CYCLIC MENDELSOHN QUADRUPLE SYSTEMS*

B.MICALE and M.PENNISI

Department of Mathematics - University of Catania

ABSTRACT

In this paper we examine the existence problem for cyclic Mendelsohn quadruple systems (briefly CMQS) and we prove that a CMQS of order v exists if and only if $v \equiv 1 \pmod 4$. Further we study the maximum number $m_4(v)$ of pairwise disjoint (on the same set) CMQS's of order v each having the same v-cycle as an automorphism. We prove that, for every $v \equiv 1 \pmod 4$, $2v - 8 \leq m_4(v) \leq v^2 - 11v + z$, where $z = 32$ if $v \equiv 1$ or $5 \pmod{12}$ and $z = 30$ if $v \equiv 9 \pmod{12}$, and that $m_4(5) = 2$, $m_4(9) = 12$, $50 \leq m_4(13) \leq 58$.

1. INTRODUCTION

Let V be a finite set and let $x_1, x_2, \ldots, x_k, k \geq 3$, be distinct elements of V. The set

$$[x_1, x_2, \ldots, x_k] = \{(x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, x_1)\}$$

will be called *Mendelsohn k-tuple* on V.

Obviously:

$$[x_1, x_2, \ldots, x_k] = [x_2, \ldots, x_k, x_1] = \ldots = [x_k, x_1, \ldots, x_{k-1}].$$

A $2-(v, k, \lambda)$ *Mendelsohn design* is a pair (V, B), where $|V| = v$ and B is a collection of Mendelsohn k-tuples on V, called *blocks*, such that every ordered pair of distinct elements of V belongs to exactly λ blocks of B.

Mendelsohn designs have been objects of considerable interest in recent years (see references).

A $2-(v, k, 1)$ Mendelsohn design will be denoted by $M(k, v)$.

It is easy to see that if (V, B) is a $M(k, v)$, then

$$|B| = \frac{v(v-1)}{k}.$$

It follows that *a necessary condition for the existence of $M(k, v)'s$ is* $v(v-1) \equiv 0 \pmod k$.

A $M(3, v)$ is called *Mendelsohn triple system of order v*, briefly MTS(v). N.S.MENDELSOHN proved in [12] that the spectrum of $M(3, v)$'s is the set of all $v \equiv 0$ or $1 \pmod 3$, except $v = 6$.

* Work supported by M.U.R.S.T. and G.N.S.A.G.A.

A $M(4,v)$ will be called *Mendelsohn quadruple system of order* v and will be denoted by MQS(v). In [2] N.BRAND and W.C.HUFFMAN have shown that a MQS(v) exists if and only if $v \equiv 0$ or $1 \pmod 4$, $v > 4$.

A $M(k,v)$ is called *cyclic* if it has an automorphism consisting of a single cycle of length v. By $m_k(v)$ we will denote the maximum number of pairwise disjoint (on the same set) cyclic $M(k,v)$'s each having the same v-cycle as an automorphism.

In [4] C.J.COLBOURN and M.J.COLBOURN proved that a cyclic MTS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$, $v \neq 9$. Further, they showed that $m_3(v) \leq v-5$ and $m_3(13) = 8$, $12 \leq m_3(19) \leq 14$, $17 \leq m_3(25) \leq 20$.

One purpose of this paper is to study the existence problem of cyclic MQS(v)'s. In what follows a cyclic MQS(v) will be denoted by CMQS. We prove that a CMQS(v) exists if and only if $v \equiv 1 \pmod 4$.

Another purpose is to study the number $m_4(v)$. We show that for every $v \equiv 1 \pmod 4$, $2v-8 \leq m_4(v) \leq v^2 - 11v + z$, where $z = 32$ if $v \equiv 1$ or $5 \pmod{12}$ and $z = 30$ if $v \equiv 9 \pmod{12}$, and that $m_4(5) = 2$, $m_4(9) = 12$, $50 \leq m_4(13) \leq 58$.

