Two Sufficient Conditions for Pancyclic Graphs

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Abstract. This paper gives two sufficient conditions for a 2-connected graph to be pancyclic. The first one is that the degree sum of every pair of nonadjacent vertices should not be less than $n/2+\delta$. The second is that the degree sum of every triple of independent vertices should not be less than $n+\delta$, where n is the number of vertices and δ is the minimum degree of the graph.

1. Introduction

All graphs considered in this paper are simple. The terminology and notation used here are standard except as indicated. A good reference for any undefined term is Bondy and Murty's book [4], and the sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and the degree of a vertex v of G is denoted by $d_G(v)$, or $d(v)$ for simplicity. For $D \subseteq V(G)$, $G[D]$ denotes the subgraph of G induced by set D. The neighbourhood of the vertex v in G is $N_G(v)$, $N_G^*(v) = N_G(v) \cup \{v\}$, and $N_G(D) = \bigcup_{x\in D} N_G(v)$. The minimum degree, the independence number and the connectivity of G are denoted by δG, $\alpha(G)$ and $\kappa(G)$, respectively. Following Bauer et al [1], define

$$\sigma_k = \min\left\{\sum_{i=1}^{k} d(v_i) \,|\, \{v_1, v_2, \ldots, v_k\} \quad \text{is an independent set of vertices in } G\right\},$$

where $1 \le k \le \alpha(G)$. Obviously $\sigma_1 = \delta(G)$. If C is a directed cycle in G and $v \in V(C)$, v^+ and v^- denote, respectively, the successor and predecessor of the vertex v along C. The closure of a graph G, denoted by $\overline{G}$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair remains, where n is the number of vertices of G.

Bondy [2] has conjectured that all sufficient degree conditions for a graph to be Hamiltonian ensure the graph to be pancyclic. Examples are Ore's condition [2], Chvátal's degree sequence condition [7] and Fan's condition [8]. In this paper, two sufficient conditions ensuring graphs to be pancyclic are given. These conditions can be regarded as generalizations of Ore's condition. The first one is that the degree sum of every pair of nonadjacent vertices should be at least $n/2+\delta(G)$; the second is that the degree sum of every triple of independent vertices should be at least $n+\delta(G)$, where n is the number of vertices of the graph G, which is assumed 2-connected.

The following known results will be needed.

Theorem A [3]. *A graph G is Hamiltonian if and only if $\overline{G}$ is Hamiltonian.*

Theorem B [5]. *If G is κ-connected and $\alpha(G) \leq \kappa$, then G is Hamiltonian.*

Theorem C [1]. *Let G be a simple graph with n vertices and connectivity $\kappa(G) \geq 2$. If $\sigma_3 \geq n + \kappa(G)$, then G is Hamiltonian.*

Theorem D [2]. *Let G be a simple graph with n vertices. If $\sigma_2 \geq n$, then G is either a pancyclic graph or a balanced complete bipartite graph.*

Theorem E[6]. *Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamilton cycle. If the consecutive vertices v_n and v_1 on C satisfy $d(v_1) + d(v_n) \geq n$ with $d(v_1) \leq d(v_n)$, then G is either*

(i) *pancyclic,*
(ii) *bipartite, or*
(iii) *misses only cycles of length $n-1$.*

Moreover, if (iii) holds, then $d(v_{n-2}), d(v_{n-1}), d(v_2), c(v_3)$ all are less than $n/2$, and G has one of two possible adjacency structures near v_1 and v_n: The first structure is that $\{v_{n-2}, v_{n-1}, v_n, v_1, v_2, v_3\}$ is an independent set of vertices in $G - E(C)$, and $(v_n, v_3), (v_n, v_{n-4}), (v_1, v_4), (v_1, v_5) \in E(G)$. The second structure which can occur only if $d(v_1) < d(v_n)$ is identical to the first one except that $(v_n, v_3) \in E(G)$ and $(v_1, v_5) \notin E(G)$.

Lemma F. *Let G be a simple graph on n vertices, and $C = (v_1, v_2, \ldots, v_{n-1})$ be a cycle of G. If $d(v_n) \geq n/2$, then G is pancyclic.*

Lemma G. *If G is a simple graph on n vertices with $n > 2\delta(G)$, then G contains either a path of length at least $2\delta(G)$ or two vertex-disjoint paths of length at least $\delta(G)$.*

The proofs of these two lemmas are very easy, so they will be omitted.

