TRUNCATIONS OF THE PASCAL TRIANGLE AND WEIGHTED-PRODUCT THEOREMS

Hazel Perfect

University of Sheffield
Sheffield, ENGLAND

1. Introduction.

Our purpose here is to obtain some simple properties of tables constructed by the method of 'truncation' of the familiar Pascal triangle. The tables are intimately related to the symmetric-chain partitions of a power set. As is well known, the existence of a symmetric-chain partition provides one of the simplest and most transparent proofs of Sperner's theorem; and it was a two-part Sperner theorem of Kleitman [3] and Katona [2] (see Section 3) which particularly motivated the investigation of the weighted-product theorems for these tables which we present in Section 2 below.

Throughout, all sets considered are finite.

We refer the reader to page 27 of [1] for the definitions of a symmetric chain and a symmetric-chain partition in an arbitrary ranked poset. Let $S_0 = \emptyset$, $S_m = \{1, \dots, m\}, m = 1, 2, \dots$. The well-known inductive procedure for a symmetric-chain partition of the power set $\mathcal{P}(S_m)$ depends upon the construction, for each symmetric chain C in such a partition of $\mathcal{P}(S_{m-1})$, of two symmetric chains C_1, C_2, of a partition of $\mathcal{P}(S_m)$, where $|C_1| = |C| + 1, |C_2| = |C| - 1$ ($C_2 = \emptyset$ if $|C| = 1$). Thus, symmetric-chain partitions of $\mathcal{P}(S_m)$ are constructed by what may be described as a '± 1-procedure', and the number of non-empty chains of cardinality n is evidently given by the entry in row m and column n of the table

m \ n	1	2	3	4	...			
0	1							
1	·	1						
2	1	·	1					
3	·	2	·	1				
4	2	·	3	·	1			
5	·	5	·	4	·	1		
6	5	·	9	·	5	·	1	
7	·	14	·	14	·	6	·	1
	...							

Here each entry is equal to the sum of those (one or two) entries immediately diagonally above it.

Since the total number of non-empty chains in a symmetric-chain partition of $\mathcal{P}(S_m)$ is equal to the number of subsets of S_m of cardinality $[\frac{m}{2}]$, Sperner's

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theorem [7] (see also [1] page 2) follows at once from the existence of a symmetric-chain partition.

Let us abstract the idea of a ± 1-procedure and call any partition of a set of 2^m elements *admissible* if it is constructed inductively by a ± 1-procedure, where we start with a single block of size 1 in the case $m = 0$. As so often, the abstract approach is fruitful. A mention of just two applications will suffice to illustrate this. In [4] (see also [1] page 183) Kleitman gave an elegant proof of a generalized form of the Littlewood-Offord theorem, viz

Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space and $a_1, \ldots, a_m$ members of $\mathcal{H}$ with each $\|a_i\| \geq 1$, and consider all formal sums $\sum_{i=1}^{m} \varepsilon_i a_i$, where each ε_i is 0 or 1. At most $\binom{m}{[\frac{m}{2}]}$ of these have sum-vectors which lie inside a hypersphere of diameter 1 unit.*

Kleitman established this by proving the existence of an admissible partition of the set $\{\sum_{i=1}^{m} \varepsilon_i a_i : \varepsilon_i \in \{0, 1\}\}$ with the additional property that, in any block, every two members are distant at least 1 unit apart.

The second application is to the problem of dominant sequences. We shall say that an m-sequence $(\varepsilon_1, \ldots, \varepsilon_m)$, where each $\varepsilon_i = \pm 1$, is *dominant* if every partial sum $\sum_{i=1}^{k} \varepsilon_i$ $(1 \leq k \leq m)$ is non-negative. It is well known[1] that the number of dominant m-sequences is equal to $\binom{m}{[\frac{m}{2}]}$. Perhaps an argument depending upon admissible partitions is not so well known. It is, however, simple to prove inductively the existence of an admissible partition of the set $\{(\varepsilon_1, \ldots, \varepsilon_m) : \varepsilon_i = \pm 1\}$ having the additional properties (i) each block of the partition contains just one dominant sequence, and (ii) if the dominant sequence in a block of cardinality s is $(\varepsilon_1, \ldots, \varepsilon_m)$, then $\sum_{i=1}^{m} \varepsilon_i = s - 1$. The result then follows at once.

