Ramsey Numbers for Monotone Paths and Cycles

Hanno Lefmann*†

Fakultät für Mathematik, Universität Bielefeld
Postfach 8640
W-4800 Bielefeld 1
Germany

Abstract. In this paper we will consider the Ramsey numbers for paths and cycles in graphs with unordered as well as ordered vertex set.

Introduction.

Let $G_0, G_1, \ldots, G_{r-1}$ be finite graphs. The generalized Ramsey number $r(G_0, G_1, \ldots, G_{r-1})$ denotes the least positive integer n such that for every coloring $\Delta : E(K_n) \to \{0, 1, \ldots, r-1\}$ of the edges of the complete graph on n vertices with r colors there exists a nonnegative integer $i < r$ and a subgraph G of K_n which is isomorphic to G_i such that $\Delta(e) = i$ for all edges $e \in E(G)$. Clearly, for any finite graphs G_i the numbers $r(G_0, G_1, \ldots, G_{r-1})$ exist by Ramsey's theorem.

In this paper, we will consider the Ramsey numbers for paths and cycles in the ordered and unordered case and compare their growth. The study of these monotone graphs are done in connection with results of Paris and Harrington [PH 77] on independence statements in Peano Arithmetic. It turns out that, although the Ramsey numbers for monotone cycles on k vertices are small compared to those for complete graphs with the same number of vertices, the corresponding Paris-Harrington functions for monotone paths grow fast, in particular, do not belong to the class of primitive recursive functions.

Unordered graphs.

Throughout this paragraph the vertex sets of the graphs under consideration are not endowed with an ordering. A path $P_\ell = (V, E)$ of length ℓ is a graph on ℓ vertices, such that there exists an enumeration of the vertices of V, $V = \{v_0, v_1, \ldots, v_{\ell-1}\}$, where $\{v_{i-1}, v_i\} \in E$ for all integers $1 \le i \le \ell - 1$ and there are no other elements in E. If one adds the edge $e = \{v_0, v_{\ell-1}\}$ to the set of edges E of P_ℓ then one obtains for $E' = E \cup \{e\}$ a cycle $C_\ell = (V, E')$ of length ℓ.

For colorings of the edges of complete graphs with two colors the Ramsey number $r(P_k, P_\ell)$ for paths was determined by Gerencsér and Gyárfás:

---

*This research was done during the author's stays at the IBM Scientific Center in Heidelberg, Germany, and at the Department of Mathematics and Computer Science of Emory University in Atlanta, Georgia, USA.

† partially supported by DFG-Deutsche Forschungsgemeinschaft.

Theorem 1 [GG 67]. *Let k, ℓ be positive integers with $2 \le k \le \ell$. Then*

$$r(P_k, P_\ell) = \ell - 1 + \left\lfloor \frac{k}{2} \right\rfloor .$$

Thus, for two-colorings the path Ramsey numbers $r(P_k, P_k)$ grow linearily in k and ℓ. In general, for colorings with more than two colors the growth rate of the path Ramsey numbers is similar, but until now these numbers are not completely determined for all possible arguments. A lower bound has been given by Faudree and Schelp:

Lemma [FS 75]. *Let $\ell_0, \ell_1, \dots, \ell_{r-1} \ge 2$ be positive integers. Then*

$$r(P_{\ell_0}, P_{\ell_1}, \dots, P_{\ell_{r-1}}) \ge \ell_0 - (r-1) + \sum_{i=1}^{r-1} \left\lfloor \frac{\ell_i}{2} \right\rfloor .$$

Faudree and Schelp have shown that for certain arguments $(\ell_0, \ell_1, \dots, \ell_{r-1})$ this lower bound gives the exact value of the corresponding path Ramsey numbers:

Theorem 2 [FS 75]. *Let $\ell_0, \ell_1, \dots, \ell_{r-1}$ be positive integers with $\ell_0 \ge 6 \cdot \left(\sum_{i=1}^{r-1} \ell_i\right)^2$. Then for $\delta = 0$ or $\delta = 1$ it is:*

$$r(P_{\ell_0}, P_{2\ell_1+\delta}, P_{2\ell_2}, \dots, P_{2\ell_{r-1}}) = 1 - r + \sum_{i=0}^{r-1} \ell_i .$$

On the other hand, equality in the Lemma does not always hold, as Irving proved for the constant arguments $(3, 3, \dots, 3)$:

Theorem 3 [Ir 74]. *For colorings with r colors it is valid:*

$$r(P_3, P_3, \dots, P_3) = \begin{cases} r+1 & \text{if } r \text{ is even} \\ r+2 & \text{if } r \text{ is odd.} \end{cases}$$

Now we will consider cycle Ramsey numbers $r(C_{\ell_0}, C_{\ell_1}, \dots, C_{\ell_{r-1}})$, where $\ell_i \ge 3$ for $i = 0, 1, \dots, r-1$. For colorings of edges of complete graphs with two colors these were completely determined by Rosta and Faudree and Schelp:

Theorem 4 [Ro 73], [FS 74]. *For all integers k, ℓ with $k, \ell \ge 3$ it is valid:*

$$r(C_k, C_\ell) = \begin{cases} 2k-1 & \text{if } \ell \text{ is odd and } (k,\ell) \ne (3,3) \\ k + \frac{\ell}{2} - 1 & \text{if } k, \ell \text{ are even and } (k,\ell) \ne (4,4) \\ \max\{\ell - 1, k - 1 + \frac{\ell}{2}\} & \text{if } k \text{ is odd and } \ell \text{ is even} \\ 6 & \text{if } (k,\ell) \text{ is } (3,3) \text{ or } (4,4). \end{cases}$$

In any case, for two-colorings the cycle Ramsey numbers $r(C_k, C_\ell)$ grow linearily. For colorings with more than two colors the situation is different from that for paths. While $r(C_3, C_3, \ldots, C_3)$ grows exponentially in the number r of colors, the number $r(C_4, C_4, \ldots, C_4)$ grows only quadratically in r, *cf.* [CG 75]. The reason for this difference is that every graph on n vertices and with $\lfloor 0.5n^{1.5} + 0.25n \rfloor$ edges contains a C_4 as a subgraph, while a graph on n vertices can have $\lfloor 0.25n^2 \rfloor$ edges without containing a C_3.

For the general situation of colorings with r colors and cycles of odd length Erdös and Graham obtained the following lower and upper bounds:

Theorem 5 [EG 75]. *Let ℓ be a positive integer. Then for colorings with r colors it is valid:*

$$\ell \cdot 2^r \leq r(C_{2\ell+1}, C_{2\ell+1}, \ldots, C_{2\ell+1}) \leq (r+2)! \cdot 2\ell.$$

Ordered graphs.

While until now we considered graphs with the vertex set being unordered, in this section we will investigate the ordered case, in particular, we will give lower and upper bounds for the ordered Ramsey numbers for monotone paths and cycles. That is, the set V of vertices is endowed with a total order, say V is a subset of the set ω of nonnegative integers, and the subgraphs under consideration, namely, paths and cycles induce a monotone sequence in the following sense:

A monotone path $P_\ell^m = (V, E)$ of length ℓ is a subgraph of G on ℓ vertices $v_0 < v_1 < \ldots < v_{\ell-1}$ such that $\{v_i, v_{i+1}\} \in E$ for all $i < \ell - 1$ and there are no other edges in E. If one adds the edge $\{v_0, v_{\ell-1}\}$ to the set of edges of the monotone path P_ℓ^m, then we obtain a monotone cycle C_ℓ^m of length ℓ. Moreover, let K_ℓ denote the complete graph on ℓ vertices.

We will consider at first the ordered Ramsey numbers for complete graphs versus monotone paths:

Theorem 6. *Let $\ell_0, \ell_1, \ldots, \ell_{r-1}$ be positive integers. Then it is valid*

$$r(K_{\ell_0}, P_{\ell_2}^m, \ldots, P_{\ell_{r-1}}^m) = 1 + \prod_{i=0}^{r-1} (\ell_i - 1).$$

For coloring with two colors, that is, $r = 2$, this was proved by Erdös in [Er 47].