Lemma H. *Let $C = (u_0, u_1, u_2, \ldots, u_{n-1})$ be a Hamiltonian cycle of a graph G. If there is an index i such that, $d(u_i) + d(u_{i+1}) > n$ then G is pancyclic.*

A proof of this lemma can be read out directly from the proof of Theorem E given in [6].

2. Result 1

In this section we shall prove two theorems concerning graphs with large σ_2.

Theorem 1. *Let G be a 2-connected graph on $n \geq 3$ vertices. G satisfies $\sigma_2 \geq n/2 + \delta(G)$, then G is Hamiltonian.*

Proof: Let $x \in V(G)$ with $d(x) = \delta(G)$. Put $S = V(G) - N^*(x)$. By Theorem D, we may assume that $\delta(G) < n/2$, and hence $|S| = n - \delta(G) - 1 \geq (n-1)/2$ and $d(v) \geq n/2$ for all $v \in S$. Therefore $\overline{G}[S]$ is a complete subgraph of

$\overline{G}$. By the connectivity of G there exist $v_1, v_2 \in N(x)$, $u_1, u_2 \in S$ such that $(v_i, u_i) \in E(G), i = 1, 2$. Put

$$R = \{v \in N(x) | d_G(v) \geq \delta(G) + 1\}, \quad r = |R|.$$

Three cases will be considered:

Case 1: $r \geq 2$. In this case $d_{\overline{G}}(u_i) \geq n - \delta(G), i = 1, 2$, and $d_{\overline{G}}(x) = n - 1$. Furthermore, for all $u \in S$, $d_{\overline{G}}(u) \geq n - \delta(G)$, thus $\overline{G}$ is complete. By Theorem A, the graph G is Hamiltonian.

Case 2: $r = 1$. Let $v_0 \in R$, $d(v_0) = \delta(G) + 1$. Then $d(v) = \delta(G)$ for all $v \in N^*(x) - \{v_0\}$. Therefore $G[N^*(x) - \{v_0\}]$ is complete, since otherwise $\delta(G) \geq n/2$. By the connectivity of G and the fact that $G[N^*(x) - \{v_0\}]$ and $\overline{G}[S]$ are complete, we conclude that the graph G is Hamiltonian, and so is G by Theorem A.

Case 3: $r = 0$. From the assumption that $\delta(G) < n/2$, it can be seen that $G[N^*(x)]$ is complete. Since $G[N^*(x)]$ and $\overline{G}[S]$ are complete, the graph $\overline{G}$ is hamiltonian by the connectivity of G. Thus G is Hamiltonian.

The proof of Theorem 1 is complete.

Theorem 2. *Let G be a 2-connected graph on $n \geq 3$ vertices. If G satisfies $\sigma_2 \geq n/2 + \delta(G)$, then either G is pancyclic or $G = K_{n/2,n/2}$.*

Proof: By Theorem D we may assume that $\delta(G) < n/2$. From Theorem 1 the graph G is Hamiltonian. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a Hamilton cycle of G. The proof of the theorem will be divided into two steps.

Claim 1: There exist consecutive vertices v_i and v_{i+1} on C such that $d(v_i) + d(v_{i+1}) \geq n$.

Assume Claim 1 does not hold. Then $|S| = n - \delta(G) - 1 \leq n/2$ by the fact that $d(v) \geq n/2$ for any $v \in S$ and thus $\delta(G) \geq n/2 - 1$. Without loss of generality, assume that $v_1 \in S$, so that $d(v_1) \geq n/2$. If $(v_2, v_n) \in E(G)$, then G is pancyclic by Lemma F. So we may assume that $(v_2, v_n) \notin E(G)$. Since $d(v_2) + d(v_n) \geq n/2 + \delta(G)$ and $d(v_2), d(v_n) < n/2$, we have $d(v_i) \geq \delta(G) + 1, i = 2, n$. Hence $n/2 > d(v_i) \geq \delta(G) + 1, i = 2, n$. This implies that $\delta(G) < n/2 - 1$, a contradiction, and hence Claim 1 does hold.

Without loss of generality, we may assume that $d(v_1) + d(v_n) \geq n$ with $d(v_1) \leq d(v_n)$. It is sufficient, by Theorem E, to prove the following.