2. Truncations of the Pascal triangle.

In the Pascal triangle a typical entry is equal to the sum of the two entries immediately diagonally above it. If the rows are labelled $0, 1, 2, \ldots$ from the apex, then the m-th row sum is equal to 2^m and, as a consequence, the p-th row sum multiplied by the q-th row sum is equal to the $(p+q)$-th row sum. The analogue of this simple multiplicative property of row sums for our truncated tables is stated in the weighted-product theorems below.

Let us, then, consider the Pascal triangle blocked or 'truncated' on the left-hand side, in such a way that the blocking line is to the left of the apex and has just r 1's immediately to the right including the apex. We write out the case $r = 4$ to

[1] See Amer. Math. Monthly *70* page 1005, *71* page 797.

clarify this and to indicate that the table is

m\n	1	2	3	4	5	...					
0				1							
1			1	·	1						
2		1	·	2	·	1					
3	1	·	3	·	3	·	1				
4	·	4	·	6	·	4	·	1			
5	4	·	10	·	10	·	5	·	1		
6	·	14	·	20	·	15	·	6	·	1	
7	14	·	34	·	35	·	21	·	7	·	1
	...										

blocking line

constructed in a similar way to the Pascal triangle itself: specifically, each entry is the sum of those (one or two) entries immediately diagonally above it. In the case $r = 1$, the table constructed is the one given in Section 1 above. Blocking lines to right of the apex (with the truncation as before on the *left*) only reproduce the $r = 1$ table and will, therefore, not concern us further. For convenience, we define the table $r = 0$ to have every entry equal to zero.

The main properties of our tables may be established by a consideration of set-theoretic interpretations (see Section 3). Instead, so that this section shall be self-contained, we indicate simple ad hoc proofs.

Lemma 2.1. *In table r, $r \geq 0$, the only possibly non-zero entries occur in the $m+r-2k$ positions in row m, where $0 \leq k < \frac{1}{2}(m+r)$, and the $(m, m+r-2k)$ entry is equal to*

$$\binom{m}{k} - \binom{m}{k-r}.$$

The proof is by straightforward induction with respect to m. (The usual conventions for binomial coefficients with negative entries are adopted.)

In table r we shall denote by $a_{mn}^{(r)}$ the entry in row m and column $n(m \geq 0, n \geq 1)$ and by $R_m^{(r)}$ the m-th row sum.

Lemma 2.2. *For $r \geq 2$*

$$a_{mn}^{(r)} = a_{m+1,n}^{(r-1)} - a_{mn}^{(r-2)} \tag{2.1}$$

(and consequently

$$R_m^{(r)} = R_{m+1}^{(r-1)} - R_m^{(r-2)}. \tag{2.2}$$

The proof depends upon Lemma 2.1 and proceeds by induction with respect to r.

Repeated application of the relations (2.2) leads to equations (for $r \geq 1$)

$$R_m^{(r)} = c_1^{(r)} R_{m+r-1}^{(1)} + c_2^{(r)} R_{m+r-3}^{(1)} + \dots + \begin{cases} c_{\frac{r+1}{2}}^{(r)} R_m^{(1)} & (r \text{ odd}) \\ c_{r/2}^{(r)} R_m^{(1)} & (r \text{ even}) \end{cases} \tag{2.3}$$

with similar ones for the $a_{mn}^{(r)}$. In the table below $c_k^{(r)}$ appears in column k of row r; and for $r \geq 3, k \geq 2, c_k^{(r)} = c_k^{(r-1)} - c_{k-1}^{(r-2)}$;

r \ k	1	2	3	...	
1	1				
2	1				
3	1	−1			
4	1	−2			
5	1	−3	1		
6	1	−4	3		
7	1	−5	6	−1	
8	1	−6	10	−4	
9	1	−7	15	−10	1

(2.4)

and from this we obtain inductively

Lemma 2.3. *The (r, k) entry $c_k^{(r)}$ in the table (2.4) is equal to*

$$(-1)^{k-1} \binom{r-k}{k-1}$$

provided $r \geq 2k - 1$ and is zero otherwise.

We turn now to a consideration of the weighted-product theorems.

