Blocking sets in G-designs

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Abstract. We give necessary and sufficient conditions for the existence of 2-colorable G-designs for each G that is connected, simple and has at most 5 edges.

1. Introduction.

Let G and K be graphs with G simple; that is, G is a subgraph of K_n, the complete undirected graph on n vertices. A *G-design* of K is a pair (P, B), where P is the vertex set of K and B is an edge-disjoint decomposition of K into copies of the graph G. A G-design of K_n is also called a *G-design of order n*. For example, a Steiner triple system of order n is a K_3-design of order n and a block design with block size 4 is a K_4-design.

For $m \leq n$ let $K_n \setminus K_m$ be the subgraph of K_n induced by $E(K_n) \setminus E(K_m)$; call $K_n \setminus K_m$ a complete graph with a *hole* of size m.

Let K_x^t be the complete t-partite graph with exactly x vertices in each part.

In recent years, much attention has been focused on G-designs and on G-designs with additional properties. Perhaps the most natural question to ask about G-designs is what is their *spectrum* (that is, for what orders do they exist?)? Surprisingly enough, the spectrum for the decomposition of K_n into relatively uncomplicated graphs, such as, for example, cycles, has yet to be determined, despite having been considered for at least 25 years (see [10] for example). More recently, the existence problem has been determined in the cases where G is a path [14], where G is a star [13], and has nearly been settled when G is a graph with at most 5 vertices [1].

Related to this problem is the existence problem for G-designs that satisfy additional properties, such as those admitting a 2-coloring. If (P, B) is a G-design, a subset X of P is called a *blocking set* of (P, B) if for each $g \in B$, $V(g) \cap X \neq \emptyset$ and $V(g) \cap (P \setminus X) \neq \emptyset$, where $V(g)$ is the vertex set of the graph g. The partition $\{X, P \setminus X\}$ is also called a 2-coloring. We shall use these 2 terms interchangeably.

It is easy to see that the only K_3-designs admitting a 2-coloring have order 3 [11]. However, numerous papers have been written on determining the spectrum for 2-colorable t-designs, (see, for example, [12]), 2-colorable projective planes [2], 2-colorable symmetric designs [4] and 2-colorable block designs [5]. Recently, a complete solution (modulo a few possible exceptions) of the problem of constructing K_4-designs which can be 2-colored has been given by D.G. Hoffman, C.C. Lindner, and K.T. Phelps [8, 9] (they also consider the problem of 2-coloring

G-designs of λK_n) and the spectrum for 2-colorable $(K_4 - e)$-designs has also been found in [7].

In this paper, we give a complete solution of the existence problem of G-designs which admit a 2-coloring for each of the 12 graphs G that is connected, simple, and has 5 edges (the result is formally stated in Theorem 2.1). We remark that the same technique solves this 2-coloring problem when G has less than 5 edges [3] and should do when G is disconnected.

As was mentioned, the existence problem (ignoring 2-colorings) for six of these 12 graphs has been considered previously [1]. However, the construction we use here is not only a unified approach of settling the spectrum problem for all 12 of the graphs, but also is extremely versatile, allowing for the 2-coloring property to be easily considered at the same time.

The construction we use makes use of idempotent symmetric latin squares and of symmetric latin squares with holes of size 2. A *latin square* of order n on the symbols $\{1, \ldots, n\}$ is an $n \times n$ array, each cell of which contains exactly one symbol and each symbol occurs exactly once in each row and in each column. A latin square is *idempotent* if each cell (x, x) contains the symbol x. A latin square is *symmetric* if the symbol in the cell (x, y) is the symbol in the cell (y, x) for all x and y.

Define $h_i = \{2i - 1, 2i\}$ for $1 \leq i \leq n$, and let $S = \{1, 2, \ldots, 2n\}$. A *latin square with holes of size* 2 and of order $2n$ is a $2n \times 2n$ array in which

(1) for $1 \leq i \leq n$, the cells of $h_i \times h_i$ contain no symbols, and each other cell contains 1 symbol; and
(2) for $1 \leq i \leq n$, each row and column in h_i contains each symbol in $S - h_i$ exactly once.

The latin squares we make use of in this paper are symmetric idempotent latin squares and symmetric latin squares with holes of size 2. It is well known that symmetric idempotent latin squares of order n exist for all odd n, and symmetric latin squares with holes of size 2 having order n exist for all even $n \geq 4$ (see [6]). Let $Z_n = \{0, 1, \ldots, n-1\}$.

2. Some small cases.

There are 12 connected, simple graphs with 5 edges. We begin with some notation to describe these 12 graphs.

$G_1 = (\{a, b, c, d\}; \{ab, bc, ac, cd, ad\})$. Denote this graph by $G_1(a, b, c, d)$.
$G_2 = (\{a, b, c, d, e\}; \{ab, bc, cd, de, ae\})$. Denote this graph by $G_2(a, b, c, d, e)$.
$G_3 = (\{a, b, c, d, e\}; \{ab, bc, cd, ad, ae\})$. Denote this graph by $G_3(a, b, c, d, e)$.
$G_4 = (\{a, b, c, d, e\}; \{ab, bc, cd, ce, de\})$. Denote this graph by $G_4(a, b, c, d, e)$.
$G_5 = (\{a, b, c, d, e\}; \{ab, bc, bd, cd, de\})$. Denote this graph by $G_5(a, b, c, d, e)$.
$G_6 = (\{a, b, c, d, e\}; \{ab, ac, bc, cd, ce\})$. Denote this graph by $G_6(a, b, c, d, e)$.
$G_7 = (\{a, b, c, d, e, f\}; \{ab, bc, cd, cf, de\})$. Denote this graph by $G_7(a, b, c, d, e, f)$.
$G_8 = (\{a, b, c, d, e, f\}; \{ac, bc, cd, de, df\})$. Denote this graph by $G_8(a, b, c, d, e, f)$.
$G_9 = (\{a, b, c, d, e, f\}; \{ab, bc, cd, ce, cf\})$. Denote this graph by $G_9(a, b, c, d, e, f)$.