Claim 2: The graph G contains a cycle of length $n - 1$.

Assume G does not contain a cycle of length $n - 1$. Then by Theorem E, we have $d(v_i) < n/2, i = 2, 3, n-1, n-2$, and $\{v_2, v_3, v_{n-1}, v_{n-2}\}$ is an independent set in $G - E(C)$. By the assumption that $\sigma_2 \geq n/2 + \delta(G)$, we can conclude that $d(v_i) \geq \delta(G) + 1, i = 2, 3, n-1, n-2$. Hence $r \geq 4$, where $r = |R| = |\{v \in N^*(x) | d(v) \geq \delta(G) + 1\}|$, $d(x) = \delta(G)$.

Next we prove that $G_1 = G - x$ is Hamiltonian, and this will be a contradiction. Indeed, since $\delta(G_1) \geq \delta(G) - 1$ and $r \geq 4$, the graph $\overline{G}_1$ contains a K_m where $m \geq n - \delta(G) + 3$, that is, $\overline{G}_1$ has at least $n - \delta(G) + 3$ vertices with degree at least $n - \delta(G) + 2$, therefore $\overline{G}_1$ is complete, and hence G_1 is Hamiltonian by Theorem A.

3. Result 2

In this section, the following theorem will be proved.

Theorem 3. *Let F_0 be the class of complete balanced bipartite graphs, F_1 be the class of complete balanced bipartite graphs missing one edge, and C_5 be a cycle of length 5. Then if G is a 2-connected graph on n vertices satisfying $\sigma_3 \geq n + \delta(G)$, then either G is pancyclic or $G \in F_2$, where $F_2 = F_0 \cup F_1 \cup \{C_5\}$.*

To prove the theorem the following results will be needed.

Theorem 4. *Let G be a 2-connected graph on n vertices. If G satisfies $\sigma_3 \geq n + \delta(G)$, and $\delta(G) \geq 3$, then there exists a vertex $x \in V(G)$ such that $d(x) = \delta(G)$ and $G' = G - x$ is 2-connected.*

Proof: Let $x \in V(G)$ with $d(x) = \delta(G)$. If $G' = G - x$ is not 2-connected, there exists $x_1 \in V(G')$ such that $G' - x_1$ is not connected. Let $G_1, G_2, \ldots, G_k$, $k \geq 2$, be the components of $G' - x_1$. Let $S = V(G) - N^*(x)$. If there exist i and j, $1 \leq i < j \leq k$, such that there exist u_i in $V(G_i) \cap S$ and u_j in $V(G_j) \cap S$, then $\{u_i, u_j, x\}$ is a triple of independent vertices, and hence by the assumption on G, we have $d(u_i) + d(u_j) \geq n$. Therefore $|V(G_i)| + |V(G_j)| \geq n \geq |V(G_i)| + |V(G_j)| + 2$, a contradiction. Hence $S - x_1$ is contained in the vertex set of one component of $G' - x_1$, say G_1. From the connectivity of G, it can be seen that $N(x) \cap V(G_1) \neq \emptyset$, $N(x_1) \cap V(G_1) \neq \emptyset$ and $V(G_i) \subseteq N(x)$, $i \neq 1$. Put $t = |N(x) \cap V(G_1)|$, then we have $|V(G_1)| = n - \delta(G) - 1 + t$, if $x_1 \notin S$, $|V(G_1)| = n - \delta(G) - 2 + t$ if $x_1 \in S$, and for other components G_i, $|V(G_i)| \geq \delta(G) - 1$. Therefore

$$n = |V(G)| \geq |V(G_1)| + |V(G_2)| + |\{x, x_1\}| \geq n + t - 1 \geq n.$$

Thus $t = 1$, $x_1 \in S$, $|V(G_2)| + \delta(G) - 1$, and $|V(G_1)| = n - \delta(G) - 1$, which implies that G_2 is a complete graph, and that every vertex in G_2 is adjacent to x_1 and has degree $\delta(G)$ in G. By the fact that $\delta(G) \geq 3$ and $|V(G_2)| \geq 2$, $G - u$ is 2-connected for any $u \in V(G_2)$. The proof is complete.