Definition: The *weighted product* (WP) of row p of table r and row q of table s is equal to

$$\sum_{i,j} a_{pi}^{(r)} a_{qj}^{(s)} \min(i, j).$$

Theorem 2.1. *(The first WP theorem) For table $r = 1$, the WP of rows p, q is equal to the $(p + q)$-th row sum (and so depends only upon $p + q$).*

Proof: Without loss of generality we may assume that $p \geq q$. We give the details only for the case when p, q are both even. (It is fairly typical.) So, from Lemma 2.1,

$$\sum_{i,j} a_{pi}^{(1)} a_{qj}^{(1)} \min(i,j)$$

$$= \sum_{k=0}^{p\backslash 2} \sum_{\ell=0}^{q\backslash 2} \left(\binom{p}{k} - \binom{p}{k-1}\right)\left(\binom{q}{\ell} - \binom{q}{\ell-1}\right) \min(p+1-2k,\ q+1-2\ell)$$

$$= \sum_{k=0}^{p\backslash 2} \sum_{\ell=0}^{q\backslash 2} \left(\binom{p}{k} - \binom{p}{k-1}\right)\left(\binom{q}{\ell} - \binom{q}{\ell-1}\right)$$

$$+ 2 \sum_{k=0}^{p/2-1} \sum_{\ell=0}^{q/2-1} \left(\binom{p}{k} - \binom{p}{k-1}\right)\left(\binom{q}{\ell} - \binom{q}{\ell-1}\right) + \dots$$

$$+ 2 \sum_{k=0}^{p/2-q/2} \sum_{\ell=0}^{0} \left(\binom{p}{k} - \binom{p}{k-1}\right)\left(\binom{q}{\ell} - \binom{q}{\ell-1}\right)$$

$$= \binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}} + 2\binom{p}{\frac{p}{2}-1}\binom{q}{\frac{q}{2}-1} + \dots + 2\binom{p}{\frac{p-q}{2}}\binom{q}{0}$$

$$= \binom{p}{\frac{p+q}{2}}\binom{q}{0} + \binom{p}{\frac{p+q}{2}-1}\binom{q}{1} + \dots + \binom{p}{\frac{p}{2}}\binom{q}{\frac{q}{2}}$$

$$+ \dots + \binom{p}{\frac{p}{2}-1}\binom{q}{\frac{q}{2}+1} + \dots + \binom{p}{\frac{p-q}{2}}\binom{q}{q}$$

$= \binom{p+q}{\frac{p+q}{2}}$ (by Vandermonde convolution), and the $(p+q)$-th row sum

$$= \sum_{k=0}^{(p+q)/2} \left(\binom{p+q}{k} - \binom{p+q}{k-1}\right) = \binom{p+q}{\frac{p+q}{2}}.$$

Theorem 2.2. *(The second* WP *theorem)* *The* WP *of row p of table r and row q of table 1 is equal to the $(p+q)$-th row sum of the table r (and so depends only upon $p+q$).*

Proof: For $r = 0$ the result ia trivially true and for $r = 1$ it follows from Theorem 2.1. We shall suppose $r \geq 2$ and assume the result for $r-1, r-2$. Then, from Lemma 2.2,

$$\sum_{i,j} a_{pi}^{(r)} a_{qj}^{(1)} \min(i,j)$$

$$= \sum_{i,j} a_{p+1,i}^{(r-1)} a_{qj}^{(1)} \min(i,j) - \sum_{i,j} a_{pi}^{(r-2)} a_{qj}^{(1)} \min(i,j)$$

$$= R_{p+1+q}^{(r-1)} - R_{p+q}^{(r-2)} = R_{p+q}^{(r)}.$$

Theorem 2.3. *(The third WP theorem) The WP of row p of table r and row q of table s is equal to*

$$\sum_{k=1}^{[\frac{s+1}{2}]} (-1)^{k-1} \binom{s-k}{k-1} R^{(r)}_{p+q+s+1-2k} \tag{2.5}$$

(dependent only upon $p+q$).

Proof: To cover the case $s = 0$ we assume that a vacuous sum is zero. For $s \neq 0$, we have

$$\begin{aligned} &\sum_{i,j} a^{(r)}_{pi} a^{(s)}_{qj} \min(i,j) \\ &= \sum_{i,j} a^{(r)}_{pi} \left(c^{(s)}_1 a^{(1)}_{q+s-1,j} + c^{(s)}_2 a^{(1)}_{q+s-3,j} + \ldots \right) \min(i,j) \\ &+ c^{(s)}_1 R^{(r)}_{p+q+s-1} + c^{(s)}_2 R^{(r)}_{p+q+s-3} + \ldots ; \end{aligned}$$

which, by Lemma 2.3, is equal to (2.5).

Since this sum is dependent only upon $p+q$, evidently r and s may be interchanged in (2.5).

3. Interpretations of the tables.

We have already seen what is the connection between table $r = 1$ and the symmetric-chain partitions of $\mathcal{P}(S_m)$. The entry in the m-th row and the n-th column is equal to the number of chains of size n in a symmetric-chain partition of $\mathcal{P}(S_m)$. We turn now to Kleitman and Katona's two-part Sperner theorem [3, 2] (see also [1] page 179) and view it in the light of the first WP theorem.