Proof of Theorem 6: Let $n = \prod_{i=0}^{r-1} (\ell_i - 1)$ and let $n^* = \prod_{i=0}^{r-2} (\ell_i - 1)$. We will show first by induction on r the lower bound $r(K_{\ell_0}, P_{\ell_1}^m, \ldots, P_{\ell_{r-1}}^m) \geq 1 + n$. For $r = 1$ this is trivial, thus, let us assume that $r(K_{\ell_0}, P_{\ell_2}^m, \ldots, P_{\ell_{r-2}}^m) \geq 1 + n^*$. Partition the set $\{0, 1, \ldots, n-1\}$ into subintervals $I_j = \{jn^* - 1, jn^*, \ldots, (j+1)n^* - 1\}$ for $j < \ell_{r-1} - 1$. For each j there exists by assumption a coloring

Δ_j: $[I_j]^2 \to \{0,1,\ldots,r-2\}$ such that there exists neither a monochromatic K_{ℓ_0} in color 0 nor a monotone path of length ℓ_i in color i for some $i \geq 1$. Then, considering the coloring $\Delta: [\{0,1,\ldots,n-1\}]^2 \to \{0,1,\ldots,r-1\}$ defined by

$$\Delta(\{x,y\}<) = \begin{cases} \Delta_j(\{x,y\}_<) & \text{if } x,y \in I_j \\ r-1 & \text{else} \end{cases}$$

gives the desired lower bound.

In order to prove the upper bound, consider at first the case $r = 2$. We use induction on ℓ_1. Put $I_0 = \{0,1,\ldots,(\ell_0-1)(\ell_1-1)\}$ and let $\Delta: [I_0]^2 \to \{0,1,\ldots,r-1\}$ be a coloring. By the induction hypothesis there exist in the set I_0 a monochromatic K_{ℓ_0} in color 0, or a monotone path $P^m(0)$ of length $\ell_1 - 1$, which is monochromatic in color 1. In the first case we are ready, thus, let us assume that there exists a monotone path $P^m(0)$ of length $\ell_1 - 1$ in color 1. Put $I_1 = I_0 \setminus \{\min P^m(0)\}$, where the minimum is taken over the set of vertices of $P^m(0)$. In I_1 we find by induction hypothesis w.l.o.g. also such a monotone path $P^m(1)$ of length $\ell_1 - 1$, which is monochromatic in color 1. Iterating this argument, we find finally ℓ_0 many monotone paths $P^m(j)$ of length $\ell_1 - 1$, which are monochromatic in color 1 and have pairwise distinct minima. Then either there exists some pair (i,j) such that $\Delta(\min P^m(i), \min P^m(j)) = 1$, and we have a monotone path of length ℓ_1 in color 1 or the complete graph of order ℓ_0 given by the minima of the paths $P^m(j)$ is monochromatic in color 0.

If the number r of colors is at least three, we will use a double induction on r and on ℓ_{r-1}. By induction assumption for r and $\ell_{r-1} - 1$ we find as before in the set $\{0,1,\ldots,\prod_{i=0}^{r-1}(\ell_i-1)\}$ at least $1 + \prod_{i=1}^{r-2}(\ell_i - 1)$ many monotone paths $P^m(j)$, $j \leq n^\star$, w.l.o.g. of length $\ell_{r-1} - 1$ and color $r-1$, which have pairwise distinct minima. Consider the restriction of the coloring Δ to the set $[X]^2$, where $X = \{\min P^m(j) \mid j \leq n^\star\}$. If for two elements $x,y \in X$, $x<y$, it is $\Delta(\{x,y\}) = r-1$, then $x, P^m(j)$ with $y = \min P^m(j)$ determines a monochromatic monotone path of length ℓ_{r-1} in color $r-1$. Therefore, assume that this set $[X]^2$ is colored with $r-1$ colors. Then the induction hypothesis for $r-1$ colors gives the desired result. ∎

As an immediate consequence we obtain the following Theorem of Rado [Ra 77]:

Corollary [Ra 77]. *Let $\ell_0, \ell_1, \ldots, \ell_{r-1}$ be positive integers. Then it is valid*

$$r(P_{\ell_0}, P^m_{\ell_1}, \ldots, P^m_{\ell_{r-1}}) = 1 + \prod_{i=0}^{r-1}(\ell_i - 1).$$

Rado also investigated the corresponding Ramsey numbers for hypergraphs. For monotone paths in 3-uniform monotone paths he gives the exact value of these

Ramsey numbers for two-colorings, compare [ES 35], [Ra 77] and [Le 89] for related results.