Theorem 5. *Let G be a 2-connected graph on n vertices with $\delta(G) \geq 3$. If G satisfies $\sigma_3 \geq n + \delta(G)$ and $G \notin F_2$, then G contains a cycle of length $n - 1$.*

Proof: By Theorem D, we may assume that $\delta(G) < n/2$. Suppose the graph G contains no cycle of length $n - 1$.

By Theorem 4, there is a vertex $x \in V(G)$ with $d(x) = \delta(G)$ such that $G' = G - x$ is 2-connected. By the assumption on G, the graph G' is not Hamiltonian, and so neither is $\overline{G}'$, the closure of G', by Theorem A. By Theorem B there exists a triple $\{u_1, u_2, u_3\}$ of independent vertices in $\overline{G}'$. Let $S = V(G) - N^*(x)$, since $\overline{G}'[S]$ is complete, we have $|\{u_1, u_2, u_3\} \cap S| \leq 1$.

Case 1: $|\{u_1, u_2, u_3\} \cap S| = 1$.

Let $u_3 \in S$, then $u_1, u_2 \in N(x)$. To prove the theorem in this case we will prove a series of claims.

Claim 1.1: $d(u_1) = d(u_2) = \delta(G) + 1$, and $\delta(G) \leq n/2 - 1$. Indeed, from the condition that $\sigma_3 \geq n + \delta(G)$, we have that $d_{\overline{G}'}(u_1) + d_{\overline{G}'}(u_2) + d_{\overline{G}'}(u_3) \geq d(u_1) - 1 + d(u_2) - 1 + d(u_3) \geq n + \delta(G) - 2$. If $d(u_1) = \delta(G)$, then $d_{\overline{G}'}(u_2) + d_{\overline{G}'}(u_3) \geq n - 1$, which implies that $u_2 u_3 \in E(\overline{G}')$, a contradiction. Thus $d(u_1) \geq \delta(G) + 1$.

If $d(u_1) \geq \delta(G) + 2$, then $d_{\overline{G}'}(u_1) + d_{\overline{G}'}(u_3) \geq d(u_1) - 1 + d_{\overline{G}'}(u_3) \geq \delta(G) + 1 + n - \delta(G) - 2 = n - 1$, which implies that $u_1 u_3 \in E(\overline{G}')$, a contradiction. Hence $d(u_1) = \delta(G) + 1$. By symmetry, $d(u_2) = \delta(G) + 1$.

Since $u_1 u_2 \notin E(\overline{G}')$, we have $n - 1 > d_{\overline{G}'}(u_1) + d_{\overline{G}'}(u_2) \geq \delta(G) + \delta(G)$ and thus $\delta(G) < (n-1)/2$, or $\delta(G) \leq n/2 - 1$.

Claim 1.2: $d(u) = \delta(G) + 1$ for all $u \in N(x)$.

Since $u_1 u_3 \notin E(\overline{G}')$, by Theorem A we have $d_{\overline{G}'}(u_1) + d_{\overline{G}'}(u_3) < n - 1$, which implies that $d_{\overline{G}'}(u_3) = n - \delta(G) - 2$, and thus $N^*_{\overline{G}'}(u_3) = S$ by the fact that $\overline{G}'[S]$ is a complete graph with $n - \delta(G) - 1$ vertices.

Since $uu_3 \notin E(\overline{G}')$ for any $u \in N(x)$, we have $d_{\overline{G}'}(u) + d_{\overline{G}'}(u_3) < n - 1$, that is, $d_{\overline{G}'}(u) < n - 1 - d_{\overline{G}'}(u_3) = n - 1 - n + \delta(G) + 2 = \delta(G) + 1$. Therefore $d(u) - 1 \leq d_{\overline{G}'}(u) \leq \delta(G)$, i.e. $d(u) \leq \delta(G) + 1$ and $d_{\overline{G}'}(u) \leq \delta(G)$.