$G_{10} = (\{a,b,c,d,e,f\}; \{ab, bc, cd, de, df\})$. Denote this graph by $G_{10}(a,b,c,d,e,f)$.
$G_{11} = (\{a,b,c,d,e,f\}; \{ab, bc, cd, de, ef\})$. Denote this graph by $G_{11}(a,b,c,d,e,f)$.
$G_{12} = (\{a,b,c,d,e,f\}; \{ab, bc, bd, be, bf\})$. Denote this graph by $G_{12}(a,b,c,d,e,f)$.

Since 5 must divide $|E(K_n)| = \frac{n(n-1)}{2}$, having $n \equiv 0,1 \pmod 5$ is a necessary condition for the existence of G_i-designs of K_n. We will prove the following result.

Theorem 2.1. *For $3 \le i \le 12$, there exists a 2-colorable G_i-design of order n if and only if $n \equiv 0,1 \pmod 5$, if $i \neq 5$, then $n \neq 5$, and if $i \notin \{4,5,6,8\}$, then $n \neq 6$. A 2-colorable G_2-design of K_n exists if and only if $n \equiv 1,5 \pmod{10}$.*

A complete solution of the existence problem of G_1-designs which admit a 2-coloring was given by M. Gionfriddo, C.C. Lindner and C.A. Rodger [7]. So we will restrict ourselves to considering G_2 through G_{12}.

We begin by finding some necessary small 2-colorable G_i-designs, and some small G_i-designs of K_n with a hole.

Lemma 2.2. *There exists a 2-colorable G_2-design of K_n for $n \in \{5, 11, 21\}$.*

Proof: The following are 2-colorable G_2-designs of K_n.

$n = 5$: $(Z_5; \{G_2(0,1,2,3,4), (G_2(0,2,4,1,3)\})$,
$\{0,2\}$ is a blocking set.
$n = 11$: $(Z_{11}; \{G_2(0,1,6,9,2) + i \mid i \in Z_{11}\})$,
$\{0,3,6,9\}$ is a blocking set.
$n = 21$: $(Z_{21}; \{G_2(0,1,10,20,4)+i, G_2(0,7,5,2,8)+i \mid i \in Z_{21}\})$,
$\{0,3,4,6,9,12,15,18\}$ is a blocking set.

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Lemma 2.3. *There exists a G_3-design of K_n for $n \in \{10, 11, 15\}$ and a G_3-design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_3-designs of K_n.

$n = 10$: $(\{\infty\} \cup Z_9 \{G_3(0,1,5,2,\infty) + i \mid i \in Z_9\})$,
$\{0,3,6,\infty\}$ is a blocking set.
$n = 11$: $(Z_{11}; \{G_3(1,5,2,0,6) + i \mid i \in Z_{11}\})$,
$\{0,3,6,9\}$ is a blocking set.

$n = 15:$ $(\{\infty\} \cup Z_2 \times Z_7);$
$\{G_3((0,0),(1,0),(1,1),(0,2),\infty) + (0,i),$
$G_3((0,0),(0,1),(1,2),(0,4),(1,5)) + (0,i),$
$G_3((1,0),(1,2),(0,0),(1,3),\infty) + (0,i) \mid i \in Z_7\}),$
$\{\infty,(0,0),(0,2),(0,4),(1,1),(1,3)\}$ is a blocking set.

The following is a 2-colorable G_3-design of $K_{11} \setminus K_6$ with hole $\{\infty_i | 0 \le i \le 5\}$.

$K_{11} \setminus K_6 : (\{\infty_i \mid 0 \le i \le 5\} \cup Z_5;$
$\{(G_3(1,3,2,4,0), G_3(\infty_2,0,3,4,1), G_3(0,\infty_0,1,\infty_1,4),$
$G_3(2,1,\infty_3,0,\infty_1), G_3(\infty_0,3,\infty_1,4,2),$
$G_3(\infty_3,3,\infty_2,2,4), G_3(\infty_4,0,\infty_5,1,2),$
$G_3(\infty_5,4,\infty_4,3,2)\}),$
$\{0,2,\infty_1,\infty_4\}$ is a blocking set.

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Lemma 2.4. *There exists a G_4-design of K_n for $n \in \{6,10,11,15,20,21,25\}$ and a G_4-design of $K_{15} \setminus K_5$, each having a 2-coloring.*

Proof: The following are 2-colorable G_4-designs of K_n.

$n = 6:$ $(Z_6; \{G_4(0,1,2,3,5), G_4(3,0,5,4,1), G_4(1,3,4,0,2)\}),$
$\{0,2\}$ is a blocking set.
$n = 10:$ $(\{\infty\} \cup Z_9; \{G_4(\infty,2,0,1,4) + i \mid i \in Z_9\}),$
$\{0,3,6,\infty\}$ is a blocking set.
$n = 11:$ $(Z_{11}; \{G_4(7,2,0,1,4) + i \mid i \in Z_{11}\}),$
$\{0,3,6,9\}$ is a blocking set.
$n = 15:$ $(\{\infty\} \cup Z_2 \times Z_7);$
$\{G_4((1,0),\infty,(0,0),(1,1),(1,2)) + (0,i);$
$G_4((0,1),(1,5),(1,3),(1,0),(0,0)) + (0,i);$
$G_4((0,6),(0,3),(0,1),(0,2),(1,0)) + (0,i) \mid i \in Z_7\}),$
$\{\infty,(0,0),(0,2),(0,4),(0,6),(1,0)\}$ is a blocking set.

