be verified that for any $B \subseteq V(T)$,

$$G[A \cup B] \in \mathcal{R}(\mathbb{R}^\infty) \Longleftrightarrow T[B] \in \mathcal{R}(\mathbb{R}^\infty).$$

In fact if η is a RPN of $T[B]$ such that $< e_i, \eta(x) >= 0$ for all $i \leq 4$ and $x \in T[B]$, then $G[A \cup B]$ has a RPN ψ given by

$$\sqrt{2}\psi(x) = \begin{cases} 2e_i & \text{if } x = x_i \text{ for some } i \leq 3 \\ (e_1 + e_2 + e_3 + e_4) & \text{if } x = p \\ (e_i + e_4 + \eta(x)) & \text{if } x \in N(x_i) \cap B \text{ for some } i \leq 3. \end{cases}$$

It is enough to show that either for some $B \subseteq V(T), T[B] \in \mathcal{R}_0(\mathbb{R}^\infty)$ or $T \in \mathcal{R}(\mathbb{R}^\infty)$. If $|V(T)| > 4$, by using (c), it can be seen that there exists $B \subseteq V(T)$ such that $T[B]$ is SE to some $T_i, 3 \leq i \leq 6$ and therefore first possibility holds. Otherwise $T \in \mathcal{R}(D_\infty)$ or a subgraph of T is SE to T_3.

This completes the proof.

Remark 3.5 If the first part of the conclusion holds and A is a subset of $V(G)$ such that $p \in A, G[A]$ is isomorphic to T_1 or T_2 and (b) is satisfied, then H can be chosen such that $A \subseteq V(H)$.

Lemma 3.6 Let S be a sigraph and T, a subgraph of S, SE to T_1 or T_2 and for all $x, y \in X$, $S[V(T) \cup \{x,y\}] \in \mathcal{R}^*(\mathbb{R}^\infty)$ and $d(x,T) \leq 1$. Then $S \in \mathcal{R}(\mathbb{R}^\infty)$ or a subgraph of S containing T is in $\mathcal{R}_0(\mathbb{R}^\infty)$.

Proof: By hypothesis,

(a) for all $x, y \in X$ and $z \in V(T), S[x,y,z]$ is not SE to T_3.

Let $p \in V(T)$ such that $\deg p = 3$; we can assume that for any x in $X - p$, $\phi(p,x) = \pm 1$ for otherwise it suffices to prove the conclusion for $S(q : x)$ where $q \in V(T)$ such that $\phi(q,x) = \pm 1$ (note that $S(q : x)$ exists by (a).) and also that $\phi(p,x) \geq 0$, for all $x \in X$. Then by (a), for all $x, y \in X, \phi(x,y) \geq 0$; i.e. S is a graph. Now the conclusion follows from the above remark.

One more preparatory lemma for proving Theorem 1.2:

Lemma 3.7 Let S be a connected sigraph of order ≤ 8, having an LDR ξ such that for some $a \in X, S - a$ is connected and has an LIR and a subgraph of $S - a$ is SE to T_1 or T_2. Then $\xi(a) \in \mathcal{I}(\xi(X - a))$.

Proof: It is enough to show that both S and $S - a$ generate the same root system. Let $|X| = n + 1$. When $n \leq 5$, they generate D_n, for a subgraph of $S - a$ is SE to T_1 or T_2. Suppose $n \geq 6$. If $S - a$ generates E_n, obviously S also does so. Therefore assume that $S - a$ generates D_n. Then $S - a$ is AE to Q_n by Theorem 1.5 and therefore by (2.11.2), $S - a \notin \mathcal{R}(E_n)$ for $Q_n \notin \mathcal{R}(E_n)$. Therefore $S \notin \mathcal{R}(E_n)$. Hence S also generates D_n.

4 Characterization of The Sigraphs in $\mathcal{R}(\Omega)$

In this section we shall prove Theorems 1.2, 1.3, 1.8, 1.9 and 1.10. We start with a lemma which gives a sufficient condition for a sigraph to have a RPN in D_∞.