On the other hand, by the connectivity of G', there exist $v_1, v_2 \in S$ and $y_1, y_2 \in N(x)$, such that $y_i v_i \in E(G')$, $i = 1, 2$. Thus $d_{\overline{G}'}(v_i) \geq |S| = n - \delta(G) - 1$ and by Theorem A we have $y_i v_i \in E(\overline{G}')$, for $1 \leq i, j \leq 2$. This shows that $d_{\overline{G}'}(v_j) \geq n - \delta(G)$, which implies that $uv_j \in E(\overline{G}')$ for any $u \in N(x)$ and $j = 1, 2$. Since $d_{\overline{G}'}(u) \leq \delta(G)$, $uv_j \in E(\overline{G}')$ and $|N(x)| = \delta(G)$, there is a vertex $u' \in N(x)$ such that $uu' \notin E(G')$. Replacing u_1 and u_2 by u and u', respectively, we get $d(u) = d(u') = \delta(G) + 1$ by the same argument. Hence $d(u) = \delta(G) + 1$ for all $u \in N(x)$. Moreover, since $\delta(G) \geq d_{\overline{G}'}(u) \geq d(u) - 1 = \delta(G)$, we have $d_{\overline{G}'}(u) = \delta(G)$ for all $u \in N(x)$, that is, $d_{\overline{G}'}(u) = d_{G'}(u)$ for all $u \in N(x)$).

Let $S' = N_S(N(x))$, then it is easy to see that $uv \in E(\overline{G}')$ for all $u \in N(x)$ and $v \in S'$. Clearly $|S'| \leq \delta(G)$, and $G'[N(x)]$ is a k-regular graph, where $k = \delta(G) - |S'|$.

Claim 1.3: $k \neq 0$.

If $k = 0$, then $N(x)$ is an independent set of vertices in G'. Let

$$N(x) = \{u_1, u_2, \ldots, u_g\}, \quad S' = \{v_1, v_2, \ldots, v_g\},$$

where $g = \delta(G)$. By the assumption of the theorem we have

$$n + \delta(G) \leq d(u_1) + d(u_2) + d(u_3) = 3\delta(G) + 3,$$

thus $\delta(G) \geq (n-1)/2 - 1$. From Claim 1.1, we have $n/2 - 1 \geq \delta(G) \geq (n-1)/2 - 1$. For every vertex $v \in S - S'$, $\{u_1, u_2, v\}$ is an independent triple of G, so $d(v) \geq n + \delta(G) - d(u_1) - d(u_2) = n - \delta(G) - 2$. Therefore for any $v, v' \in S$, if one of them is in $S - S'$, then $vv' \in E(G)$. It is easy to check the theorem is true for this case. Hence $k \neq 0$.

Since $k > 0$, the graph $G'[N(x)]$ is a k-regular graph. Let $N(x) = \{u_1, u_2, \ldots, u_g\}$, $S' = \{v_1, v_2, \ldots, v_{g-k}\}$, and $S - S' = \{v_{g-k+1}, v_{g-k+2}, \ldots, v_{n-g-1}\}$.

If $k \geq (\delta(G) - 1)/2$, then $G'[N(x)]$ contains a Hamilton path, and thus $\overline{G'}$ is Hamiltonian by the fact that $\overline{G'}[S]$ is complete. Hence G' is Hamiltonian by Theorem A, a contradiction.

If $1 \leq k \leq \delta(G)/2 - 1$ and $G'[V(x)]$ does not contain a Hamilton path, then $G'[N(x)]$ contains either a path P of length at least $2k$ or two vertex-disjoint paths, P_1 and P_2, each of them having length at least k by Lemma G.

In the first case, without loss of generality, let $P = (u_1, u_2, \ldots, u_p)$, $2k + 1 \leq p \leq \delta(G) - 1$. The following is a Hamiltonian cycle of $\overline{G'}$:

$$\begin{aligned} C =& (v_1, u_1, u_2, \ldots, u_p, v_2, u_{p+1}, v_3, \ldots, \\ & u_g, v_{g-p+2}, v_{g-p+3}, \ldots, v_{g-k}, v_{g-k+1}, \ldots, v_{n-g-1}). \end{aligned}$$

In the second case, without loss of generality, let

$$\begin{aligned} P_1 &= (u_1, u_2, \ldots, u_t), \\ P_2 &= (u_{t+1}, u_{t+2}, \ldots, u_{t+g}), \end{aligned}$$

$t + q \leq \delta(G)$ and $t, q \geq k + 1$. The following is a Hamiltonian cycle of $\overline{G'}$:

$$\begin{aligned} C =& (v_1, u_1, u_2, \ldots, u_t, v_2, u_{t+1}, u_{t+2}, \ldots, u_{t+q}, v_3, u_{t+q+1}, v_4, u_{t+q+2}, v_5, \ldots, \\ & u_g, v_{g-t+3-q}, v_{g-t-q+4}, \ldots, v_{g-k}, v_{g-k-1}, \ldots, v_{n-g-1}). \end{aligned}$$

In both cases G' is Hamiltonian by Theorem A. Hence G contains a cycle of length $n - 1$, which contradicts the hypothesis on G. The proof of the theorem for Case 1 is complete.