$n = 20:$ $(\{\infty\} \cup Z_{19}; \{G_4(5,2,0,1,7)+i, G_4(\infty,0,8,13,17)+i | i \in Z_{19}\})$,
$\{0,1,4,7,10,13,16,\infty\}$ is a blocking set.

$n = 21:$ $(Z_{21}; \{G_4(17,8,0,6,7) + i, G_4(14,4,0,2,5) + i \mid i \in Z_{21}\})$,
$\{0,3,4,6,9,12,15,18\}$ is a blocking set.

$n = 25:$ $(\{\infty\} \cup Z_2 \times Z_{12})$;
$\{G_4((0,4),(1,2),(0,0),(0,6),(1,7)) + (0,i)$,
$G_4(\infty,(0,0),(1,0),(1,6),(1,7)) + (0,i)$,
$G_4(\infty,(0,6),(1,6),(1,1),(0,0)) + (0,i)$,
$G_4((0,10),(1,8),(0,6),(1,0),(1,1)) + (0,i)$,
$G_4(\infty,(1,8),(0,3),(0,1),(0,0)) + (0,j)$,
$G_4((1,6),(0,9),(0,4),(0,0),(1,3)) + (0,j)$,
$G_4((1,7),(1,5),(1,8),(1,0),(0,4)) + (0,j) |$
$0 \leq i \leq 5, 0 \leq j \leq 11\})$,
$\{0\} \times Z_{12}$ is a blocking set.

The following is a 2-colorable G_4-design of $K_{15} \setminus K_5$ with hole $\{\infty_i | 0 \leq i \leq 5\}$.

$K_{15} \setminus K_5 : (\{\infty_i \mid 0 \leq i \leq 4\} \cup (Z_2 \times Z_5)$;
$\{G_4((0,3),(0,1),(0,0),(1,0),(1,1)) + (0,i)$ for $0 \leq i \leq 3\}$.,
$G_4((0,1),\infty_1,(1,2),(0,0),\infty_0)) + (0,i)$ for $0 \leq i \leq 4$,
$G_4((0,1),\infty_3,(1,3),(0,0),\infty_2) + (0,i)$ for $0 \leq i \leq 4$,
$G_4((1,2),(1,4),(1,1),(0,2),\infty_4)$,
$G_4((1,0),(0,4),(1,4),(0,0),\infty_4)$,
$G_4((1,4),(1,0),(1,2),(0,3),\infty_4)$,
$G_4((1,1),(1,3),(1,0),(0,1),\infty_4)$,
$G_4((0,2),(0,0),(0,4),(1,3),\infty_4)\})$,
$\{\infty_0,\infty_2,(1,1),(1,2),(1,3),(1,4)\}$ is a blocking set.

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Lemma 2.5. *There exists a G_5-design of K_n for $n \in \{5,6,10,11,20,21\}$, having a 2-coloring.*

Proof: The following are 2-colorable G_5-designs of K_n.

$n=5:$ $(Z_5; \{G_5(0,2,4,1,3), G_5(1,0,4,3,2)\})$,
$\{0,2\}$ is a blocking set.

$n=6:$ $(Z_6; \{G_5(0,5,1,4,2), G_5(0,2,5,3,4), G_5(2,1,3,0,4)\})$,
$\{0,2\}$ is a blocking set.

$n=10$: ($\{\infty\} \cup Z_9; \{G_5(\infty,0,3,1,5) + i \mid i \in Z_9\}$),
$\{0,3,6,\infty\}$ is a blocking set.

$n=11$: ($Z_{11}; \{G_5(10,1,4,0,5) + i \mid i \in Z_{11}\}$),
$\{0,3,6,9\}$ is a blocking set.

$n=20$: ($\{\infty\} \cup Z_{19}; \{G_5(\infty,0,4,1,8)+i, G_5(9,0,8,2,7)+i \mid i \in Z_{19}\}$),
$\{0,3,6,9,12,15,17,\infty\}$ is a blocking set.

$n=21$: ($Z_{21}; \{G_5(2,0,5,1,7)+i, G_5(9,0,10,3,11)+i \mid i \in Z_{21}\}$),
$\{0,3,6,9,12,15,18,20\}$ is a blocking set.

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Lemma 2.6. *There exists a G_6 design of K_n for $n \in \{6,10,11,15,20,21,25\}$ and a G_6-design of $K_{15} \setminus K_5$, each having a 2-coloring.*

Proof: The following are 2-colorable G_6-designs of K_n.

$n=6$: ($Z_6; \{G_6(0,2,1,3,4), G_6(2,4,3,5,0), G_6(0,4,5,1,2)\}$),
$\{0,2\}$ is a blocking set.

$n=10$: ($\{\infty\} \cup Z_9; \{G_6(1,3,0,4,\infty) + i \mid i \in Z_9\}$),
$\{0,3,6,\infty\}$ is a blocking set.

$n=11$: ($Z_{11}; \{G_6(1,3,0,4,5) + i \mid i \in Z_{11}\}$),
$\{0,3,6,9\}$ is a blocking set.

$n=15$: ($\{\infty\} \cup (Z_2 \times Z_7); \{G_6((1,0),(1,1),(0,0),(0,2),\infty)+(0,i)$,
$G_6((0,2),(0,3),(1,0),(1,3),\infty) + (0,i)$,
$G_6((0,1),(0,4),(1,0),(0,5),(1,2)) + (0,i) \mid i \in Z_7\}$),
$\{(0,0),(0,2),(0,4),\infty\}$ is a blocking set.