2. EXISTENCE AND CONSTRUCTION OF CMQS(v)'s.

If (V,B) is a CMQS(v), then we may assume that $V = \mathbf{Z}_v$ and that if $b = [b_1, b_2, b_3, b_4] \in B$, then also every block

$$b + n = [b_1 + n, b_2 + n, b_3 + n, b_4 + n],$$

$n \in \mathbf{Z}_v$, belongs to B. We call *orbit* of b the set

$$0(b) = \{b + n : n \in \mathbf{Z}_v\}.$$

Given an ordered pair (a,b) of distinct elements of $\mathbf{Z}_v$, the number $b-a$, belonging to $\mathbf{Z}_v - \{0\}$, will be called the *difference* of (a,b). With each block $b = [b_1, b_2, b_3, b_4] \in B$ one can associate a (cyclically ordered) quadruple of differences:

$$d(b) = (b_2 - b_1, b_3 - b_2, b_4 - b_3, b_1 - b_4),$$

which will be called the *difference quadruple* (briefly *d-quadruple*) of b.

Observe that for $b, b' \in B$ we have $d(b') = d(b)$ if and only if $b' \in 0(b)$.

The set of d-quadruples

$$\overline{B} = \{d(b) : b \in B\}$$

will be called the *difference family* of $(\mathbf{Z}_v, B)$.

THEOREM 1. A CMQS(v) *exists if and only if there exists a set* D *of cyclically ordered quadruples of elements belonging to* $\mathbf{Z}_v - \{0\}$ *such that:*

(1) *every* $a \in \mathbf{Z}_v - \{0\}$ *is contained in exactly one quadruple of* D;
(2) *for every* $(a_1, a_2, a_3, a_4) \in D$:

$$\sum_{i=1}^{4} a_i = 0 \quad \textit{and} \quad \sum_{i=1}^{m} a_i \neq 0 \quad \textit{for every} \quad m = 1, 2, 3.$$

Proof. Let $(\mathbf{Z}_v, B)$ be a CMQS(v) and let $\overline{B}$ be its difference family.

For every $a \in \mathbf{Z}_v - \{0\}$ there exists in B a block $[0, a, x, y]$ and, therefore, there exists in $\overline{B}$ the d-quadruple $(a, x - a, y - x, -y)$. Further, if there exist in B two d-quadruples having a difference in common, $d = (a_1, a_2, a_3, a_4)$ and $d' = (a_1, a_2', a_3', a_4')$, then $b = [0, a_1, a_1 + a_2, a_1 + a_2 + a_3]$ and $b' = [0, a_1, a_1 + a_2', a_1 + a_2' + a_3']$ belong to B; since b and b' have a pair in common, $b = b'$ and therefore $d = d'$. Hence (1) holds.

Now, let $b = (a_1, a_2, a_3, a_4)$ be any d-quadruple of $\overline{B}$ and let

$$(a_1, a_2, a_3, a_4) = d([b_1, b_2, b_3, b_4]).$$

For every $j = 1, 2, 3, 4$, $a_j = b_{j+1} - b_j$, where the indices are taken modulo 4, hence

$$\sum_{j=1}^{m} a_j = b_{m+1} - b_1, \quad \text{for every} \quad m = 1, 2, 3, 4.$$

From this it follows that (2) holds.

Suppose now that there exists a set D of cyclically ordered quadruples of elements belonging to $\mathbf{Z}_v - \{0\}$ such that (1) and (2) hold.

For every $d = (a_1, a_2, a_3, a_4) \in D$ we consider the Mendelsohn quadruples

$$b(n) = [n, a_1 + n, a_1 + a_2 + n, a_1 + a_2 + a_3 + n], \quad n \in \mathbf{Z}_v.$$

From (2) it follows that the elements contained in $b(n)$ are pairwise distinct.

Let B be the set of all blocks $b(n), n \in \mathbf{Z}_v$, obtained when d varies in D. In order to verify that $(\mathbf{Z}_v, B)$ is a CMQS(v) it suffices to prove that every ordered pair (b_1, b_2) of distinct elements of $\mathbf{Z}_v$ belongs to exactly one block of B.

From (1) there exist a_2, a_3, a_4 such that $(b_2 - b_1, a_2, a_3, a_4) \in D$ and, therefore, $(b_1, b_2) \in [b_1, b_2, a_2 + b_2, a_2 + a_3 + b_2] \in B$.

Further, if there exist in B two blocks having a pair in common, $b = [b_1, b_2, b_3, b_4]$ and $b' = [b_1, b_2, b_3', b_4']$, then $d = (b_2 - b_1, b_3 - b_2, b_4 - b_3, b_1 - -b_4) \in D$ and $d' = (b_2 - b_1, b_3' - b_2, b_4' - b_3', b_1 - b_4') \in D$. From (1) it follows $d = d'$ and, therefore, $b_3 = b_3'$, $b_4 = b_4'$, and $b = b'$. ∎

Before determining the spectrum of CMQS(v)'s we prove the following lemmas.