Case 2: $|\{u_2, u_2, u_3\} \cap S| = 0$ for any independent vertex set $\{u_1, u_2, u_3\}$.

First we show that in this case $uv \in E(\overline{G'})$ for any $u \in N(x)$ and $v \in S$.

Since $\{u_1, u_2, u_3\} \cap S = \emptyset$ for any triple of independent vertices in $V(\overline{G'})$, there are u_i and u_j in $\{u_1, u_2, u_3\}$ for any $v \in S$, such that $u_iv, u_jv \in E(G')$ and thus $d_{\overline{G'}}(v) \geq |S| - 1 + 2 = n - \delta(G)$. Therefore $d_{\overline{G'}}(u) + d_{\overline{G'}}(v) \geq \delta(G) - 1 + n - \delta(G) = n - 1$ for any $u \in N(x)$. Hence $uv \in E(\overline{G'})$ for any $u \in N(x)$ and $v \in S$ by the definition of closure $\overline{G'}$ of G'.

Since $u_1u_2 \notin E(\overline{G'})$, we have that $n - 2 \geq d_{\overline{G'}}(u_1) + d_{\overline{G'}}(u_2) \geq 2|S| = 2(n - \delta(G) - 1)$. Hence $\delta(G) \geq n/2$, which contradicts the hypothesis that $\delta(G) < n/2$. The proof of the theorem is complete.

Let n be odd and $F = F_n = (V, E)$ be the graph with vertex set $V = \{x\} \cup N \cup S' \cup \{x_1, x_2\}$ and edge set $E = E_1 \cup E_2 \cup E_3 \cup E_4$, where $|N| = |S'| = (n-3)/2$, $E_1 = \{xu : u \in N\}$, $E_2 = \{uv : u \in N, , v \in S'\}$, $E_3 \subseteq \{v_1v_2 : v_1, v_2 \in S'\}$, and $E_4 = \{vx_i : v \in S', i = 1, 2\} \cup \{x_1x_2\}$. The following corollary can be derived from the proof of Theorem 5.

Corollary. *Let G be 2-connected graph on n vertices with $\delta(G) \geq 3$. If G satisfies $\sigma_3 \geq n + \delta(G)$ and $G \notin F_2$, then either there exists a vertex x with $d(x) = \delta$ such that $G' = G - x$ is Hamiltonian, or n is odd and G is the graph F_n.*

Now we proceed to prove Theorem 3. It is clear that G is Hamiltonian by Theorem C. As usual we assume that $\delta(G) < n/2$, since otherwise the theorem holds for G by Theorem D. Also, the theorem holds if $n \leq 6$. Obviously, F is pancyclic. Two cases will be considered.

Case (A): $\delta(G) \geq 3$ and $G \notin F_2 \cup \{F\}$.

By the corollary above there is $x \in V(G)$ with $d(x) = \delta(G)$ such that $G' = G - x$ is Hamiltonian. Let C be a Hamiltonian cycle of G' with a fixed orientation, $S = V(G) - N^*(x)$ and $N^*(x) = N(x) \cup \{x\}$.

Subcase A.1: There exists $y \in S$ such that $d(y) \geq n/2$.

(1) If $y^+ \in S$ (or $y^- \in S$) and $d(y^+) + d(y) \geq n$ (or $d(y^-) + d(y) \geq n$), then G' is pancyclic by Lemma H, and hence so is G.

(2) If $y^+, y^- \in S$, $d(y^+) < n/2$, and $d(y^-) < n/2$, then $y^+y^- \in E(G)$. Therefore G' is pancyclic by Lemma F, and hence so is G.

(3) If $y^+, y^- \in N(x)$, then G is pancyclic by Lemma F.

(4) Thus we may assume that $y^- \in N(x)$, $y^+ \in S$ and $d(y^+) < n/2$.