Lemma 4.1 If S is a sigraph such that no subgraph is SE to any of the sigraphs $T_i, i = 1,2,3$, then $S \in \mathcal{R}(D_\infty)$.

Proof: By Van Rooij and Wilf's characterization of line graphs in [RW], S_u is a line graph. Let R be a *root graph* of S_u (i.e. S_u is the line graph of R.). We assume that $V(R) \subseteq \mathcal{B}$ (an orthonormal basis for $\mathbb{R}^\infty$).

Let $t \in V(R)$ and E_t be the set of edges incident with t. Define a function $\sigma_t : E_t \to \{-1, 1\}$ such that

(**) for all distinct $f, g \in E_t$, $\sigma_t(f)\sigma_t(g) = \phi(f,g)$.

Such a function σ_t can be constructed as follows: for some $e \in E_t$, set $\sigma_t(e) = 1$; for any $f \in E_t - e$, let $\sigma_t(f) = \phi(e,f)$. Now for any two distinct

$f, g \in E_t - e$, $\phi(e,f)\phi(f,g)\phi(g,e) = 1$ for no subgraph of S is SE to T_3, and $(**)$ follows.

Now define a map $\psi : X \to D_\infty$ as follows: for any $e \in X$, if a and b are ends of e in R, then $\psi(e) = \sigma_a(e)a + \sigma_b(e)b$. It is easy to verify by using $(**)$, that ψ is a RPN of S in D_∞.

Lemma 4.2 Suppose S is a sigraph and $A \subseteq X$ such that the following hold:

(1) $S[A]$ is isomorphic to T_1.

(2) For some $a \in A$, $S - a \in \mathcal{R}(D_\infty)$.

(3) For some $b \in A - a$, $N(a) = N(b)$.

(4) For all $x \in X$, no subgraph of $S[A \cup x]$ is in $\mathcal{R}_0(\mathbb{R}^\infty)$.

Then $S \in \mathcal{R}(D_\infty)$.

Proof: Let $A = \{p, a, b, c\}$, where $a, b, c \in N(p)$. By (2), $S - a$ has a RPN ξ in D_∞. Assume $\xi(b) = e_1 - e_2$ and $\xi(p) = e_1 + e_3$. If for any x in $X - a$, $\xi(x)$ is $\pm(e_1 + e_2)$, then x is c for otherwise by (3), $S[A \cup x]$ would be SE to T_5 and (4) would not hold. Again by (3), $N(a) = N(b) = N(c)$. By (4), $S[A]$ is a component of S and the conclusion follows from (2). Therefore suppose $\pm(e_1 + e_2) \notin \xi(X - a)$; then it can be verified by using (3) and (4), that the function $\psi : X \to D_\infty$ defined by

$$\psi(x) = \begin{cases} e_1 + e_2 & \text{if } x = a \\ \xi(x) & \text{else} \end{cases}$$

is a RPN of S in D_∞.

Lemma 4.3 If S is a sigraph in $\mathcal{M}(D_\infty)$ such that no subgraph of order ≤ 5 is in $\mathcal{R}_0(\mathbb{R}^\infty)$, then $|X| = 6$.

Proof: By Lemma 4.1 and hypothesis, there exists $A \subseteq X$ such that $S[A]$ is SE to T_1 or T_2; we can assume by Remark 2.7, $S[A]$ is isomorphic to T_1 or T_2.

Case (1) $S[A] = T_1$. Since minimality of $S \implies$ for all $a \in A$, $S - a \in \mathcal{R}(D_\infty)$ and hypothesis $\implies$ (4.2.4), it follows from Lemma 4.2 that, for any two distinct $a, b \in A$, $N(a) \neq N(b)$. Therefore there exist x, y in X such that in $T = S[A \cup \{x, y\}]$, vertices of A have mutually different neighbourhoods and $T \notin \mathcal{R}(D_\infty)$, by Remark 2.8.