$n=20$: ($\{\infty\} \cup Z_{19}; \{G_6(1,4,0,5,\infty)+i, G_6(2,8,0,7,9)+i \mid i \in Z_{19}\}$),
$\{0,3,6,9,12,15,17,\infty\}$ is a blocking set.

$n=21$: ($Z_{21}; \{G_6(3,10,0,8,9) + i, G_6(1,5,0,2,6) + i \mid i \in Z_{21}\}$),
$\{0,3,6,9,12,15,18,20\}$ is a blocking set.

$n = 25 : (\{\infty\} \cup Z_2 \times Z_{12};$

$\{G_6(0,0),(0,6),(1,1),(0,1),\infty) + (0,i),$

$G_6((0,1),(1,0),(1,6),(1,1),(0,0)) + (0,i),$

$G_6((1,6),(0,7),(1,7),(0,0),\infty) + (0,i),$

$G_6((0,6),(1,7),(1,0),(1,1),(0,7)) + (0,i),$

$G_6((0,0),(0,4),(1,2),(1,4),(1,5)) + (0,j),$

$G_6((1,0),(1,8),(0,4),(0,9),(1,7)) + (0,j),$

$G_6((0,0),(0,1),(0,3),(1,0),\infty) + (0,j)$

$| 0 \le i \le 5, 0 \le j \le 11 \}), \{0\} \times Z_{12}$ is a blocking set.

The following is a 2-colorable G_6-design of $K_{15} \setminus K_5$ with hole $\{\infty_i \mid 0 \le i \le 4\}$.

$K_{15} \setminus K_5 : (\{\infty_i \mid 0 \le i \le 4\} \cup Z_{10};$

$\{ (G_6(1,2,0,3,9), G_6(2,5,4,3,9),$

$G_6(3,7,6,5,9), G_6(1,7,8,5,9),$

$G_6(9,\infty_0,1,5,6), G_6(0,7,\infty_0,5,6), G_6(3,8,\infty_0,2,4),$

$G_6(7,\infty_1,2,6,8), G_6(1,4,\infty_1,6,8), G_6(3,5,\infty_1,0,9),$

$G_6(6,\infty_2,0,5,8), G_6(3,9,\infty_2,5,8), G_6(4,7,\infty_2,1,2),$

$G_6(5,\infty_3,9,2,7), G_6(1,3,\infty_3,2,7), G_6(6,8,\infty_3,0,4),$

$G_6(8,\infty_4,4,0,6), G_6(2,3,\infty_4,0,6), G_6(5,7,\infty_4,1,9) \}),$

$\{\infty_0,\infty_1,0,3,7,9\}$ is a blocking set.

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Lemma 2.7. *There exists a G_7-design of K_n for $n \in \{10, 11, 15\}$ and a G_7 design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_7-designs of K_n.

$n = 10 : (\{\infty\} \cup Z_9; \{G_7(4,1,0,2,6,\infty) + i \mid i \in Z_9\}),$

$\{0,3,6,\infty\}$ is a blocking set.

$n = 11 : (Z_{11}; \{G_7(4,1,0,2,6,5) + i \mid i \in Z_{11}\}),$

$\{0,3,6,9\}$ is a blocking set.

$n = 15 : (\{\infty\} \cup (Z_2 \times Z_7);$

$\{G_7((0,0),(1,3),(1,0),(0,3),(1,5),(0,6)) + (0,i),$

$G_7((0,5),(0,2),(1,0),(1,1),(1,3),\infty) + (0,i),$

$G_7((0,1),(1,0),(0,0),(0,2),(0,3),\infty) + (0,i) \mid i \in Z_7 \}),$

$\{\infty,(0,0),(0,2),(0,4),(1,1),(1,3)\}$ is a blocking set.

The following is a 2-colorable G_7-design of $K_{11} \setminus K_6$ with hole $\{\infty_i | 0 \leq i \leq 5\}$.

$K_{11} \setminus K_6$: ($\{\infty_i \mid 0 \leq i \leq 5\} \cup Z_5$; { $G_7(0,\infty_0,2,\infty_2,4,1)$,
$G_7(\infty_1,0,\infty_2,3,2,1)$, $G_7(0,3,\infty_1,4,1,2)$,
$G_7(0,4,\infty_0,1,\infty_1,3)$, $G_7(1,0,\infty_4,2,4,3)$,
$G_7(1,3,\infty_3,4,\infty_4,0)$, $G_7(1,\infty_3,2,\infty_5,4,0)$,
$G_7(\infty_4,1,\infty_5,3,4,0)$ }) , $\{0,2,\infty_1,\infty_4\}$ is a blocking set.

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Lemma 2.8. *There exists a G_8-design of K_n for $n \in \{6,10,11,15\}$, each having a 2-coloring.*

Proof: The following are 2-colorable G_8-designs of K_n.

$n=6$: (Z_6; $\{G_8(0,4,5,2,1,3), G_8(0,2,4,3,1,5), G_8(2,3,0,1,4,5)\}$) ,
$\{0,2\}$ is a blocking set.

$n=10$: ($\{\infty\} \cup Z_9$; $\{G_8(1,\infty,0,2,5,6) + i \mid i \in Z_9\}$) ,
$\{0,3,6,\infty\}$ is a blocking set.

$n=11$: (Z_{11}; $\{G_8(1,2,0,3,7,8) + i \mid i \in Z_{11}\}$) ,
$\{0,3,6,9\}$ is a blocking set.