The classical result of Erdös and Szekeres on monotone subsequences, which has applications in sorting problems in theoretical computer science, *cf.* [HHS 89], follows easily from Theorem 6:

Corollary [ES 35]. *Let m, n be positive integers. Then for every one-to-one mapping $f\colon \{0, 1, \dots, mn\} \to \omega$ there exists in f*

either an ascending subsequence with $(m+1)$ terms
or a descending subsequence with $(n+1)$ terms.

For the proof of this Corollary consider the coloring $\Delta : [\{0, 1, \dots, mn\}]^2 \dot{\to} \{0, 1\}$ defined by $\Delta(\{i, j\} <) = 0$ if $f(i) < f(j)$ and $\Delta(\{i, j\} <) = 1$ if $f(i) > f(j)$.

Moreover, by Theorem 6 the result of Erdös and Szekeres can be extended to arbitrary partial orders in the following way:

Corollary. *Let k, ℓ, m, n be positive integers and let $(X, \leq)$ be a partially ordered set. Then for every mapping $f\colon \{0, 1, \dots, k\ell mn\} \to X$ there exists in f*

either an antichain with $(k+1)$ terms
or a strongly ascending subsequence with $(\ell+1)$ terms
or a strongly descending subsequene with $(m+1)$ terms
or a constant subsequence with $(n+1)$ terms.

Next, we will consider the Ramsey numbers for monotone cycles.

Theorem 7. *Let k, ℓ with $k, \ell \geq 3$ be positive integers. Then*

$$1 + (k-1)(\ell-1) \leq r(C_k^m, C_\ell^m) \leq 2k\ell - 3k - 3\ell + 6.$$

Proof of Theorem 7: As the graph P_k^m is a subgraph of the graph C_k^m it follows that $r(C_k^m, C_\ell^m) \geq r(P_k^m, P_\ell^m)$, and, thus, we get the lower bound by Theorem 6.

For proving the upper bound, put $n = r(P_{k-1}^m, C_\ell^m) + r(C_k^m, P_{\ell-1}^m)$ and let $\Delta [\{0, 1, \dots, n-1\}]^2 \to \{0, 1\}$ be a coloring. Then at the element 0 the number of incident edges is at least either $r(P_{k-1}^m, C_\ell^m)$ in color 0 or $r(C_k^m, P_{\ell-1}^m)$ in color 1. By Theorem 6, as C_k^m is a subgraph of K_k we have $r(K_k, P_\ell^m) \geq r(C_k^m, P_\ell^m)$. Therefore, we conclude:

$$\begin{aligned} r(C_k^m, C_\ell^m) &\leq r(P_{k-1}^m, C_\ell^m) + r(P_k^m, C_{\ell-1}^m) \\ &= (k-2)(\ell-1) + 1 + (k-1)(\ell-2) + 1 \\ &= 2k\ell - 3k - 3\ell + 6. \end{aligned}$$

∎

It is not clear whether the lower or the upper bound is the better approximation of the numbers $r(C_k^m, C_\ell^m)$. In particular, for $k = \ell = 3$ we have $r(C_3^m, C_3^m) = 6$, which is just the upper bound in Theorem 7.

In order to obtain upper bounds for the general numbers $r(C_{\ell_0}^m, C_{\ell_1}^m), \ldots, C_{\ell_{r-1}}^m)$ we will look at the numbers $r(K_k, C_\ell^m)$. By a similar argument as in the proof of Theorem 7 we obtain

$$r(K_k, C_\ell^m) \leq r(K_{k-1}, C_\ell^m) + r(K_k, P_{\ell-1}^m)$$

and evaluating this recursion gives

$$\begin{aligned} r(K_k, C_\ell^m) &\leq \ell + \sum_{i=2}^{k-1} (i(\ell - 2) + 1) \\ &= \frac{k(k-1)(\ell-1)}{2} + 1 \\ &< \frac{k^2(\ell-1)}{2}. \end{aligned}$$

Moreover, using a color mixing argument, that is, for a given coloring $\Delta : [\{0, 1, \ldots, n-1\}]^2 \to \{0, 1, \ldots, r-1\}$ define a new coloring $\Delta^*: [\{0, 1, \ldots, n-1\}]^2 \to \{0, 1\}$ in the following way: $\Delta^*(\{x, y\}) = 0$ if and only if $\Delta(\{x, y\}) = r - 1$. Thus, we conclude

$$r(C_{\ell_0}^m, C_{\ell_1}^m, \ldots, C_{\ell_{r-1}}^m) \leq r(K_{r(C_{\ell_0}^m, C_{\ell_1}^m, \ldots, C_{\ell_{r-2}}^m)}, C_{\ell_{r-1}}^m$$

and, therefore, by induction we obtain with this recurrence relation the following