LEMMA 1. *If* $\overline{B}$ *is the difference family of a* CMQS(v), $(\mathbf{Z}_v, B)$, *then every* $d \in \overline{B}$ *is one of the following forms:*

- (a, a, a, a);
- (a, a', a, a'), *with* $a \neq a'$;
- (a_1, a_2, a_3, a_4), *with* $a_i \neq a_j$ *for* $i, j = 1, 2, 3, 4$ *and* $i \neq$ j.

Proof. Let $d = (a_1, a_2, a_3, a_4) \in \overline{B}$. We prove that if $a_1 = a_2$, then $a_2 = a_3 = a_4$, and if $a_1 = a_3$, then $a_2 = a_4$.

Consider $b = [0, a_1, a_1 + a_2, a_1 + a_2 + a_3] \in B$. If $a_1 = a_2$, then $b + (a_2 + a_3 + a_4) = [0, a_1, a_1 + a_3, a_1 + a_3 + a_4] = b$, hence $a_2 = a_3 = a_4$. If $a_1 = a_3$, then $b + (a_3 + a_4) = [0, a_1, a_1 + a_4, 2a_1 + a_4] = b$ and $a_2 = a_4$. ∎

LEMMA 2. *If* $\overline{B}$ *is the difference family of a* CMQS(v), $(\mathbf{Z}_v, B)$, *with* $v \equiv 1 \pmod 4$, *then every d-quadruple of* $\overline{B}$ *is made up of pairwise distinct elements.*

Proof. If $(a, a, a, a) \in \overline{B}$, then every block belonging to 0(b), with $b = [0, a, 2a, 3a]$, contains exactly four pairs having a as difference. Since there are exactly v (ordered) pairs of elements of $\mathbf{Z}_v$ having a as difference and since a is not contained in another d-quadruple of $\overline{B}$, we have $\mid 0(b) \mid = \frac{v}{4}$, and $v \equiv 0 \pmod 4$.

If $(a, a', a, a') \in \overline{B}$, $a \neq a'$, then every block belonging to 0(b), where $b = [0, a, a + a', 2a + a']$, contains exactly two pairs having a as difference. It follows that $\mid 0(b) \mid = \frac{v}{2}$ and therefore $v \not\equiv 1 \pmod 4$. ∎

THEOREM 2. *A* CMQS(v) *exists if and only if* $v \equiv 1 \pmod 4$.

Proof. Suppose that $(\mathbf{Z}_v, B)$ is a CMQS(v) and let $\overline{B}$ be its difference family.

In relation to the classification of the d-quadruples of $\overline{B}$ determined by Lemma 1, for every $d \in \overline{B}$ we set

$$P(d) = \begin{cases} a & \text{if} \quad d = (a, a, a, a) \\ a + a' & \text{if} \quad d = (a, a', a, a') \\ a_1 + a_2 + a_3 + a_4 & \text{if} \quad d = (a_1, a_2, a_3, a_4) \end{cases}$$

where + is the usual addition between integers.

From (2) of Theorem 1 it follows that

$$P(d) = \begin{cases} \dfrac{v}{4} \text{ or } \dfrac{3v}{4} & \text{if} \quad d = (a, a, a, a) \\ \dfrac{v}{2} \text{ or } \dfrac{3v}{2} & \text{if} \quad d = (a, a', a, a') \\ v \text{ or } 2v \text{ or } 3v & \text{if} \quad d = (a_1, a_2, a_3, a_4). \end{cases}$$

Since the elements contained in d belong to $\mathbf{Z}_v - \{0\}$, if $v \equiv 0 \pmod 4$, then

$$\sum_{d \in \overline{B}} P(d) = \frac{tv}{4},$$

with t odd.

On the other hand, from (1) of Theorem 1 it follows that

$$\sum_{d \in \overline{B}} P(d) = \frac{v(v-1)}{2},$$

therefore $t = 2(v-1)$, and t must be even.

Hence, if there exists a CMQS(v), then necessarily $v \equiv 1 \pmod 4$.