Case (2) $S[A] = T_2$. Let $A = \{p, x_1, x_2, x_3\}$ where $\phi(x_2, x_3) = 0$. Since no subgraph of S is SE to T_3, $R = S(p : x_1)$ exists. Now by minimality of S, $S - x_i \in \mathcal{R}(D_\infty), i = 1, 2, 3$; therefore, $R - x_i, i = 2, 3$ and $R - (p - x_1)$ are in $\mathcal{R}(D_\infty)$, by (2.11.2). Also for any $x \in V(R)$, no subgraph T of $R[p, p - x_1, x_2, x_3, x]$ is in $\mathcal{R}_0(\mathbb{R}^\infty)$ for otherwise $p \in V(T)$ and a subgraph of $S[p, x_1, x_2, x_3, x]$ would be in $\mathcal{R}_0(\mathbb{R}^\infty)$ by (2.11.3), contradicting the hypothesis. Since $R \notin \mathcal{R}(D_\infty)$ by (2.11.2), as in the first case, there exist $x, y \in V(R)$ such that $R[p, p - x_1, x_2, x_3, x, y] \notin \mathcal{R}(D_\infty)$ and again by (2.11.2), $S[A \cup \{x, y\}] \notin \mathcal{R}(D_\infty)$.

Hence in both cases $|X| = 6$. This completes the proof.

Now we are in a position to prove Theorem 1.2: It is enough to prove this for any sigraph S in $\mathcal{M}(I\!R^\infty)$.

Case (1) $|X| \le 9$. By Lemma 4.1, A subgraph T of S is SE to T_i, for some $i \le 3$. Let

$$\sigma(S) = \sum_{x \in X} d(x, T).$$

We can assume that

(a) for any sigraph S' in $\mathcal{M}(I\!R^\infty)$ such that $|V(S')| < |X|$ the conclusion holds and

(b) among the sigraphs in $\mathcal{M}(I\!R^\infty)$ for which the conclusion does not hold, S has been chosen such that $\sigma(S)$ is as minimum as possible.

Then by (b)

(c) No subgraph of S is SE to $T_i, 3 \le i \le 6$.

Let $\max_{x \in X} d(x, T) = n+1$. By Lemma 3.6 and (b), $n > 0$. Let $a_1, a_2 \in X$ such that $d(a_1, T) + 1 = d(a_2, T) = n + 1$ and $\phi(a_1, a_2) = \pm 1$. Let us prove that the conclusion does not hold for $R = S(a_1 : a_2)$. Suppose $A \subseteq V(R)$ such that $R[A] \in \mathcal{R}_0(I\!R^\infty)$. Assume that $\phi(a_1, a_2) = 1$. Then $a_3 = a_1 - a_2 \in A$. $a_1 \notin A$ for otherwise $S[(A - a_3) \cup a_2] \in \mathcal{R}_0(I\!R^\infty)$. Now it follows that $A = V(R) - a_1$ and $R[A]$ is connected for otherwise the conclusion would hold for S.

If $R - \{a_1, a_3\}$ is disconnected, then there is a vertex $a \in V(R)$ adjacent to a_3 but not to any one of $\{x \in (X - a_1) | d(x, T) \le n\}$. Therefore it follows that $\phi(a_1, a) = \pm 1$ and $\phi(a_2, a) = 0$. Let P be a shortest path joining a_1 to T. By using (c), it can be verified that $S[V(T) \cup V(P) \cup \{a, a_2\}] \in \mathcal{R}_0(D_\infty)$ either directly or by showing that this subgraph is AE to the first graph in Figure 3.1; i.e. the conclusion holds for S. A contradiction. Therefore

(d) $R - \{a_1, a_3\}$ is connected.

