On Small Triangle-Free Graphs

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Abstract. We construct all four-chromatic triangle-free graphs on twelve vertices, and a triangle-free, uniquely three-colourable graph.

Our terminology and notation is consistent with that of Bondy and Murty [2].

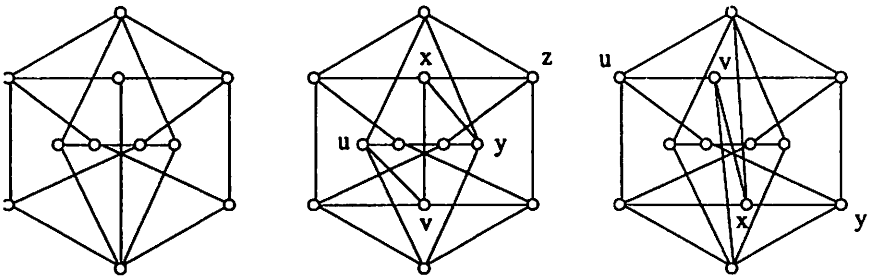
In [4] Chvátal proved that the Grötzsch graph (see Figure 1 (a)) is the unique four-chromatic, triangle-free graph on at most eleven vertices (the reasoning behind the unusual drawings of the graphs in Figure 1 will become apparent later, *cf.* Figure 2). Avis [1] used the uniqueness of the Grötzsch graph to prove that the minimum number of vertices in a five-chromatic triangle-free graph is at least 19. Recently, Jensen and Royle [private communication] have used an exhaustive computer search to determine that there exist five-chromatic triangle-free graphs on 22 vertices, but none smaller. For an interesting look at this and similar problems, the reader is directed to Toft's book *Graph Colouring Problems*, [11, chapter 6] (see also [10]).

In [3] Chvátal exhibits a four-chromatic, four-regular graph of girth four on 12 vertices (see Figure 1 (b)). A careful consideration of such graphs implies that, in fact, this graph is the unique four-chromatic, four-regular, triangle-free graph of diameter two on 12 vertices.

We constructively identify all four-chromatic, triangle-free graphs on twelve vertices (*cf.* Theorem 1). We also point out how a slight extension of Avis' method can be used to give a direct proof that a five-chromatic triangle-free graph must have at least 20 vertices.

At least three different erroneous examples of uniquely three-colourable, triangle-free graphs on 12 vertices have appeared in the literature [7, both the first and third printing], [8]. We exhibit a uniquely three-colourable, triangle-free graph on 12 vertices.

Define a graph to be **3-saturated** (in context, simply **saturated**) if it does not contain a complete subgraph on 3 vertices but does so upon the addition of any new edge (a diameter two condition). The four-chromatic, triangle-free graphs on 12 vertices are subgraphs of four-chromatic, saturated graphs (saturating the graph will not increase the chromatic number since we know that a five chromatic



(a) Grötzsch graph; (b) Chvátal graph; (c) The graph G_2.

Figure 1

(The labels in this figure are used only in Corollary 1 and following.)

triangle-free graph must have at least 22 vertices). Our first problem is then, what are the possible 12 vertex graphs which are both four-chromatic and saturated?

Let G be a subgraph of H, $u \in V(G)$, $v \in V(H)$, and $uv \notin E(G)$. We say that v is **G-duplicated** if $N_H(u) \supseteq N_G(v)$. We say that u **G-duplicates** v. If the graphs G and H are clear from the context, we drop the "G", and say that v is duplicated or that u duplicates v, respectively. Put $H = G$ in the above definition, and suppose that u duplicates v. Then $\chi(G - v) = \chi(G)$. Thus, a critical graph admits no duplicated vertices.