Suppose now that $v \equiv 1 \pmod 4$ and let $v = 4h + 1$.

Let $d_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$, $i = 1, 2, \ldots, h$, where $a_{i1} = i$, $a_{i2} = h + {} + i$, $a_{i3} = 4h - i + 1$, $a_{i4} = 3h - i + 1$ and let $D = \{d_i : i = 1, 2, \ldots, h\}$. We verify that (1) and (2) of Theorem 1 hold for D.

In fact, for every $a \in \mathbf{Z}_v - \{0\}$ we have: if $1 \le a \le h$, then $a \in d_a$, if $h + 1 \le a \le 2h$, then $a \in d_{a-h}$, if $2h + 1 \le a \le 3h$, then $a \in d_{3h-a+1}$, if $3h + 1 \le a \le 4h$, then $a \in d_{4h-a+1}$. From this it follows that (1) holds.

Further, for every $i = 1, 2, \ldots, h$, we have:

$$\sum_{j=1}^{4} a_{ij} = 8h + 2, \quad h + 2 \le a_{i1} + a_{i2} \le 3h,$$

$$a_{i2} + a_{i3} = 5h + 1, \; 5h + 2 \le a_{i3} + a_{i4} \le 7h, \; a_{i4} + a_{i1} = 3h + 1,$$

for that also (2) holds.

Hence, from Theorem 1, for every $v \equiv 1 \pmod 4$ there exists a CMQS(v) and this complete the proof. ∎

3. DISJOINT CMQS(v)'s

Two MQS(v)'s (on the same set) are *disjoint* if they have no block in common.

Let $v \equiv 1 \pmod 4$ and let $m_4(v)$ be the maximum number of pairwise disjoint CMQS(v)'s having the same v-cycle as an automorphism.

Since there are $\frac{v(v-1)(v-2)(v-3)}{4}$ Mendelsohn quadruples on v elements and since a MQS(v) contains exactly $\frac{v(v-1)}{4}$ blocks, it follows that

$$m_4(v) \le (v-2)(v-3).$$

In this section we intend to study $m_4(v)$.

It is easy to prove that:

THEOREM 3. *If* (V, B) *and* (V, B') *are two* CMQS(v)'s *having the same v-cycle as an automorphism and if* $\overline{B}$ *and* $\overline{B}'$ *are the respective difference families, then* (V, B) *and* (V, B') *are disjoint if and only if* $\overline{B} \cap \overline{B}' = \emptyset$.

THEOREM 4. *For every* $v \equiv 1 \pmod 4$, $m_4(v) \geq 2v - 8$.

Proof. Let $v = 4h + 1$. Consider the families

$$D_{11} = \{(a_{i1}, a_{i2}, a_{i3}, a_{i4}) : i = 1, 2, \ldots, h\},$$

$$D_{12} = \{(a_{i1}, a_{i4}, a_{i3}, a_{i2}) : i = 1, 2, \ldots, h\},$$

where $a_{i1} = i,\ a_{i2} = h + i,\ a_{i3} = 4h - i + 1,\ a_{i4} = 3h - i + 1$.

The family D_{11} has been studied in Theorem 2, and D_{11} and D_{12} determine two CMQS(v)'s.

Now, for every $j = 1, 2, \ldots, h - 1$, consider the families

$$D^j_{21} = \{(a^j_{i1}, a_{i2}, a^j_{i3}, a_{i4}) : i = 1, 2, \ldots, h\},$$

$$D^j_{22} = \{(a^j_{i1}, a_{i4}, a^j_{i3}, a_{i2}) : i = 1, 2, \ldots, h\},$$

where $a^j_{i1} = i \oplus j$ and $a^j_{i3} = 4h + 1 - (i \oplus j)$, $i \oplus j$ being a sum modulo h.

From Theorem 1 it follows that D^j_{21} and D^j_{22} determine CMQS(v)'s.

In fact, $a^j_{i1} + a_{i2} + a^j_{i3} + a_{i4} = 8h + 2$.