Let ξ be a RPN of $S - \{a_1, a_2\} = R - \{a_1, a_3\}$. Since $S - a_i, i = 1, 2$ and $R - a_1 \in \mathcal{R}(I\!R^\infty)$, by Proposition 2.2 there exist vectors $v_i, i = 1, 2, 3$ of norm $\sqrt{2}$ such that for all $x \in X - \{a_1, a_2\}$, $< v_i, \xi(x) >= \phi(a_i, x)$. Then for all $x \in X - \{a_1, a_2\}$, $< v_1 - v_2, \xi(x) >=< v_3, \xi(x) >$. Since $v_3 \in \mathcal{L}(\xi(X - \{a_1, a_2\}))$, we get

(e) $\quad < v_1 - v_2, v_3 >=< v_3, v_3 >= 2.$

By (d) and Lemma 3.7, $v_3 \in \mathcal{I}(\xi(X - \{a_1, a_2\}))$. Therefore $< v_1, v_3 >$ is in $\mathbb{Z}$. $< v_1, v_3 >\neq \pm 2$ for otherwise $v_1 \in \mathcal{L}(\xi(X - \{a_1, a_2\}))$ and the conclusion would hold for S. Further, since $R \notin \mathcal{R}(I\!R^\infty)$, it follows that $< v_1, v_3 >= 0$ or -1. Now by (e), $< v_2, v_3 >= -2$ or -3. Therefore $v_2 = -v_3$ and $v_2 \in \mathcal{L}(\xi(X - \{a_1, a_2\}))$. A contradiction. Therefore the conclusion does not hold for R.

Now by (a), $R \in \mathcal{M}(I\!R^\infty)$; but $\sigma(R) = \sigma(S) - 1$, contradicting the choice of S in (b). Therefore the conclusion holds in this case.

Case (2) $|X| \geq 10$. We can assume, by Lemma 4.3, that there exists $A \subseteq X$ such that $|A| = 6$ and $S[A] \in \mathcal{M}(D_\infty)$. Let $B \subseteq X$ such that $A \subseteq B, |B| = 9$ and $S[B]$ is connected. By minimality of $S, S[B] \in \mathcal{R}(I\!R^\infty)$ and it follows from Theorem 1.1, that $S[B]$ has an LDR in E_8 for $S[B] \notin \mathcal{R}(D_\infty)$ and $dimE_8 = 8$. This completes the proof.

Theorem 1.3 is a consequence of the following two propositions and Theorem 1.2.

Proposition 4.4 $S \in \mathcal{R}(I\!R^\infty) \Longrightarrow \phi^*$ is linear.

Proof: Let ξ be a RPN of S. Suppose $A \subseteq X$ and there are scalars $\alpha_x, x \in A$ such that for all $t \in A$,

$$\sum_{x\in A} \alpha_x \phi^*(x,t) = 0$$

Then

$$\begin{aligned}\Big\|\sum_{x\in A} \alpha_x\xi(x)\Big\|^2 &= \sum_{x,y\in A} \alpha_x\alpha_y < \xi(x), \xi(y) > \\ &= \sum_{x,y\in A} \alpha_x\alpha_y\phi^*(x,y) \\ &= \sum_{x\in A} \alpha_x \sum_{y\in A} \alpha_y\phi^*(x,y) \\ &= 0.\end{aligned}$$

Therefore $\sum_{x\in A} \alpha_x\xi(x) = 0$.

Hence for all $t \in X, \sum_{x\in A} \alpha_x < \xi(x), \xi(t) >= 0$;

i.e.

$$\sum_{x\in A} \alpha_x\phi^*(x,t) = 0.$$

Therefore ϕ^* is linear.

Proposition 4.5 If $S \in \mathcal{M}(I\!R^\infty)$ and $A \subseteq X$ such that $S[A] \in \mathcal{R}_0(I\!R^\infty)$, then ϕ^* is not linear and $|X| = |A| + 1$.