Let $v_0 \in N(x)$ be the first vertex of C after y^+, and let $v_0 \overrightarrow{C} y^-$ and $v_0 \overleftarrow{C} y^-$ denote two segments of C determined by v_0 and y^-, where $v_0 \overrightarrow{C} y^-$ follows the orientation of C and $v_0 \overleftarrow{C} y^-$ does not.

Subcase (4.1): There exists s_0 in $v_0 \overrightarrow{C} y^-$ such that $s_0 \in S$.

Let $s_0 \in S$ be the first vertex from v_0 to y^- along C.

If $s_0 y^+ \in E(G)$ then $C' = xy^- \overleftarrow{C} s_0 y^+ \overrightarrow{C} s_0^- x$ is a Hamilton cycle of the graph $G - y$. Therefore G is pancyclic by Lemma F.

If $s_0 y^+ \notin E(G)$, then $d(s_0) > n/2$, and thus $s_0^+ \in S$ by (3) and $d(s^+) < n/2$ by (1). Hence $s^+ y^+ \in E(G)$ and $C' = xy^- \overleftarrow{C} s_0^+ y^+ \overrightarrow{C} s_0^- x$ is a Hamilton cycle of the graph $G - \{s_0, y_0\}$. Since $d(s_0) > n/2$, $G - y$ is pancyclic by Lemma F, and hence so is G.

Subcase (4.2): . All vertices of $v_0 \overrightarrow{C} y^-$ are in $N(x)$.

From the facts above we can see that there is no vertex v in $y^= \overrightarrow{C} v_0^-$ such that $d(v) \geq n/2$ except v_0^-. Therefore, $G[V(y^+ \overrightarrow{C} v_0^{-2})]$ is complete, and hence $G[V(y^+ \overrightarrow{C} v_0^-)]$ is complete if $d(v_0^-) < n/2$, by Lemma H. Since $n/2 > d(y^+) \geq |S| - 2 = n - \delta(G) - 3$ we have $\delta(G) > n/2 - 3$, or $\delta(G) > n/2 - 2$ if $d(v_0^-) < n/2$. Therefore G contains cycles of lengths k, $k = 3, \ldots, \max\{\delta(G) + 1, |S| - 2\}$ and $k = 7, 8, \ldots, n$. If G is not pancyclic, then G contains no cycle of length 5 or 6 by $\delta(G) \geq 3$.

If G does not contains a cycle of length 5, then $\delta(G) = 3$, $|S| \leq 6$ and

$$yv_0^-, v_0^- y^-, v_0^- v_0^+, yv_0, yv_0^+ \notin E(G).$$

Hence $|S| \leq n/2 + 1$, where the equality holds only if $yv_0^{-2} \in E(G)$. However we see that $yv_0^{-2} \notin E(G)$, since otherwise $d(v_0^{-2}) \geq |S| - 1 \geq n/2$, which is a contradiction. Therefore $|S| \leq n/2 + 2$, which implies that $d(v_0^{-2}) \geq |S| - 2 \geq n/2$, which is also a contradiction.

Similarly, it can be proved that G contains a cycle of length 6.

Subcase A.2: For all $v \in S$, $d(v) < n/2$.

In this case $G[S]$ is complete. By the connectivity of G there exists $v \in S$, such that $d(v) \geq |S| = n - \delta(G) - 1$. Since $d(v) < n/2$ and $\delta(G) < n/2$, we see that $n/2 > \delta(G) > (n/2) - 1$. Therefore $\delta(G) = (n-1)/2$. Since $n/2 > d(v) \geq \delta(G)$ for all $v \in S$, we have that $d(v) = (n-1)/2$ for all $v \in S$. Hence each vertex v of S is adjacent to exactly one vertex v of $N(x)$, which means that for every vertex u in S, $d(u) = \delta(G)$. If $G[N(x)]$ is complete, then G is pancyclic. If $G[N(x)]$ is not complete, we take a vertex v of$S$ instead of the vertex x, and repeat the argument above.

Case (B): . $\delta(G) = 2$.

Let $N(x) = \{x_1, x_2\}$.

Subcase B.1: $d(x_1) \geq 3$, $d(x_2) \geq 3$.