Further, if $i + j \leq h$, then $a^j_{i1} = i + j$ and $a^j_{i3} = 4h - i - j + 1$. Hence $a^j_{i1} + a_{i2} \leq 4h - 1, 4h + 2 \leq a_{i2} + a^j_{i3} \leq 5h,\ 4h + 3 \leq a^j_{i3} + a_{i4} \leq 7h - 1$ and $a_{i4} + a^j_{i1} \leq 4\text{h}$. If $i + j > h$, then $a^j_{i1} = i + j - h$ and $a^j_{i3} = 5h - i - j + 1$. Hence $a^j_{i1} + a_{i2} \leq 3h - 1, 5h + 2 \leq a_{i2} + a^j_{i3} \leq 6h, 5h + 3 \leq a^j_{i3} + a_{i4} \leq 8h - 1$ and $a_{i4} + a^j_{i1} \leq 3\text{h}$. It follows that for D^j_{21} and D^j_{22} (1) and (2) hold.

Finally, for every $j = 1, 2, \ldots, h - 1$, we consider the families:

$$D^j_{31} = \{(a^j_{i1}, a_{i2}, a_{i3}, a^j_{i4}) : i = 1, 2, \ldots, h\},$$

$$D^j_{32} = \{(a^j_{i1}, a_{i2}, a^j_{i4}, a_{i3}) : i = 1, 2, \ldots, h\},$$

$$D^j_{33} = \{(a^j_{i1}, a_{i3}, a_{i2}, a^j_{i4}) : i = 1, 2, \ldots, h\},$$

$$D^j_{34} = \{(a^j_{i1}, a_{i3}, a^j_{i4}, a_{i2}) : i = 1, 2, \ldots, h\},$$

$$D^j_{35} = \{(a^j_{i1}, a^j_{i4}, a_{i2}, a_{i3}) : i = 1, 2, \ldots, h\},$$

$$D^j_{36} = \{(a^j_{i1}, a^j_{i4}, a_{i3}, a_{i2}) : i = 1, 2, \ldots, h\},$$

where $a^j_{i4} = 3h + 1 - (i \oplus j)$. Also the families $D^j_{3r}, j = 1, 2, \ldots, h - 1$ and $r = 1, 2, \ldots, 6$, determine CMQS(v)'s.

In fact, $a_{i1}^j + a_{i2} + a_{i3} + a_{i4}^j = 8h+2$ and $a_{i4}^j + a_{i1}^j = 3h+1$. Further, if $i+j \leq h$, then $a_{i4}^j = 3h-i-j+1$, hence $4h+2 \leq a_{i1}^j + a_{i3} \leq 5h, 3h+2 \leq a_{i2} + a_{i4}^j \leq 4h$ and $4h+3 \leq a_{i3} + a_{i4}^j \leq 7h-1$; if $i+j > h$, then $a_{i4}^j = 4h-i-j+1$, hence $5h+2 \leq a_{i1}^j + a_{i3} \leq 6h, 4h+2 \leq a_{i2} + a_{i4}^j \leq 5h$ and $5h+3 \leq a_{i3} + a_{i4}^j \leq 8h-1$.

It is easy to verify that $D_{11}, D_{12}, D_{21}^j, D_{22}^j, D_{3r}^j, j = 1, 2, \ldots, h-1$ and $r = 1, 2, \ldots, 6$, are pairwise disjoint. It follows that there exist at least $8h-6$ pairwise disjoint CMQS(v)'s, and

$$m_4(v) \geq 2v - 8. \blacksquare$$

THEOREM 5. *For every* $v \equiv 1 \pmod 4$, $m_4(v) \leq v^2 - 11v + z$, *where* $z = 32$ *if* $v \equiv 1$ or $5 \pmod{12}$ *and* $z = 30$ *if* $v \equiv 9 \pmod{12}$.

Proof. Let $(\mathbf{Z}_v, B)$ be a CMQS(v), with $v \equiv 1 \pmod 4$. If $\overline{B}$ is the difference family of $(\mathbf{Z}_v, B)$, then from Lemma 2 it follows that every d-quadruple of $\overline{B}$ is made up of pairwise distinct elements.

Let $D(v)$ be the set of all cyclically ordered quadruples of distinct elements of $\mathbf{Z}_v - \{0\}$, for which (2) of Theorem 1 holds. Let $D_1(v)$ be the set of all quadruples of $D(v)$ which contain 1 and let $M(v) =| D_1(v) |$.

From (1) of Theorem 1 it follows immediately that

$$m_4(v) \leq M(v), \text{ for every } v \equiv 1 \pmod 4 .$$

We intend to compute $M(v)$.

Let $d \in D_1(v)$ and let $1, a, b, c$, with $1 < a < b < c$, be the elements contained in d.