Theorem 1. *The four-chromatic, saturated graphs on 12 vertices are (up to isomorphism):*

(a) $\mathcal{G}_1 = \{G_1 : G_1$ *is the Grötzsch graph,* G*, with an extra vertex that duplicates a vertex of* $G\}$;

(b) G_2, *the Grötzsch graph,* G*, with an extra vertex,* p*, whose neighbourhood is a maximal independent set of vertices and* $N(p) \neq N(q)$ *for any vertex* q *of* G *(see Figure 1(c));*

(c) G_3, *the Chvátal graph (see Figure 1(b)).*

Proof: Let H be a four-chromatic saturated graph with 12 vertices.

We first suppose that the maximum degree, Δ, is at least seven. Pick a vertex, t, of degree Δ. Since $\chi(H) = 4$, $R = H - \{t\} - N(t)$ has an odd cycle, implying that H is not triangle-free. (If R is two-colourable, then a third colour can be used to colour $N(t)$ and one of the two colours used for R is available for t.)

In a similar manner, in the following cases we take t to have degree Δ and neighbours $n_1, n_2, \ldots, n_\Delta$; we let $R = H - \{t\} - N(t)$, and, in those cases where R has a five-cycle, C_5, we denote its vertices by $c_1, c_2, \ldots, c_5$. Suppose H is not four-vertex critical. Then H has a vertex, p, whose removal leaves the Grötzsch graph, G. Hence, either H belongs to $\mathcal{G}_1$ or, using the saturated property, that p is adjacent to an independent set of vertices that dominate G. This latter case leads to $H = G_2$. Thus, we assume in the following that H is four-vertex critical.

Case 1 $\Delta = 6$.

Since $\chi = 4$, R must be a five-cycle, C_5. If every vertex of C_5 is C_5-duplicated by an element of $N(t)$, then H can be constructed from the Grötzsch graph by joining a new vertex to the vertex of degree five. This implies that H belongs to $\mathcal{G}_1$. Thus, we may assume without loss of generality that c_5 is not C_5-duplicated (we call such vertices **non-duplicated** when the context is clear). Colour vertex c_5 with colour 3 and properly colour the rest of C_5 with colours 1 and 2. Now the n_i's may all be coloured with either colour 1 or 2, leaving colour 3 available for t, a contradiction.

Case 2 $\Delta = 5$.

Again, if every vertex of C_5 is C_5-duplicated by an element of $N(t)$ then H contains the Grötzsch graph, and is either G_2 or belongs to $\mathcal{G}_1$. Thus, we may assume that C_5 contains non-duplicated vertices (such as the vertex c_5 in Case 1). The remaining vertex, p, must be adjacent to all non-duplicated vertices of C_5. (Otherwise, giving such a vertex colour 3, the rest of C_5 colours 1 and 2, we may colour $N(t)$ with 1 and 2 and assign colour 3 to p and to t. Thus, $\chi(H) \leq 3$, a contradiction.)

(i) Suppose that c_5 is the only such non-duplicated vertex and that the neighbours of $c_1, \ldots, c_4$ on C_5 are C_5-duplicated by $n_1, \ldots, n_4$, respectively. The fact that H is saturated, and that n_5 does not C_5-duplicate c_5 implies that p is adjacent to n_5 (otherwise n_5 is H-duplicated by some other n_i, whence, $H - n_5$ would be four-chromatic, but H is four-vertex critical). The saturated property now restricts the possibilities for other edges. In particular, consider the neighbourhood of p. Clearly, $\delta(H) \geq 3$ and, therefore, we have at least one of the edges pn_2 and pn_3.

If both pn_2 and pn_3 are edges, consider $N(n_5)$; n_5 must be adjacent to vertices of the five-cycle but each and every choice remaining either creates a triangle, C_5-duplicates c_5 or leaves $H - n_5$ four-chromatic. Since this contradicts our assumption that H is critical, both pn_2 and pn_3 can not be present.

If pn_3 is an edge, but not pn_2, then the saturated property implies that either pc_3 is an edge, or n_5c_3 is an edge. In the first case, there is a problem with $N(n_5)$, as above. In the second case, the saturated property implies that n_5c_1 is an edge, but then $H - n_1$ is four-chromatic, again a contradiction.