Fact 8. *Let $\ell_0, \ell_1, \ldots, \ell_{r-1}$ be positive integers with $3 \leq \ell_0 \leq \ell_1 \leq \ldots \leq \ell_{r-1}$. Then*

$$1 + \prod_{i=0}^{r-1} (\ell_i - 1) \leq r(C_{\ell_0}^m, \ldots, C_{\ell_{r-1}}^m) \leq \frac{\ell_0^{2^{r-2}} \cdot \prod_{i=1}^{r-1} \ell_i^{2^{r-1-i}}}{2^{\frac{r(r-1)}{2}}}.$$

A fast growing function.

In this chapter we will indicate for monotone paths and cycles some applications related to logic. In [PH 77] Paris and Harrington gave the first concrete example for Gödels Second Incompleteness Theorem, namely, they proved that the following variant of Ramsey's Theorem is true, but not provable in first order Peano Arithmetic:

Theorem 9 [PH 77]. *For all positive integers a, k, r, there exists a positive integer n such that the following is valid:*

(*) *For every colorng* $\Delta: [\{a, a+1, \dots, n\}]^k \to \{0, 1, \dots, r-1\}$ *of the* k*-element subsets of* $\{a, a+1, \dots, n\}$ *with* r *colors there exists a subset* $X \subseteq \{a, a+1, \dots, n\}$ *which is relatively large, that is,* $|X| \geq \min X, k+1$, *and which is monochromatic, that is, the restriction* $\Delta|[X]^k$ *is a constant coloring.*

One can prove this theorem by applying the infinite version of Ramsey's Theorem. Notice, that this is not a proof within first order Peano Arithmetic.

For given positive integers a, k, r let $PH(a, k, r)$ denote the least positive integer n such that the sentence (*) in Theorem 9 is true. While Paris and Harrington used model theoretic arguments to prove their result, later Ketonen and Solovay [KS 81] showed by purely combinatorial arguments, that the diagonal function $PH(k+1, k, k)$ grows more rapidly than every function $f: \omega \to \omega$ for which first order Peano Arithmetic can prove totality. This gives another proof of the Theorem of Paris and Harrington.

The above mentioned results deal essentially with complete graphs and hypergraphs as the monochromatic subgraphs. We are interested whether also subgraphs with a small number of edges, in particular, monotone paths and cycles, imply that the corresponding Paris-Harrington functions grow fast.

Define the number $PH(C^m)(a, r)$ as the least positive integer n such that for every coloring $\Delta^*: [\{a, a+1, \dots, n\}]^2 \to \{0, 1, \dots, r-1\}$ there exists a monochromatic monotone cycle $C^m = (V, E)$, which is relatively large, that is, $|V| \geq \min V, 3$. We will show that the function $PH(C^m)(0, r)$ is not primitive recursive. In order to do so, we define a sequence $(f_i)_{i<\omega}$ of functions $f_i: \omega \to \omega$ by

$$f_0(n) = n + 1$$

and, if the functions f_i are defined for all $i < j$, let

$$f_j(n) = f_{j-1}^{n+1}(1)$$

where for integers n and functions f the n'th iteration of f is $f^n(x) = f(f(\dots(f(k))\dots))$. Finally, diagonalize and define an Ackermann function $g: \omega \to \omega$ by

$$g(n) = f_n(n).$$

Such a function g does not belong to the class of primitive recursive functions, see [Pe 67], for example.

Theorem 10. *The function* $PH(C^m)(0, r)$ *is not primitive recursive.*

Proof of Theorem 10: We will show that $n = PH(C^m)(0, 2r+3) > g(r)$ for all $r \geq 3$. As polynomial substitution in the argument does not affect the property of a function beeing not primitive recursive, this will show the desired result.