Observe that $1 + a + b + c = tv$, $t = 1$ or 2. Let $D'(v)$ be the set of all quadruples of $D_1(v)$ for which $1 + a + b + c = v$ and let $D''(v)$ be the set of all quadruples of $D_1(v)$ for which $1 + a + b + c = 2v$; let $M'(v) =| D_1'(v) |$ and $M''(v) =| D_2''(v) |$. Clearly, for $v \leq 9, M'(v) = 0$.

If $c = v-1$, then in $D_1(v)$ there are exactly two quadruples containing $1, a, b, c:$ $(1, a, c, b)$ and $(1, b, c, a)$; instead, if $c \neq v-1$, then in $D_1(v)$ there are exactly six quadruples containing $1, a, b, c$:

$$(1, a, b, c), (1, a, c, b), (1, b, a, c), (1, b, c, a), (1, c, a, b), (1, c, b, a).$$

First, suppose that $1 + a + b + c = v$, $v \geq 13$. Observe that b has the maximum value when $c = b + 1$, hence $a = v - 2b - 2$. Then $b \geq 3$ and $v - 2b - 2 \geq 2$, and

$$3 \leq b \leq \left[\frac{v-4}{2}\right] = \frac{v-5}{2}$$

($[x]$ denotes the largest integer not exceeding x).

For every b such that $3 \leq b \leq \frac{v-5}{2}$ we have $2 \leq a \leq b-1$, i.e. $2 \leq v - b - c - 1 \leq b - 1$. Hence, $v - 2b \leq c \leq v - b - 3$ and, simultaneously, $c \geq b + 1$.

It follows that

$$\begin{cases} v-2b \le c \le v-b-3 & \text{if} \quad 3 \le b \le \left[\frac{v-1}{3}\right] \\ b+1 \le c \le v-b-3 & \text{if} \quad \left[\frac{v-1}{3}\right]+1 \le b \le \frac{v-5}{2}. \end{cases}$$

Hence, for every $v > 13$,

$$(3) \qquad M'(v) = 6\left(\sum_{b=3}^{[\frac{v-1}{3}]}(b-2) + \sum_{b=[\frac{v-1}{3}]+1}^{\frac{v-5}{2}}(v-2b-3)\right),$$

and $M'(13) = 18$.

Suppose now that $1+a+b+c = 2v$, $v \ge 5$. Observe that b has the minimum value when $a = b-1$, hence $c = 2v-2b$. Then $b \le v-2$ and $2v-2b \le v-1$, and

$$\frac{v+1}{2} \le b \le v-2.$$

Further, for every b we have $b+1 \le c \le v-1$, i.e. $b+1 \le 2v-a-b-1 \le v-1$; hence $v-b \le a \le 2v-2b-2$ and, simultaneously, $a \le b-1$.

It follows that

$$\begin{cases} v-b \le a \le b-1 & \text{if} \quad \frac{v+1}{2} \le b \le \left[\frac{2v-1}{3}\right] \\ v-b \le a \le 2v-2b-2 & \text{if} \quad \left[\frac{2v-1}{3}\right]+1 \le b \le v-2 \end{cases}$$

Since for every b there exists exactly one a such that $a+b = v$, in $D_1(v)$ there are exactly $\frac{v-3}{2}$ quadruples for which $c = v-1$.

Hence, for every $v \ge 9$,

$$(4) \qquad M''(v) = 6\left(\sum_{b=\frac{v+1}{2}}^{[\frac{2v-1}{3}]}(2b-v) + \sum_{b=[\frac{2v-1}{3}]+1}^{v-2}(v-b-1)\right) - 2(v-3),$$

and $M''(5) = 2$.

Observe that

$$M(5) = M''(5) = 2, \quad M(9) = M''(9) = 12$$

and

$$M(13) = M'(13) + M''(13) = 18 + 40 = 58.$$

From (3) and (4) it follows that for every $v > 13$:

$$(5)\quad M(v) = 6\left(\sum_{b=3}^{[\frac{v-1}{3}]} (b-2) + \sum_{b=[\frac{v-1}{3}]+1}^{\frac{v-5}{2}} (v-2b-3) + \right.$$

$$\left. \sum_{b=\frac{v+1}{2}}^{[\frac{2v-1}{3}]} (2b-v) + \sum_{b=[\frac{2v-1}{3}]+1}^{v-2} (v-b-1) \right) - 2(v-3).$$

It is tedious but straightforward to compute $M(v)$ from (5) and get at the statement of the theorem. ■

Collecting together Theorems 4 and 5 gives the following theorem

THEOREM 6. *For every* $v \equiv 1 \pmod 4$, $v \geq 5$:

$$(6)\qquad 2v - 8 \leq m_4(v) \leq v^2 - 11v + z$$

where $z = 32$ *if* $v \equiv 1$ *or* $5 \pmod{12}$ *and* $z = 30$ *if* $v \equiv 9 \pmod{12}$.

