On The Closure Of A Graph With Cut Vertices

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Abstract. Let K be a maximal block of a graph G and let x and y be two nonadjacent vertices of G. If $|V(K)| \leq \frac{1}{2}(n+3)$ and x and y are not cut vertices, we show that x is not adjacent to y in the closure $c(G)$ of G. We also show that, if $x, y \notin V(K)$, then x is not adjacent to y in $c(G)$.

1. Introduction.

Let $G = (V, E)$ be a simple graph and $n = |V|$. The closure of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains. We denote the closure of G by $c(G)$. By [1, p. 56, Lemma 4.4.2], $c(G)$ is well defined.

A vertex x of G is called a cut vertex if E can be partitioned into two nonempty subsets E_1 and E_2 such that the subgraphs $G[E_1]$ and $G[E_2]$ induced by E_1 and E_2, repectively, have just the vertex x in common. A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property. Every graph is the union of its blocks.

The largest possible closure of a graph G can be the complete graph containing G (for example, see [1, p. 56, Figure 4.6]). Let z be a cut vertex which separates E into two nonempty subsets E_1 and E_2 and let G_i be the subgraph of G with $E(G_i) = E_i$ $(i = 1, 2)$. Suppose that $x \in V(G_1)$, $y \in V(G_2)$, $x \neq z$ and $y \neq z$. We show that x is not adjacent to y in $c(G)$. We also show that, if K is a maximal block of a graph G, $x, y \notin V(K)$ and x is not adjacent to y, then x is not adjacent to y in $c(G)$. Therefore, if a graph G consists of many large blocks, only edges (u, v) (where either u or v is a cut vertex of G) can be added to its closure $c(G)$ of G and so $c(G)$ cannot be a very large graph.

2. Graphs with cut vertices.

For each x in V, let $d(x)$ denote the degree of x in G.

Theorem 2.1. *Let z be a cut vertex which separates E into two nonempty subsets E_1 and E_2 and let G_i be the subgraph of G with $E(G_i) = E_i$ $(i = 1, 2)$. Suppose that $x \in V(G_1)$, $y \in V(G_2)$, $x \neq z$ and $y \neq z$. Then x is not adjacent to y in $c(G)$.*

Proof: Since z is a cut vertex, it follows that x is not adjacent to any vertex $w \in V(G_2)$ with $w \neq z$ in G and y is not adjacent to any vertex $w' \in V(G_1)$ with $w' \neq z$ in G. In particular, x is not adjacent to y in G. We use contradiction.

Suppose that x is adjacent to y in $c(G)$. Also we may assume, without loss of generality, that x and y are the first such pair of vertices adjacent in $c(G)$. Let $n_i = |V(G_i)|$ $(i = 1, 2)$. It is clear that $d(x) \leq n_1 - 1$ and $d(y) \leq n_2 - 1$. Since $V(G_1) \cap V(G_2) = \{z\}$, we have $n = n_1 + n_2 - 1$ and so $d(x) + d(y) \leq n_1 + n_2 - 2 = n - 1$ in G. But x is adjacent to y in $c(G)$. This means either x is adjacent to some $u \in V(G_2)$ (with $u \neq z$) before x is adjacent to y in $c(G)$ or y is adjacent to some $u' \in V(G_1)$ (with $u' \neq z$) before y is adjacent to x in $c(G)$. But x and y are assumed to be the first such pair of vertices, which is a contradiction. Therefore, x is not adjacent to y in $c(G)$. This completes the proof. ∎

The following corollary follows immediately from Theorem 2.1.

Corollary 2.2. *Let K_1 and K_2 be two blocks of G. Suppose that $x \in V(K_1)$, $x \notin V(K_2)$, $y \in V(K_2)$, and $y \notin V(K_1)$. Then x is not adjacent to y in $c(G)$.*

A block K of G is called a maximal block, if $|V(K)| \geq |V(K')|$ for each block K' of G.

Remark 1: Let K be a block of G. If $|V(K)| > \frac{n}{2}$, then it is clear that K is the only maximal block of G. If G has more than one maximal block, then $2|V(K)| \leq n + 1$ and so $|V(K)| \leq \frac{1}{2}(n+1)$ $\left(< \frac{1}{2}(n+3)\right)$.

Theorem 2.3. *Let K be a maximal block and let x and y be two nonadjacent vertices of G.*

(a) *If $|V(K)| \leq \frac{1}{2}(n+3)$ and x and y are not cut vertices, then x is not adjacent to y in $c(G)$.*

(b) *If $x, y \notin V(K)$, then x is not adjacent to y in $c(G)$.*

Proof: (a) If x and y are in different blocks, then it follows from Corollary 2.2 that x is not adjacent to y in $c(G)$. If x and y are in the same block, say K', then $d(x) \leq |V(K')| - 2$ and $d(y) \leq |V(K')| - 2$. Hence,

$$d(x) + d(y) \leq 2|V(K')| - 4 \leq 2|V(K)| - 4 \leq (n+3) - 4 = n - 1.$$

Since x or y cannot be adjacent to another vertex $u \notin V(K')$ in $c(G)$ (Corollary 2.2), it follows that x and y are not adjacent in $c(G)$. This proves (a).