If $|N_S(N^*(x))| \geq 3$ or $x_1 x_2 \in E(G)$, then it is easy to prove that $G - x$ has a Hamilton cycle, and we proceed as in Case A. Therefore we may assume that $d(x_1) = d(x_2) = 3$ and $x_1 x_2 \notin E(G)$. Let $N_S(N^*(x)) = \{s_1, s_2\}$. Then for every $s \in S - \{s_1, s_2\}$, $\{s, x_1, x_2\}$ is an independent triple of G, and so $d(x_1) + d(x_2) + d(s) \geq n + 2$, or $d(s) \geq n - 4$. Therefore $G[S]$ is either $K_{n-\delta(G)-1}$ or $K_{n-\delta(G)-1} - s_1 s_2$. Hence G is pancyclic since $n \geq 7$.

Subcase B.2: $d(x_1)) = 2$. Let $N(x_1) = \{x, x_3\}$ and C be a Hamilton cycle of G. Put $x^- = x_2$, $x^+ = x_1$, and $x^{+2} = x_3$. If there exists $s_0 \in V(G) - \{x_1, x_2, x_3, x\}$ such that $d(s_0) \geq n/2$, then $d(s_0^-) + d(s_0^+) \geq n$ or $s_0^- s_0^+ \in E(G)$. Therefore G is pancyclic by Lemmas F and E.

If for all $s \in V(G) - \{x_1, x_2, x_3\}$ we have $d(s) < n/2$, then $G[S - \{x_3\}]$ is a complete graph. In this case, it is easy to see that G is pancyclic since $n \geq 7$. Indeed, if $d(x_2) \leq n/2$ then from $d(x_1) = 2$ we know that for all $v \in S - \{x_3\}$, $vx_2 \in E(G)$. Therefore $n/2 \geq d(x_2) \geq |S| = n - 3$, and so $n < 6$. Similarly, from $d(x_3) \leq n/2$ we can deduce $n \leq 6$.

Therefore $d(x_2) > n/2$, $d(x_3) > n/2$ and there exists $s \in S - \{x_3\}$ such that $x_2 s, x_3 s \in E(G)$. Hence $n - 3 = |S| = d(s) < n/2$, i.e. $n < 6$, which is a contradiction. This completes the proof of Theorem 3.

Remark: Theorem 2 and Theorem 3 are incomparable in the sense that one does not imply the other.

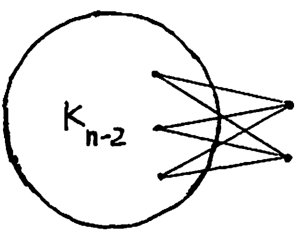

Figure 1

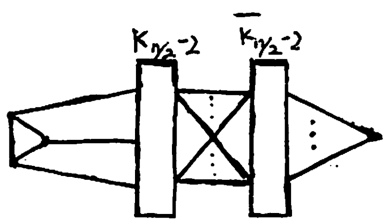

Figure 2

The graph in Fig. 2 satisfies the hypothesis in Theorem 2, but not hypothesis in Theorem 3. The graph in Fig. 1 satisfies the hypothesis in Theorem 3 but not the hypothesis in Theorem 2.

Acknowledgments

The authors would like to thank the referees and Dr. Kohayakawa for their valuable suggestions. This work was supported in part by the Natural Science Foundation of China.

References

1. D. Bauer et al., *A Generaligation of a Result of Haggkvist and Nicoghossian*, J. Combinatorial Theory (B) (to appear).
2. J. A. Bondy, *Pancyclic Graphs II*, J. Combinatorial Theory (B) **11** (1971), 80–84.
3. J. A. Bondy and V. Chvátal, *A Method in Graph Theory*, Discrete Math. **15** (1976), 111–135.
4. J. A. Bondy and U.S.R. Murty, "Graph Theory with Applications", MacMillan Press, London, 1976.
5. V. Chvátal, P. Erdös, *A Note on Hamiltonian Circuits*, Discrete Math. **2** (1972), 111–135.
6. S. L. Hakimi and E. Schmeichel, *A Cycle Structure Theorem for Hamiltonian Graphs*, J. Combinatorial Theory (B) 45 (1988), 99–107.
7. E. F. Schmeichel, S. L. Hakimi, *Pancycl ic Graphs and a Conjecture of Bondy and Chvátal*, J. Combinatorial Theory (B) 17 (1974), 22–34.
8. F. Tian, R. H. Shi, *A New Kind of Pancyclic Graphs*, J. Systems Sci. Math. Sci. (1986), 258–262.