$n=15$: ($\{\infty\} \cup (Z_2 \times Z_7)$; $\{G_8(\infty, (0,1),(0,0),(1,0),(1,1),(0,2),)$,
$G_8((0,0),(0,1),(0,3),(1,0),(1,3),(0,6)) + (0,i) \mid i \in Z_7\}$) ,
$\{\infty,(0,0),(0,1),(0,3),(0,4),(1,0)\}$ is a blocking set.

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Lemma 2.9. *There exists a G_9-design of K_n for $n \in \{10,11,15\}$ and a G_9-design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_9-designs of K_n.

$n=10$: ($\{\infty\} \cup Z_9$; $\{G_9(\infty,0,1,3,4,5) + i \mid i \in Z_9\}$) ,
$\{\infty,0,3,6\}$ is a blocking set.

$n=11$: (Z_{11}; $\{G_9(0,1,3,6,7,8) + i \mid i \in Z_{11}\}$) ,
$\{0,3,6,9\}$ is a blocking set.

$n=15$: ($\{\infty\} \cup (Z_2 \times Z_7)$;
$\{G_9(\infty, (0,0),(1,0),(0,1),(0,2),(1,3)) + (0,i)$,
$G_9(\infty,(1,0),(0,3),(0,4),(0,5),(0,6)) + (0,i)$,
$G_9((1,1),(1,0),(1,2),(0,0),(0,1),(0,6)) + (0,i) \mid i \in Z_7\}$) ,
$\{\infty,(0,0),(1,0),(1,1),(1,3),(1,4)\}$ is a blocking set.

The following is a 2-colorable G_9-design of $K_{11} \setminus K_6$ with hole $\{\infty_i | 0 \leq i \leq 5\}$.

$K_{11} \setminus K_6$: ($\{\infty_i \mid 0 \leq i \leq 5\} \cup Z_5$; { ($G_9(3,1,\infty_0,0,2,4)$,
$G_9(4,0,\infty_2,1,2,3)$, $G_9(0,1,\infty_3,2,3,4)$, $G_9(0,3,\infty_4,1,2,4)$,
$G_9(2,1,\infty_5,0,3,4)$, $G_9(\infty_3,0,\infty_1,1,3,4)$,
$G_9(\infty_0,3,4,1,2,\infty_2)$,
$G_9(\infty_4,0,2,3,\infty_1,\infty_5)$ }) , $\{0,1,\infty_1,\infty_4\}$ is a blocking set.

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Lemma 2.10. *There exists a G_{10}-design of K_n for $n \in \{10,11,15\}$ and a G_{10}-design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_{10}-designs of K_n.

$n = 10$: ($\{\infty\} \cup Z_9$; $\{G_{10}(\infty,0,1,3,6,7) + i \mid i \in Z_9\}$) ,
$\{\infty,0,3,6\}$ is a blocking set.
$n = 11$: (Z_{11}; $\{G_{10}(0,3,1,2,6,7) + i \mid i \in Z_{11}\}$) ,
$\{0,3,6,9\}$ is a blocking set.

$n = 15$: ($\{\infty\} \cup (Z_2 \times Z_7)$;
$\{G_{10}(\infty, (0,0),(1,0),(0,1),(0,2),(0,3)) + (0,i)$,
$G_{10}(\infty,(1,0),(0,2),(0,5),(1,1),(1,6)) + (0,i)$,
$G_{10}((1,1),(1,0),(1,2),(1,5),(0,1),(0,3)) + (0,i) | i \in Z_7\}$) ,
$\{\infty,(0,0),(1,0),(1,1),(1,3),(1,4)\}$ is a blocking set.

The following is a 2-colorable G_{10}-design of $K_{11} \setminus K_6$ with hole $\{\infty_i \mid 0 \leq i \leq 5\}$.

$K_{11} \setminus K_6$: ($\{\infty_i \mid 0 \leq i \leq 5\} \cup Z_5$; { ($G_{10}(3,\infty_0,4,\infty_1,0,1)$,
$G_{10}(3,\infty_1,2,\infty_0,0,1)$, $G_{10}(3,\infty_2,4,\infty_3,0,1)$,
$G_{10}(3,\infty_3,2,\infty_2,0,1)$, $G_{10}(3,4,0,\infty_4,1,2)$,
$G_{10}(2,4,\infty_4,3,0,1)$,
$G_{10}(2,1,0,\infty_5,3,4)$, $G_{10}(4,1,\infty_5,2,0,3)$ }) ,
$\{0,2,\infty_1,\infty_4\}$ is a blocking set.

∎

Lemma 2.11. *There exists a G_{11}-design of K_n for $n \in \{10, 11, 15\}$ and a G_{11}-design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_{11}-designs of K_n.

$n = 10 : (\ \{\infty\} \cup Z_9; \{G_{11}(\infty, 0, 4, 7, 8, 1) + i \mid i \in Z_9\}\ ),$
$\{\infty, 0, 3, 6\}$ is a blocking set.

$n = 11 : (\ Z_{11}; \{G_{11}(0, 1, 3, 6, 10, 4) + i \mid i \in Z_{11}\}\ ),$
$\{0, 3, 6, 9\}$ is a blocking set.

$n = 15 : (\ \{\infty\} \cup (Z_2 \times Z_7);$
$\{G_{11}(\infty, (0,0), (1,0), (0,1), (1,6), (1,1)) + (0,i),$
$G_{11}(\infty, (1,0), (0,6), (0,0), (0,3), (1,6)) + (0,i),$
$G_{11}((0,0), (0,2), (1,4), (1,5), (1,1), (0,4)) + (0,i) \mid i \in Z_7\}),$
$\{\infty, (0,0), (1,0), (1,2), (1,4), (1,6)\}$ is a blocking set.