(b) Let $k = |V(K)|$.

Case 1. Suppose that x and y are not cut vertices. Then by Corollary 2.2, we can assume that both x and y belong to the same block, say K_1. Since $x, y \notin V(K)$, $K \neq K_1$. If $|K_1| = |K|$, then G has more than one maximal block and so by Remark 1, $|V(K)| < \frac{1}{2}(n+3)$. Hence, by (a), x cannot be adjacent to y in $c(G)$. If $|K_1| < |K|$, then $2|K_1| < n + 1$. As in the proof of (a), we have

$$d(x) + d(y) \leq 2|V(K_1)| - 4 < (n+1) - 4 = n - 3 < n - 1.$$

and x is not adjacent to y in $c(G)$.

Case 2. Suppose that x is a cut vertex. Let $K_1, K_2, \ldots, K_p$ be the blocks containing x and let $n_i = |V(K_i)|$ $(i = 1, 2, \ldots, p)$. If x and y are not in a common block, then by Corollary 2.2, x is not adjacent to y. Hence, we can assume that $y \in V(K_1)$. Since x is not adjacent to y, we have

$$d(x) \leq (n_1 - 2) + (n_2 - 1) + \ldots + (n_p - 1) = \sum_{i=1}^{p} n_i - p - 1. \tag{1}$$

The subgraph G_1 of G consisting of $K_1, K_2, \ldots, K_p$ has $(n_1 - 1) + (n_2 - 1) + \ldots + (n_p - 1) + 1 = \sum_{i=1}^{p} n_i - p + 1$ vertices. Let G_2 be the subgraph of G consisting of K and G_1. Then G_2 has at least $(k - 1) + \left(\sum_{i=1}^{p} n_i - p + 1\right) = \sum_{i=1}^{p} n_i + k - p$ vertices. Hence, there are at most $n - \left(\sum_{i=1}^{p} n_i + k - p\right)$ vertices left, which can be adjacent to y. Since y is not adjacent to x in K_1, we have

$$d(y) \leq (n_1 - 2) + \left(n - \sum_{i=1}^{p} n_i - k + p\right). \tag{2}$$

Therefore, by (1) and (2), we have

$$\begin{aligned} d(x) + d(y) &\leq \sum_{i=1}^{p} n_i - p - 1 + n_1 - 2 + n - \sum_{i=1}^{p} n_i - k + p \\ &= n + n_1 - k - 3. \end{aligned}$$

Since K is a maximal block, $n_1 \leq k$. Therefore, $d(x) + d(y) \leq n - 3$ and so x cannot be adjacent to y in $c(G)$. This completes the proof of the theorem. ∎

Remark 2: If $d(x) = 1$ in G, then $d(x) = 1$ in $c(G)$. In fact, let y be a vertex of G which is not adjacent to x. Then $d(y) \leq n - 2$. Hence, $d(x) + d(y) \leq n - 1$ and so $d(x) = 1$ in $c(G)$.

In [5], Khuller gives an algorithm to find $c(G)$. This algorithm can be implemented in $O(n^3)$ time, which improves the $O(n^4)$ time bound given in [4]. If G has cut vertices, we show that the running time of Khuller's algorithm can be improved by the following observations. Use Hopcroft and Tarjan's algorithm to find all blocks and cut vertices of G (see [2, p. 57] or [3, p. 24]). This algorithm can be implemented in $O(|E|)$ time. Let $x, y \in V$ and $N = \{(x, y) : (x, y) \notin E\}$. If $d(x) = 1$, remove (x, y) from N (Remark 2). Let K be a maximal block of G. If $|V(K)| \leq \frac{1}{2}(n + 3)$ and x and y are not cut vertices, remove (x, y) from N (Theorem 2.3(a)). If $x, y \notin V(K)$, remove (x, y) from N (Theorem 2.3(b)). If K_1 and K_2 are two blocks of G and $x \in V(K_1)$, $x \notin V(K_2)$, $y \in V(K_2)$ and $y \notin V(K_1)$, remove (x, y) from N (Corollary 2.2). Finally, use Khuller's algorithm on N to find $c(G)$. If G contains many cut vertices and large blocks, the running time of Khuller's algorithm can be improved. For example, suppose that G has $p (\geq 1)$ cut vertices and $|V(K)| \leq \frac{1}{2}(n + 3)$ for each block K of G. Then, it follows from Theorem 2.3 that $O(|N|) \leq O(pn)$. Hence, Khuller's algorithm now runs on $O(pn^2)$ time.

References

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