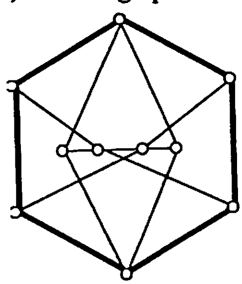
The case of pn_2 but not pn_3 is symmetrical to the above case.

(ii) Suppose there are two vertices which are not C_5-duplicated. Since H is triangle-free these vertices cannot be adjacent. Thus, we may assume that c_3 and c_5 are the non-duplicated vertices and that c_1, c_2 and c_4 are C_5-duplicated by n_1, n_2 and n_4, respectively. Then pc_3 and pc_5 are edges of H, and the presence of any of the edges pn_1, pn_2 or pn_4 creates a triangle. Hence, we may assume pn_5 is an edge. Again consider $N(n_5)$ as above; the edge n_5c_4 must exist, and so must one of n_5c_1 or n_5c_2. In either case a non-duplicated vertex becomes duplicated.

It is impossible to have three non-duplicated vertices on C_5 because each is required to be adjacent to p, and such a configuration contains a triangle.

Case 3 $\Delta = 4$. Since H is saturated, the neighbourhood of each vertex is a dominating set. Suppose there is a vertex s such that $d(s) = 3$. Let the neighbours of s be n_1, n_2 and n_3, and their neighbourhoods in $H - s$ be A_1, A_2 and A_3, respectively. Exactly two of these sets, say A_2 and A_3, may intersect, and there can be only one vertex in the intersection. Note that a non-empty intersection implies that $d(n_1) = d(n_2) = 4$. Colour the set A_1, the vertices n_2 and n_3 with colour 1; colour A_2 and n_1 with colour 2; colour the remainder of A_3 and s with colour 3. Hence, $\chi(H) = 3$. We may, therefore, assume that H is four-regular, four-chromatic, and, since the $(4,5)$-cage has 19 vertices, of girth four. Then H is exactly the Chvátal graph, G_3. (As we have stated earlier, the uniqueness of G_3 may be established in a straight-forward manner by considering $N(t)$, R and the appropriate adjacencies as in the case $\Delta = 5$. The details are not informative, and we suppress them.) ∎

It is apparent from Figure 1 that all of the graphs in Theorem 1, and also the Grötzsch graph, contain a particular 10 vertex subgraph, which we call the **10-graph.** Any three-colouring of the 10-graph always colours a particular chordless six-cycle in cyclical order, that is, the vertices are coloured $1,2,3,1,2,3$ (modulo permutation of the colours). The 10-graph and its special six-cycle are shown in Figure 2.



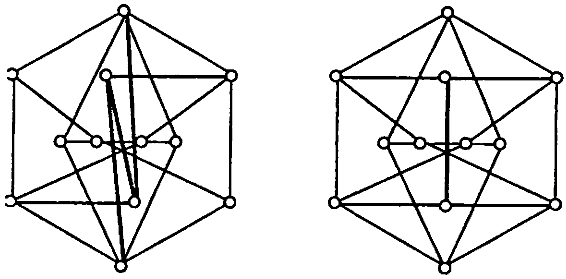
The 10-graph and its special six-cycle (emphasized)
Figure 2

Corollary 1. *The twelve-vertex, four-chromatic, triangle-free graphs are (up to isomorphism)* $G_2, G_2 - xy, G_2 - uv - xy, G_3, G_3 - xy, G_3 - uv - xy$ $(u, v, x, y$ *are the vertices indicated in Figure 1), and* $\mathcal{G}_1^* = \{G_1^*\colon G_1^*$ *is the Grötzsch graph,* G*, with an extra vertex that is duplicated by a vertex of* $G\}$.

Note that the graphs in G_1^* are those subgraphs of the graphs in G_1 (see Theorem 1) that contain the Grötzsch graph.