Theorem 3.1. *(Two-part Sperner theorem) Let P, Q be disjoint subsets of S_m with $|P| = p$, $|Q| = q$, $p + q = m$. A collection $\mathcal{F}$ of subsets of S_m containing no two distinct members F, G with $F \subseteq G$ and $G \backslash F \subseteq P$ or $G \backslash F \subseteq Q$ has cardinality at most equal to*

$$\binom{m}{[\frac{m}{2}]}.$$

This is a somewhat surprising result since the defining property of $\mathcal{F}$ is in general weaker than the antichain property. The upper bound of the theorem is of course attained. The nub of the argument is the claim that, if Π, Π' are symmetric-chain partitions of $\mathcal{P}(P)$, $\mathcal{P}(Q)$ resp., then

$$\sum_{\substack{C \in \Pi \\ C' \in \Pi'}} \min(|C|, |C'|) = \binom{m}{[\frac{m}{2}]}; \tag{3.1}$$

established by counting occurrences of $[\frac{m}{2}]$-subsets. But

$$\sum_{\substack{C\in\Pi \\ C'\in\Pi'}} \min(|C|,|C'|) = \sum_{i,j} a_{pi}^{(1)} a_{qj}^{(1)} \min(i,j)$$

$$= \binom{p+q}{[\frac{p+q}{2}]} = \binom{m}{[\frac{m}{2}]}$$

by Theorem 2.1.

There appears to be no obvious WP of three rows, p, q, t (say), of table 1 (or table r) which depends only upon the sum $p+q+t$; and so we would not expect a 3-part Sperner theorem of similar form to be valid.

Our more abstract derivation of (3.1) at once suggests other two-part generalizations not involving power sets; for example, the two-part version of Theorem 1.1, the formulation of which we leave to the reader.

It is quite simple to give a set-theoretic interpretation of table r. To this end, let us denote by $\mathcal{P}_r(S_m)$ the direct product of the totally ordered set $\{O, 1, \dots, r-1\}$ and $\mathcal{P}(S_m)$ (under inclusion), $r \geq 1$. The poset $\mathcal{P}_r(S_m)$ is, of course, isomorphic to the poset of $(m+1)$-sequences $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_m)$ under componentwise order, where $0 \leq \varepsilon_0 \leq r-1$, $0 \leq \varepsilon_i \leq 1$ for $1 \leq i \leq m$; and in particular $\mathcal{P}_1(S_m)$ is isomorphic to $\mathcal{P}(S_m)$ and $\mathcal{P}_2(S_m)$ is isomorphic to $\mathcal{P}(S_{m+1})$. Since $\{O, 1, \dots, r-1\}$ and $\mathcal{P}(S_m)$ possess symmetric-chain partitions so does $\mathcal{P}_r(S_m)$. When $m = 0$ this consists of a single chain of length r; and it is readily seen that a symmetric-chain partition of $\mathcal{P}_r(S_m)$ is constructed inductively by a ± 1-procedure but starting from a single block of size r. Therefore, the entry in the m-th row and the n-th column of table r is equal to the number of chains of size n in a symmetric-chain partition of $\mathcal{P}_r(S_m)$. This interpretation will obviously lead to alternative proofs of the results of Section 2 as already remarked upon. Also, the generalization of Theorem 3.1 to multisets, which may be proved in a similar way to Theorem 3.1 itself (the nub of the argument by counting occurrences of multisets of middle rank), implies more general WP theorems than the ones we have presented here for the truncated Pascal triangle. However, within its limits, our ad hoc approach provides an abstract framework and may suggest applications in other directions.

4. Some identities.

The tables in Section 2 allow us to establish some identities involving binomial coefficients. Some are well known, others perhaps less so. We restrict ourselves to a brief mention of two related types.