From Theorem 6 it follows, in particular, that $m_4(5) = 2$; the difference families $\overline{B}_1 = \{(1,2,4,3)\}$ and $\overline{B}_2 = \{(1,3,4,2)\}$ determine two disjoint CMQS(5)'s.

We examine the cases $v = 9$ and $v = 13$.

a) $v = 9$.

From (6) we have $10 \leq m_4(9) \leq 12$. But it is possible to construct 12 pairwise disjoint CMQS(9)'s by the following difference families:

$$\overline{B}_1 = \{(1,3,8,6),(2,4,7,5)\},\quad \overline{B}_2 = \{(1,6,8,3),(2,5,7,4)\},$$

$$\overline{B}_3 = \{(1,4,8,5),(2,3,7,6)\},\quad \overline{B}_4 = \{(1,5,8,4),(2,6,7,3)\},$$

$$\overline{B}_5 = \{(1,2,8,7),(3,4,6,5)\},\quad \overline{B}_6 = \{(1,7,8,2),(3,5,6,4)\},$$

$$\overline{B}_7 = \{(1,4,6,7),(2,3,5,8)\},\quad \overline{B}_8 = \{(1,4,7,6),(2,3,8,5)\},$$

$$\overline{B}_9 = \{(1,6,4,7),(2,5,3,8)\},\quad \overline{B}_{10} = \{(1,6,7,4),(2,5,8,3)\},$$

$$\overline{B}_{11} = \{(1,7,4,6),(2,8,3,5)\},\quad \overline{B}_{12} = \{(1,7,6,4),(2,8,5,3)\}.$$

Hence, $m_4(9) = 12$.

b) $v = 13$.

From (6) it follows that $18 \leq m_4(13) \leq 58$. However it is possible to construct at least 50 pairwise disjoint CMQS(13)'s.

In fact, we consider the quadruples of $D(13)$ (see proof of Theorem 5)

$d_1 = (1,2,3,7), d_2 = (1,2,4,6), d_3 = (1,3,4,5), d_4 = (1,4,10,11),$
$d_5 = (1,5,9,11), d_6 = (1,6,8,11), d_7 = (1,6,9,10), d_8 = (1,7,8,10),$
$d_9 = (2,3,9,12), d_{10} = (2,4,8,12), d_{11} = (2,5,7,12), d_{12} = (2,5,9,10),$
$d_{13} = (2,6,8,10), d_{14} = (2,7,8,9), d_{15} = (3,4,7,12), d_{16} = (3,4,8,11),$
$d_{17} = (3,5,6,12), d_{18} = (3,5,7,11), d_{19} = (3,6,8,9), d_{20} = (4,5,6,11),$
$d_{21} = (4,5,7,10), d_{22} = (6,10,11,12), d_{23} = (7,9,11,12), d_{24} = (8,9,10,12),$
$\overline{d}_1 = (1,2,11,12), \overline{d}_2 = (1,3,10,12), \overline{d}_3 = (1,4,9,12), \overline{d}_4 = (1,5,8,12),$
$\overline{d}_5 = (1,6,7,12), \overline{d}_6 = (2,3,10,11), \overline{d}_7 = (2,4,9,11), \overline{d}_8 = (2,5,8,11),$
$\overline{d}_9 = (2,6,7,11), \overline{d}_{10} = (3,4,9,10), \overline{d}_{11} = (3,5,8,10), \overline{d}_{12} = (3,6,7,10),$
$\overline{d}_{13} = (4,5,8,9), \overline{d}_{14} = (4,6,7,9), \overline{d}_{15} = (5,6,7,8).$

For every $d_i = (a,b,c,d), 1 \leq i \leq 24$, let $d_i^1 = d_i$, $d_i^2 = (a,b,d,c), d_i^3 = (a,c,b,d)$, $d_i^4 = (a,c,d,b)$, $d_i^5 = (a,d,b,c)$ and $d_i^6 = (a,d,c,b)$; for every $\overline{d}_i = (a,b,c,d)$ let $\overline{d}_i^1 = (a,b,d,c)$ and $\overline{d}_i^2 = (a,c,d,b)$.