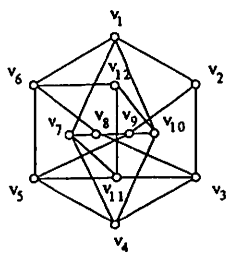
It can be directly verified that each graph in Corollary 1 is four-chromatic. There are edges other than the ones mentioned in Corollary 1 whose deletion from the graphs in Theorem 1 leave a four-chromatic graph. In each instance it is straightforward, although tedious, to check that the resultant graph belongs to our list.

The only triangle-free, four-critical graphs are $G_2 - uv - xy$, and $G_3 - uv - xy$. These are shown in Figure 3 (a) and (b), respectively.



The 12-vertex, triangle-free, four-critical graphs
Figure 3

We exhibit a triangle-free, uniquely three-colourable graph, U, with 12 vertices (see Figure 4). The 10-graph plays an important role in the proof that U is uniquely three-colourable. Observe that U is the Chvátal graph (see Figure 1(b)) with the edge xz deleted. We know of no smaller such graph.



The graph U, a triangle-free, uniquely three-colourable graph
Figure 4

Proposition 1. *The graph U shown in Figure 4 is uniquely three-colourable and triangle-free.*

Proof: The graph U is obviously triangle-free. It remains to show that it is uniquely three-colourable.

Note that U contains a copy T of the 10-graph with vertex-set $\{v_1, v_2, \ldots, v_{10}\}$. Any three-colouring of U induces a three-colouring of T. We have previously noted that the six-cycle $v_1, v_2, \ldots, v_6$ must be coloured in cyclic order, and that this order is unique up to renaming the colours.

Without loss of generality $v_1, v_2, \ldots, v_6$ are coloured $1, 2, 3, 1, 2, 3$, respectively. This forces v_{11} to have colour 1. Similarly, vertices $v_{12}, v_{10}, v_9, v_8, v_7$ are each in turn forced to have colours $2, 3, 1, 2, 3$, respectively. Thus, U is uniquely three-colourable. ∎

It is not difficult to use the 10-graph to construct other, different, triangle-free, uniquely three-colourable graphs.

Let $n(k)$ denote the minimum number of vertices in a k-chromatic triangle-free graph. A well known construction of Mycielski [6] gives a sequence of k-critical triangle-free graphs. This construction implies an upper bound on $n(k)$ of $3 \cdot 2^{k-2} - 1$. In particular, it leads to $n(4) \leq 11$ (the Grötzsch graph, see [2]) and $n(5) \leq 23$. Chvátal [4] has shown that $n(4) = 11$, and Jensen and Royle [private communication] have reported that a computer search yields $n(5) = 22$. It is known by probabilistic arguments, Erdös [5], that $n(k)$ is bounded above by $c(k \log(k))^2$ for some constant c. T. Jensen has obtained the lower bound $ck^2 \log(k)$. His proof is given in [11].

In [1], Avis shows that $n(4) \geq 19$. His approach relies heavily on the uniqueness of the the Grötzsch graph. We outline a similar direct proof that $n(4) \geq 20$.

Proposition 2. *Any five-chromatic, triangle-free graph, H, has at least 20 vertices.*

Outline of proof: We follow the threads of Avis proof in [1]. Note, first of all, that every four-chromatic, triangle-free graph on twelve vertices has a four-colouring with colours $1, 2, 3, 4$ such that exactly one vertex, v, is coloured 4, $d(v) \geq 4$, and no independent set intersects all four colour classes. (We note in passing that the Grötzsch graph also has the above property.) Secondly, as in Avis proof, observe that since the Ramsey number $r(3,6) = 18$ and H is triangle-free, $\alpha(T) \geq 6$. One then considers cases. In each case a special subset of vertices is deleted, leaving an 11 or 12 vertex, four-chromatic graph, then, using the first property noted above in place of the uniqueness of the Grötzsch graph, this four-colouring is extended to H. The cases are $\alpha \geq 8$, $\alpha = \Delta = 6$, and $\alpha = 6$ and $\Delta = 5$. The proofs are similar to [1], Lemma 1, Lemma 2, and Theorem 1, respectively, except that in the case $\alpha = 7$ the proof is similar to Lemma 2 if $\Delta = 7$.

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