Consider the following coloring $\Delta: [\{0,1,\ldots,n\}]^2 \to \{0,1,\ldots,2r+2\}$ defined by

$$\Delta(\{i,j\}<) = \begin{cases} i & \text{if } i \leq r+1 \\ r+2+y & \text{if } y = \max\{k \leq r \mid f_k(i-2) \leq j-2\}. \end{cases}$$

Suppose that the set $X = \{x_0, x_1, \ldots, x_{|X|-1}\}$ determines a relatively large monochromatic monotone cycle. Then it follows $\Delta(\{x_i, x_{i+1}\}_<) = r+2+y$ for some $y \leq r$, in particular, $x_0 \geq r+2$. Moreover, as $f_y(x_i - 2) \leq x_{i+1} - 2$, this implies

$$\begin{aligned} x_{|X|-1} - 2 &\geq f_y^{(|X|-1)}(x_0 - 2) \\ &\geq f_y^{(x_0-1)}(x_0 - 2) \qquad \text{as } |X| \geq x_0 \\ &\geq f_y^{(x_0-1)}(1) \\ &= f_{y+1}(x_0 - 2). \end{aligned}$$

By our assumption the set $X = \{x_0, x_1, \ldots, x_{|X|-1}\}$ determines a monotone monochromatic cycle and, therefore, $\Delta(\{x_0, x_{|X|-1}\}_<) = r+2+y$, that is, $f_y(x_0 - 2) \leq x_{|X|-2} - 2$. By the maximality of y it follows $y = r$. This implies

$$\begin{aligned} g(r) = f_r(r) &\leq f_r(x_0 - 2) \qquad \text{as } x_0 \geq r+2 \\ &< f_r(x_1 - 2) \\ &= x_2 - 2 \\ &\leq PH(C^m)(0, 2r+3), \end{aligned}$$

and finally proves the theorem. ∎

Concerning upper bounds, by color mixing arguments one can prove that these are also of Ackermann type; we omit the details.

It would be interesting to know whether the diagonal Paris-Harrington numbers for monotone cycles in uniform hypergraphs (defined analogously — cyclically — as in the case of graphs) has a similar growth rate as the function $PH(k+1, k, k)$ for complete hypergraphs does have.

References

CG 75. F.K.G. Chung and R.L. Graham, *Multicolor Ramsey theorems for complete bipartite graphs*, J. Comb. Theory, Ser. B **19** (1975), 164–169.

EG 75. P. Erdös and R.L. Graham, *On partition theorems for finite graphs*, in "Infinite and Finite Sets", (Haynal, Rado, Sos, eds.), Budapest, 1975, pp. 515–527.

Er 47. P. Erdös, *Some remarks on the theory of graphs*, Bull. Am. Math. Soc. **53** (1947), 292–294.

ER 50. P. Erdös and R. Rado, *On a combinatorial theorem*, J. London Math. Soc. **25** (1950), 249–255.

ES 35. P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compos. Math. **2** (1935), 463–470.

FS 74. R. Faudree and R. Schelp, *All Ramsey numbers for cycles in graphs*, Disc. Math. **8** (1974), 313–329.

FS 75. R. Faudree and R. Schelp, *Path Ramsey numbers in multicolorings*, J. Comb. Theory, Ser. B **19** (1975), 150—160.

GG 67. L. Gerencsér and A. Gyárfás, *On Ramsey-type problems*, Ann. Univ. Sci. Budap. Eötvös, Sect. Math. **10** (1967), 167–170.

HHS 89. B. Halstenberg, W. Hohberg, and H.U. Simon, *The effect of deletions in binary search trees*, Technical Report TI 1 (1989). TH Darmstadt, 1989.

Ir 74. R. Irving, *Generalized Ramsey numbers for small graphs*, Disc. Math. **9** (1974), 251–264.

KS 81. J. Ketonen and R. Solovay, *Rapidly growing Ramsey functions*, Annals of Math. **113** (1981), 267–314.

Le 89. H. Lefmann, *A note on monoton waves*, J. Comb. Theory Ser. A (1989), 316–318.

PH 77. J. Paris and L. Harrington, *A mathematical incompleteness in Peano Arithmetic*, in "Handbook of Mathematical Logic", (Barwise, ed.), Amsterdam, 1977, pp. 1133–1142.

Ra 77. R. Rado, *Weak versions of Ramsey's theorem*, J. Comb. Theory Ser. B (1977), 24–32.

Ro 73. V. Rosta, *On a Ramsey type problem of J.A. Bondy and P. Erdös I + II*, J. Comb. Theory, Ser. B **15** (1973), 94–105, 105–120.