By the following difference families we can construct 50 pairwise disjoint CMQS(13)'s: $\{d_1^j, d_{20}^j, d_{24}^j\}$, $\{d_3^j, d_{14}^j, d_{22}^j\}$, $\{d_4^j, d_{11}^j, d_{19}^j\}$, $\{d_5^j, d_{13}^j, d_{15}^j\}$, $\{d_6^j, d_9^j, d_{21}^j\}$, $\{d_7^j, d_{10}^j, d_{18}^j\}$, for $j = 1,2,\ldots,6$; $\{d_2^j, d_{23}^j, \overline{d}_{11}^j\}$, $\{d_8^j, d_{17}^j, \overline{d}_7^j\}$, $\{d_{12}^j, d_{16}^j, \overline{d}_5^j\}$, $\{\overline{d}_1^j, \overline{d}_{10}^j, \overline{d}_{15}^j\}$, $\{\overline{d}_2^j, \overline{d}_9^j, \overline{d}_{13}^j\}$, $\{\overline{d}_3^j, \overline{d}_8^j, \overline{d}_{12}^j\}$, $\{\overline{d}_4^j, \overline{d}_6^j, \overline{d}_{14}^j\}$, for $j = 1,2$.

Hence, $50 \leq m_4(13) \leq 58$.

REFERENCES

[1] F.E.BENNET, *Coniugate orthogonal latin squares and Mendelsohn designs,* Ars Combinatoria 19 (1985), 51-62.

[2] N.BRAND-W.C.HUFFMAN, *Invariants and constructions of Mendelsohn designs,* Geometriae Dedicata 22 (1987), 173-196.

[3] N.BRAND-W.C.HUFFMAN, *Mendelsohn designs admitting the affine group,* Graphs Combin. 3 (1987), n.4, 313-324.

[4] C.J.COLBOURN-M.J.COLBOURN, *Disjoint cyclic Mendelsohn triple systems,* Ars Combinatoria 11 (1981), 3-8.

[5] B.GANTER-R.A.MATHON-A.ROSA, *A complete census of (10,3,2) block designs and of Mendelsohn triple systems of order ten, I: Mendelsohn triple systems without repeated blocks,* Proc. Seventh Manitoba Conf. Num. Math. Comput. (1977), 383-398.

[6] B.GANTER-R.A.MATHON-A.ROSA, *A complete census of (10,3,2) block designs and of Mendelsohn triple systems of order ten, II: Mendelsohn triple systems with repeated blocks,* Proc. Eighth Manitoba Conf. Num. Math. Comput. (1978), 181-204.

[7] D.G.HOFFMAN-C.C.LINDNER, *Embedding of Mendelsohn triple systems,* Ars Combinatoria 11 (1981), 265-269.

[8] D.G.HOFFMAN-C.C.LINDNER, *Mendelsohn triple systems having a prescribed number of triples in common,* Europ. J. Combin. 3 (1982), n.1, 51-61.

[9] C.C.LINDNER, *On the number of disjoint Mendelsohn triple systems,* J.Combin. Theory, ser. A 30 (1981), n.3, 326-330.

[10] R.A. MATHON-A.ROSA, *A census of Mendelsohn triple systems of order nine,* Ars Combinatoria 4 (1977), 309-315.

[11] N.S.MENDELSOHN, *Combinatorial designs as models of universal algebras,* in *Recent Progress in Combinatorics* (Proc. Third Waterloo Conf. on Combinatorics, 1968), W.T.Tutte (ed.), Academic Press, New York, 1969, 123-132.

[12] N.S.MENDELSOHN, *A natural generalization of Steiner triple systems,* in *Computers in number theory* (A.O.Atkin, B.Birch, eds.), Academic Press, London, 1971, 323-338.

[13] K.T.PHELPS-C.C.LINDNER, *On the number of Mendelsohn and transitive triple systems,* Europ. J. Combin. 5 (1984), n.3, 239-242.