The following is a 2-colorable G_{11}-design of $K_{11} \setminus K_6$ with hole $\{\infty_i \mid 0 \le i \le 5\}$.

$K_{11} \setminus K_6 : (\ \{\infty_i \mid 0 \le i \le 5\} \cup Z_5; \{\ (G_{11}(\infty_0, 4, 1, 2, 0, 3),$
$G_{11}(\infty_1, 3, 1, 0, 4, 2), G_{11}(3, 2, \infty_1, 0, \infty_4, 4),$
$G_{11}(4, 3, \infty_0, 2, \infty_2, 1),$
$G_{11}(1, \infty_1, 4, \infty_2, 0, \infty_3), G_{11}(1, \infty_4, 2, \infty_3, 4, \infty_5),$
$G_{11}(2, \infty_5, 1, \infty_3, 3, \infty_4), G_{11}(1, \infty_0, 0, \infty_5, 3, \infty_2)\ \}\ ),$
$\{0, 2, \infty_1, \infty_4\}$ is a blocking set.

∎

Lemma 2.12. *There exists a G_{12}-design of K_n for $n \in \{10, 11, 15\}$ and a G_{12}-design of $K_{11} \setminus K_6$, each having a 2-coloring.*

Proof: The following are 2-colorable G_{12}-designs of K_n.

$n = 10 : (\ \{\infty\} \cup Z_9; \{G_{12}(\infty, 0, 1, 2, 3, 4) + i \mid i \in Z_9\}\ ),$
$\{\infty, 0, 3, 6\}$ is a blocking set.

$n = 11 : (\ Z_{11}; \{G_{12}(1, 0, 2, 3, 4, 5) + i \mid i \in Z_{11}\}\ ),$
$\{0, 3, 6, 9\}$ is a blocking set.

$n = 15 : (\ \{\infty\} \cup (Z_2 \times Z_7);$
$\{G_{12}(\infty, (0,0), (1,0), (1,1), (1,2), (1,3)) + (0,i),$
$G_{12}(\infty, (1,0), (1,1), (1,2), (1,3), (0,3)) + (0,i),$
$G_{12}((0,1), (0,0), (0,2), (0,3), (1,5), (1,6)) + (0,i) \mid i \in Z_7\}\ ),$
$\{\infty, (0,0), (1,0), (1,2), (1,4), (1,6)\}$ is a blocking set.

The following is a 2-colorable G_{12}-design of $K_{11} \setminus K_6$ with hole $\{\infty_i \mid 0 \le i \le 5\}$.

$K_{11} \setminus K_6$: ($\{\infty_i \mid 0 \le i \le 5\} \cup Z_5$; { $G_{12}(2,0,1,\infty_0,\infty_1,\infty_2)$,
$G_{12}(3,1,2,\infty_0,\infty_1,\infty_2)$, $G_{12}(4,2,3,\infty_0,\infty_1,\infty_2)$,
$G_{12}(4,3,0,\infty_0,\infty_1,\infty_2)$, $G_{12}(0,4,1,\infty_0,\infty_1,\infty_2)$,
$G_{12}(0,\infty_3,1,2,3,4)$, $G_{12}(0,\infty_4,1,2,3,4)$,
$G_{12}(0,\infty_5,1,2,3,4)$ }),
$\{0,2,\infty_1,\infty_4\}$ is a blocking set.

∎

3. The main construction.

We now prove Theorem 2.1.

Proof: We can write n as $10x$, $10x+1$, $10x+5$ or $10x+6$. In the following construction, we begin by finding 2-colorable G_i-designs of K_5^{2x+1} and of K_{10}^{x}. We also find 2-colorable G_j-designs of K_5^{2x} for $j \in \{3,7,9,10,11,12\}$. We then partition edges joining vertices within the parts, together with $0,1,5$ or 6 further vertices (depending upon the congruence of n modulo 10), into G_i-designs by using the small G_i-designs constructed in Section 2. This is done so that the 2-colorings of the small designs are compatible with the 2-colorings of K_5^{2x+1}, K_{10}^{x}, and K_5^{2x}.

Throughout, let $(Z_{2x+1},\cdot)$ be an idempotent symmetric quasigroup, and let $(Z_{2x},\circ)$ be a symmetric quasigroup with holes of size 2. Recall that $(Z_{2x+1},\cdot)$ exists for all x and $(Z_{2x},\circ)$ exists for all $x \ge 3$.

For $2 \le i \le 12$, the following are 2-colorable G_i-designs of K_5^{2x+1} for $x \ge 1$ and of K_{10}^{x} for $x \ge 3$, and 2-colorable G_j-designs of K_5^{2x} for $j \in \{3,7,9,10,11,12\}$ and $x \ge 1$. Throughout, let $H = \{\{0,1\},\{2,3\},\dots,\{2x-2,2x-1\}\}$.

$i=2$: ($Z_5 \times Z_{2x+1}$;{ $G_3((0,a),(0,b),(1,a),(3,a\cdot b),(1,b))+(i,0)\mid$
$0 \le i \le 4, 0 \le a < b \le 2x$ }) is a G_2-design of K_5^{2x+1}
with blocking set $Z_2 \times Z_{2x+1}$.
($Z_5 \times Z_{2x}$; { $G_3(\ (0,a),(0,b),(1,a),(3,a\circ b),(1,b)\ )+(i,0)\mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H$} is a G_2-design of
K_{10}^{x} with blocking set $Z_2 \times Z_{2x}$.