Proof: Let ξ be a RPN of $S[A]$. By hypothesis, there are scalars α_x, $x \in A$, such that $\alpha_p \neq 0$ for some $p \in A$ and

$$(a) \qquad \sum_{x\in A} \alpha_x\xi(x) = 0.$$

Therefore

$$(b) \qquad \sum_{x\in A} \alpha_x\phi^*(x,a) = 0, \text{ for all } a \in A.$$

By Proposition 2.2, $S-p$ has a RPN η such that η and ξ coincide on $A-p$. Since $S \notin \mathcal{R}(\mathbb{R}^\infty)$, for some $q \in (X-p)$,

$$(c) \qquad < \xi(p), \eta(q) > \neq \phi(p,q).$$

Now by (a), $\sum_{x \in A} \alpha_x < \xi(x), \eta(q) >= 0$

i.e. $\sum_{x \neq p} \alpha_x \phi^*(x,q) + \alpha_p < \xi(p), \eta(q) >= 0$ (since ξ and η coincide on $A-p$.).

Since $\alpha_p \neq 0$, by (c) we get $\sum_{x \in A} \alpha_x \phi^*(x,q) \neq 0$.

Therefore by (b), linearity does not hold for $S[A \cup q]$ and by the last proposition and minimality of $S, |X| = |A| + 1$. This completes the proof.

Now we describe $\mathcal{M}(\mathbb{R}^\infty)$. Let $\mathcal{F}$ be the collection of all sigraphs S satisfying the following:
(a) $|X| \leq 10$.
(b) ϕ^* is not linear.
(c) For some $a \in X, S - a \in \mathcal{R}_0(\mathbb{R}^\infty)$.

Then by the above proposition and Theorem 1.2, clearly $\mathcal{M}(\mathbb{R}^\infty)$ is the subcollection of sigraphs S in $\mathcal{F}$ such that no proper subgraph of S is in $\mathcal{F}$.

Now we prove Theorem 1.8: By Remark 2.8, $R_6 \notin \mathcal{R}(D_\infty)$. Therefore for any sigraph S in $\mathcal{R}(D_\infty)$, by (2.11.2), (1) holds and (2) follows from Theorem 1.3.

Next it is enough to show for a sigraph S in $\mathcal{M}(D_\infty)$, (1) or (2) does not hold. If there exists $Y \subseteq X$ such that $|Y| \leq 5$ and $S[Y] \in \mathcal{R}_0(\mathbb{R}^\infty)$, then by Corollary 3.2 and minimality of $S, S \in \mathcal{M}(\mathbb{R}^\infty)$ and by Proposition 4.5, $|X| = |Y| + 1 \leq 6$. This bound holds, otherwise also by Lemma 4.3. Now if $S \in \mathcal{R}^*(\mathbb{R}^\infty)$, by Theorem 1.5, S is AE to R_6 and (1) does not hold. Otherwise $S \in \mathcal{M}(\mathbb{R}^\infty)$. Therefore by Theorem 1.3, linearity does not hold for S; further by Lemma 4.1, a subgraph $S[A]$ is SE to some $T_i, 1 \leq i \leq 3$ and (by Proposition 4.5, when $i = 3$) $|X - A| \leq 2$. Hence (2) does not hold. This completes the proof.

Corollary 4.6 Any sigraph in $\mathcal{M}(D_\infty)$ has at most 6 vertices.

Next we prove Theorem 1.9: Since $P_9, Q_9 \in \mathcal{R}^*(\mathbb{R}^\infty)$ and $dim E_8 = 8$, it follows that $P_9, Q_9 \notin \mathcal{R}(E_8)$. Therefore for any sigraph in $\mathcal{R}(E_8)$, by (2.11.2), (1) holds and (2) follows from Theorem 1.3.