From Lemma 2.1 we may evidently write down a simple formula for the m-th row sum in table r. The use of Lemma 2.3, together with equation (2.3), leads to a different expression. We consider this, and give details only when $m+r$ is odd. When r is odd, this expression for the m-th row is

$$\sum_{k=1}^{\frac{r+1}{2}}(-1)^{k-1}\binom{r-k}{k-1}R^{(1)}_{m+r+1-2k}$$
$$=\sum_{k=1}^{\frac{r+1}{2}}(-1)^{k-1}\binom{r-k}{k-1}\binom{m+r+1-2k}{\frac{1}{2}(m+r+1-2k)}.$$

In terms of s,p, where $m+r=2s+1, r=2p+1$ $(s\geq p)$, this becomes

$$\sum_{k=1}^{p+1}(-1)^{k-1}\binom{2p+1-k}{k-1}\binom{2s+2-2k}{s+1-k}$$

or, slightly more conveniently,

$$\sum_{k=0}^{p}(-1)^{k}\binom{2p-k}{k}\binom{2s-2k}{s-k}.$$

When r is even, the formula for the m-th row sum is

$$\sum_{k=1}^{\frac{r}{2}}(-1)^{k-1}\binom{r-k}{k-1}\binom{m+r+1-2k}{\frac{1}{2}(m+r+1-2k)}.$$

In terms of s,p, where $m+r=2s+1, r=2p+2$ $(s>p)$, this becomes

$$\sum_{k=1}^{p+1}(-1)^{k-1}\binom{2p+2-k}{k-1}\binom{2s+2-2k}{s+1-k}$$
$$=\sum_{k=0}^{p}(-1)^{k}\binom{2p+1-k}{k}\binom{2s-2k}{s-k)}.$$

Since the m-th row sum in table r is equal to 2^m if $m<r$, we have

$$\sum_{k=0}^{p}(-1)^{k}\binom{2p-k}{k}\binom{2s-2k}{s-k}=2^{2s-2p}$$

for $p\leq s\leq 2p$, and

$$\sum_{k=0}^{p}(-1)^k\binom{2p+1-k}{k}\binom{2s-2k}{s-k} = 2^{2s-2p-1}$$

for $p+1 \le s \le 2p+1$. These combine to give the identity

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n-k}{k}\binom{2s-2k}{s-k} = 2^{2s-n} \tag{4.1}$$

for $\left[\frac{n+1}{2}\right] \le s \le n$. For the case $s = n$, see, for example, page 37 and page 66 of [6].

The companion to (4.1), corresponding to the case when $m + r$ is even, is

$$\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n-k}{k}\binom{2s+1-2k}{s-k} = 2^{2s-n+1} \tag{4.2}$$

for $\left[\frac{n}{2}\right] < s \le n$.

Next, we turn to some identities derivable from the third WP theorem. We recall that the WP of row p of table r and row q of table s is equal to

$$\sum_{k=1}^{[\frac{s+1}{2}]}(-1)^{k-1}\binom{s-k}{k-1}R^{(r)}_{p+q+s+1-2k}.$$

Since, given r, s, this depends only on $p+q$, therefore, with $p+q=m$ it is equal to the WP of row m of table r and row 0 of table s, which in turn is equal to

$$\sum_{i=0}^{[\frac{1}{2}(m+r-1)]}\left[\binom{m}{i}-\binom{m}{i-r}\right]\min(m+r-2i, s)$$

(from Lemma 2.1). We substitute for the row sums in table r (from (2.3)) and look at the special cases when $m = 0, 1$. First, then, when $m = 0$ we have

$$\begin{aligned}
&\sum_{k=1}^{[\frac{s+1}{2}]}(-1)^{k-1}\binom{s-k}{k-1}\sum_{j=1}^{[\frac{r+1}{2}]}(-1)^{j-1}\binom{r-j}{j-1}R^{(1)}_{r+s+2-2(j+k)}\\
&= \sum_{i=0}^{[\frac{1}{2}(r-1)]}\left[\binom{0}{i}-\binom{0}{i-r}\right]\min(r-2i, s)\\
&= \min(r, s).
\end{aligned}$$

Simple changes of notation yield the identity in the slightly more convenient form

$$\sum_{j=0}^{[r/2]}\sum_{k=0}^{[s/2]}(-1)^{j+k}\binom{r-j}{j}\binom{s-k}{k}\binom{r+s-2(j+k)}{\left[\frac{r+s-2(j+k)}{2}\right]} = \min(r+1, s+1).$$

It is not difficult to deal with the case $m = 1$ similarly and obtain

$$\sum_{j=0}^{[r/2]}\sum_{k=0}^{[s/2]}(-1)^{j+k}\binom{r-j}{j}\binom{s-k}{k}\binom{r+s+1-2(j+k)}{\left[\frac{r+s+1-2(j+k)}{2}\right]} = 2\min(r+1, s+1) - \delta_{r,s}.$$

There are evident connections with well-known identities associated with Chebyshev polynomials. Page 6 and page 59 of [6], and Problem 39 of [5] are relevant.

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