$i=3$: ($Z_5 \times Z_y$; { $G_3(\ (2,a),(0,b),(0,a),(1,b),(4,b)\ )+(i,0)\mid$
$0 \le i \le 4, 0 \le a < b \le y-1$ }) is a G_3-design of K_5^{y}
with blocking set $Z_2 \times Z_y$.
($Z_5 \times Z_{2x}$; { $G_3(\ (2,a),(0,b),(0,a),(1,b),(4,b)\ )+(i,0)\ )\mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H$ } is a G_3-design of
K_{10}^{x} with blocking set $Z_2 \times Z_{2x}$.

$i = 4:$ $(Z_5 \times Z_{2x+1}; \{ G_4 ((4,a),(2,b),(0,a),(0,b),(1,a\cdot b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x \})$ is a G_4-design of
K_5^{2x+1} with blocking set $Z_2 \times Z_{2x}$.
$(Z_5 \times Z_{2x}; \{ G_4 ((4,a),(2,b),(0,a),(0,b),(1,a \circ b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x-1, \{a,b\} \notin H \})$ is a
G_4 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 5:$ $(Z_5 \times Z_{2x+1}; \{ G_5 ((3,a),(0,b),(1,a\cdot b),(0,a),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x \})$ is a G_5-design of
K_5^{2x+1} with blocking set $Z_2 \times Z_{2x+1}$.
$(Z_5 \times Z_{2x}; \{ G_5 ((3,a),(0,b),(1,a \circ b),(0,a),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x-1, \{a,b\} \notin H \})$ is a
G_5 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 6:$ $(Z_5 \times Z_{2x+1}; \{ G_6 ((0,a),(0,b),(1,a\cdot b),(3,a),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x \})$ is a G_6-design of
K_5^{2x+1} with blocking set $Z_2 \times Z_{2x+1}$.
$(Z_5 \times Z_{2x}; \{ G_6 ((0,a),(0,b),(1,a \circ b),(3,a),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x-1, \{a,b\} \notin H \})$ is a
G_6 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 7:$ $(Z_5 \times Z_y; \{ G_7 ((2,b),(1,a),(0,b),(2,a),(4,b),(0,a)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq y-1 \})$ is a G_7-design of
$K_5^y$with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x}; \{ G_7 ((2,b),(1,a),(0,b),(2,a),(4,b),(0,a)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x-1, \{a,b\} \notin H \})$ is a
G_7 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 8:$ $(Z_5 \times Z_y; \{ G_8 ((1,a),(3,a),(0,b),(0,a),(1,b),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq y-1 \})$ is a G_8-design of
K_5^y with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x}; \{ G_8 ((1,a),(3,a),(0,b),(0,a),(1,b),(3,b)) + (i,0) |$
$0 \leq i \leq 4, 0 \leq a < b \leq 2x-1, \{a,b\} \notin H \})$ is a
G_8 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 9:$ $(Z_5 \times Z_y;\{\ G_9\,((4,b),(3,a),(2,b),(4,a),(2,a),(0,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le y-1\ \})$ is a G_9-design of
K_5^y with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x};\{\ G_9\,((4,b),(3,a),(2,b),(4,a),(2,a),(0,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H\ \})$ is a
G_9 – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 10:$ $(Z_5 \times Z_y;\{\ G_{10}\,((4,a),(1,b),(0,a),(0,b),(1,a),(2,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le y-1\ \})$ is a G_{10}-design of
K_5^y with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x};\{\ G_{10}\,((4,a),(1,b),(0,a),(0,b),(1,a),(2,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H\ \})$ is a
G_{10} – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 11:$ $(Z_5 \times Z_y;\{\ G_{11}\,((3,a),(1,b),(0,a),(0,b),(1,a),(3,b)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le y-1\ \})$ is a G_{11}-design of
K_5^y with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x};\{\ G_{11}\,((3,a),(1,b),(0,a),(0,b),(1,a),(3,b)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H\ \})$ is a
G_{11} – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

$i = 12:$ $(Z_5 \times Z_y;\{\ G_{12}\,((0,a),(0,b),(1,a),(2,a),(3,a),(4,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le y-1\ \})$ is a G_{12}-design of
K_5^y with blocking set $Z_2 \times Z_y$.
$(Z_5 \times Z_{2x};\{\ G_{12}\,((0,a),(0,b),(1,a),(2,a),(3,a),(4,a)) + (i,0) \mid$
$0 \le i \le 4, 0 \le a < b \le 2x-1, \{a,b\} \notin H\ \})$ is a
G_{12} – design of K_{10}^x with blocking set $Z_2 \times Z_{2x}$.

Unify these designs with the following notation. For $2 \le i \le 12$, define $(Z_5 \times Z_{2x+1}, B_{i,10x+5})$ to be a G_i-design of K_5^{2x+1} and $(Z_5 \times Z_{2x}, B_{i,10x})$ to be G_i-design of K_{10}^x. For $j \in \{3,7,9,10,11,12\}$ define $(Z_5 \times Z_{2x}, B'_{j,10x})$ to be a G_j-design of K_5^{2x}. For $3 \le i \le 12, i \notin \{4,5,6\}$ let $(Z_5 \times \{a,b\}, C_{i,10}(a,b))$ be a G_i-design of K_5^2 with blocking set $Z_2 \times \{a,b\}$.