Now suppose (1) and (2) hold for a sigraph S. We can assume that $S \in \mathcal{R}(D_\infty)$ by Theorems 1.1 and 1.3 and for all $a \in X, S - a \in \mathcal{R}(E_8)$. Then S is connected. If $S \in \mathcal{R}_0(\mathbb{R}^\infty)$, then by Theorem 1.4 and Proposition 2.5, $S \in \mathcal{R}(E_8)$. Otherwise, by Theorem 1.5 S is AE to P_n or $Q_n, n = |X|$. Further $n \leq 8$ for otherwise any connected subgraph of S of order 9 would be AE to P_9 or Q_9. Since $P_8, Q_8 \in \mathcal{R}(E_8)$, by (2.11.2) $S \in \mathcal{R}(E_8)$. This completes the proof.

Corollary 4.7 A sigraph is in $\mathcal{M}(E_8)$ if and only if either it is in $\mathcal{M}(\mathbb{R}^\infty)$ and no subgraph is an AE of P_9 or Q_9 or it is an AE of P_9 or Q_9.

Note that any sigraph in $\mathcal{M}(\mathbb{R}^\infty)$ of order < 10 is also in $\mathcal{M}(E_8)$.

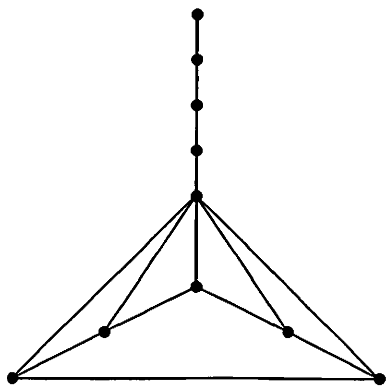

Figure 4.8 A graph in $\mathcal{M}(\mathbb{R}^\infty) \cap \mathcal{M}(E_8)$ of order 10

Finally we prove Theorem 1.10: Let S be a sigraph generating E_n (n is 7 or 8). By Theorem 1.4, a subgraph $S[Y]$ in $\mathcal{R}^*(\mathbb{R}^\infty)$ generates E_n. Clearly $S[Y] \notin \mathcal{R}(D_\infty)$ and by Lemma 4.3, there exists $Z \subseteq Y$ such that $|Z| = 6$ and $S[Z] \in \mathcal{M}(D_\infty)$. Since $\mathcal{I}(E_n)$ is irreducible, $S[Y]$ is connected and therefore we can choose a set A such that $Z \subseteq A \subseteq Y, |A| + 1 = |Y|$ and $S[A]$ is connected. Then $S[A] \in \mathcal{R}^*(\mathbb{R}^\infty) - \mathcal{R}(D_\infty)$ and $|A| = n - 1$; therefore by Witt's theorem, it generates E_{n-1}. This completes the proof.

Concluding Remarks

Motivation for formulating the central concept of this paper has come from [$V3$] (its main result is that any sigraph in $\mathcal{M}(E_8)$ has at most 10 vertices.). While the author has been writing that and developing related concepts and notions, the problem of giving computer-free proof of the fact that a graph G with least eigenvalue < -2 contains a subgraph of order ≤ 9 with least eigenvalue $= -2$ has been brought to his attention, by [BN] and led to the formation of Theorems 1.2 and 1.3.

The notion of linearity is not new. It is a modified form of linear relational property introduced in [V2]. Theorem 1.8 is an improvement of the main theorem in [V2]. Propositions 4.4 and 4.5 have been proved there in different forms.

It has been observed in [CDS] that there is no minimal forbidden graph having least eigenvalue -2 for the family of generalized line graphs. The same result holds for $\mathcal{R}(\Lambda)$ also. Motivated by this fact, the reason for this has been found: That is Theorem 1.4.

To see how closely the graphs of two hereditary families resemble in structure, one approach is to determine whether the intersection of their families of minimal forbidden graphs is sufficiently large or not. In this sense, the significance of Corollary 4.7 is that to investigate the properties of the family of graphs with all eigenvalues ≥ -2, the natural candidates are not generalized line graphs but the graphs represented by E_8 (Because of this reason, deriving Theorem 1.9 has been easier than proving Theorem 1.8.).

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