The names of the vertices in the small designs are now changed to suit the construction: we introduce more unifying notation. (The following designs were constructed in Section 2.) For $i \in \{2,5\}$ let $(Z_5 \times \{a\}; B_{i,5}(a)$ be a G_i-design of K_5, with blocking set $\{0,1\} \times \{a\}$. For $i \in \{4,5,6,8\}$ let $(\{\infty\} \cup (Z_5 \times \{a\});$ $B_{i,6}(a))$ be a G_i-design of K_6, with blocking set $\{0,1\} \times \{a\}$. For $3 \le i \le 12$ let $(Z_5 \times \{a,b\}; B_{i,10}(a,b))$ be a G_i-design of K_{10}, with blocking set $\{0,1\} \times$

$\{a,b\}$. For $2 \le i \le 12$ let $(\{\infty\} \cup (Z_5 \times \{a,b\}); B_{i,11})$ be a G_i-design of K_{11}, with blocking set $\{0,1\} \times \{a,b\}$. For $i \in \{4,6\}$ let $(\{\infty_i \mid 0 \le i \le 4\} \cup (Z_5 \times Z_2); B_{1,15}(a,b))$ be a G_i-design of K_{15}, with blocking set $\{\infty_0, \infty_1\} \cup (\{0,1\} \times \{a,b\})$. For $3 \le i \le 12$, $i \notin \{4,5,6\}$ let $(Z_5 \times \{a,b,c\}, B_{i,15}(a,b,c))$ be a G_i-design of K_{15} with blocking set $Z_2 \times \{a,b,c\}$. For $i \in \{4,6\}$ let $(\{\infty_i \mid 0 \le i \le 4\} \cup (Z_5 \times Z_2); B_{i,15\setminus 5}(a,b))$ be a G_i-design of $K_{15} \setminus K_5$, with blocking set $\{\infty_0, \infty_1\} \cup (\{0,1\} \times \{a,b\})$. For $i \in \{3,7,9,10,11,12\}$ let $(\{\infty_i \mid 0 \le i \le 5\} \cup (Z_5 \times \{a\}); B_{i,11\setminus 6}(a))$ be a G_i-design of $K_{11} \setminus K_6$, with blocking set $\{\infty_0, \infty_1\} \cup (\{0,1\} \times \{a\})$.

Finally, we put the small designs together with the designs of the relevant t-partite graphs. Notice that if $i \in \{2,4,5,6\}$ $B_{i,10x}$ is only defined for $x \ge 3$ (since there is no symmetric quasigroup with $x = 2$ holes of size 2).

For $3 \le i \le 12$, define a 2-colorable G_i-design of K_n by taking the following blocks B. (Consider the cases $n = 10x$, $10x + 1$, $10x + 5$ and $10x + 6$ in turn).

(1) $n = 10x$. If $x = 1$ or if $x = 2$ and $i \in \{4,5,6\}$, then see Lemma 2.i. Otherwise, let $B = \bigcup_{\{a,b\} \in H} B_{i,10}(a,b) \cup B_{i,10x}$, with blocking set $Z_2 \times Z_{2x}$.

(2) $n = 10x + 1$. If $x = 1$ or if $x = 2$ and $i \in \{4,5,6\}$, then see Lemma 2.i. Otherwise, let $B = \bigcup_{\{a,b\} \in H} B_{i,11}(a,b) \cup B_{i,10x}$, with blocking set $Z_2 \times Z_{2x}$.

(3) $n = 10x + 5$.

(a) If $i = 5$, then let $B = \bigcup_{a=0}^{2x} B_{i,5}(a) \cup B_{i,10x+5}$, with blocking set $Z_2 \times Z_{2x+1}$.

(b) If $i \ne 5$, then: if $x = 1$ or $x = 2$ and $i \in \{4,6\}$ then see Lemma 2.i; otherwise

(i) if $i \in \{4,6\}$, then let $B = \bigcup_{\{a,b\} \in H \setminus \{0,1\}} B_{i,15\setminus 5}(a,b) \cup B_{i,15}(0,1) \cup B_{i,10x}$, with blocking set $Z_2 \times Z_{2x+1}$, and

(ii) if $i \notin \{4,5,6\}$ then let $B = \bigcup_{\{a,b\} \in H \setminus \{0,1\}} B_{i,10}(a,b) \cup B_{i,15}(0,1,2x) \cup_{\substack{0 \le a < b \le 2x \\ \{a,b\} \notin H'}} C_{i,10}(a,b)$ (where $H' = H \cup \{\{0,2x\}, \{1,2x\}\}$) with blocking set $Z_2 \times Z_{2x+1}$.

(4) $n = 10x + 6$.

(a) If $i \in \{4,5,6,8\}$ (so there is G_i-design of K_6), then $B = \bigcup_{a=0}^{2x} B_{i,6}(a) \cup B_{i,10x+5}$ with blocking set $Z_2 \times Z_{2x+1}$, and otherwise

(b) $B = \bigcup_{a=1}^{2x} B_{i,11\setminus 6}(a) \cup B_{i,11}(0) \cup B'_{i,10x}$ with blocking set $Z_2 \times Z_{2x+1}$.

We finally define 2-colorable G_2-designs of K_n. Since all degrees in G_2 are even, the necessary conditions for the existence of a G_2-design of K_n are $n \equiv 1,5 \pmod{10}$. We will show that these necessary conditions are sufficient for the existence of a 2-colorable G_2-design of K_n.

$n = 10x + 1$: If $x \in \{1, 2\}$, then see Lemma 2.2. Otherwise,

$$\left(\{\infty\} \cup (Z_5 \times Z_{2x});\ \bigcup_{\{a,b\} \in H} B_{2,11}(a,b) \cup B_{2,10x}\right)$$

is a G_2-design of K_n, with blocking set $Z_2 \times Z_{2x}$.
$n = 10x + 5$: If $x = 1$, then see Lemma 2.2. Otherwise, $(Z_5 \times Z_{2x+1}; \cup_{a=0}^{2x} B_{2,5}(a) \cup B_{2,10x+5})$ is a G_2-design of K_n, with blocking set $Z_2 \times Z_{2x+1}$.

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