A Planar Poset Which Requires Four Pages

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1. Introduction.

A book consists of a spine and a number of pages.

Let $P = (P, \leq)$ be a partially ordered set (simply, *poset*) and $\mathcal{L}(P)$ be the set of all linear extensions of P. The Hasse diagram of P is denoted by $H(P)$. Every *edge* $\{x, y\}$ in $H(P)$ corresponds to the covering pair $x \leq y$ $(x \neq y)$ in P, and we denote it by xy. The edge set of $H(P)$ is denoted by $E(P)$.

A book embedding of a poset P with respect to $L \in \mathcal{L}(P)$ is the embedding of $H(P)$ with its elements placed on the spine in accordance with L and edges assigned to pages in such a way that the edges assigned to one page do not cross. The *page number* $pn(P, L)$ *of* P *with respect to* L is the smallest number k of pages such that $H(P)$ has a book embedding on k pages. The *page number* $pn(P)$ of P is defined as follows

$$pn(P) = \min\{pn(P, L): \quad L \in \mathcal{L}(P)\}.$$

The page number was first defined for graphs by Bernhart and Kainen [1], where the vertices of a graph can be put on the spine in arbitrary order. They conjectured that planar graphs may require an arbitrary large number of pages. In a series of attempts, it was finally established by Yannakakis [6], that $pn(G) \leq 4$ for every planar graph G. He provided also a quite involved construction of a planar graph, which need 4 pages.

The page number for posets has been introduced by R. Nowakowski and then considered by Syslo [5]. It is unknown how the page number behaves for planar posets (see Kelly [2] for some considerations on such posets, and it is conjectured (Nowakowski [4]) that the page number may be rather unbounded for such posets.

The purpose of this note is to provide a family of quite simple planar posets which require 4 pages and to show that the planar posets defined in [4] are of bounded page number.

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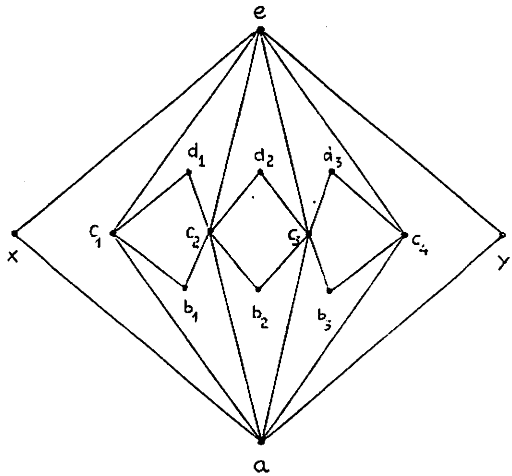


Figure 1

2. The result.

Our planar poset that requires four pages will contain a poset P shown in Figure 1 which has the following property:

Theorem 1. *If L is a linear extension of P in which $a < x < c_1, c_2, c_3, c_4 < y < e$ then $pn(P, L) \geq 4$.*

Proof: Let a linear extension L of P satisfy the assumptions of the theorem and suppose, that $pn(P, L) \leq 3$. Without loss of generality, we can place edge ax on page 1 and edge ye on page 2. We now show that in such an arrangement, elements c_1, c_2, c_3, c_4, have to appear in L in a certain order. ∎

Lemma 1. *There is no $c \in \{c_1, c_2, c_3, c_4\}$ such that the relations $c_i < c < c_{i+1}$ or $c_{i+1} < c < c_i$ for $i = 1, 2, 3$, hold in L.*

Proof of Lemma 1: Let us suppose that for some $c \in \{c_1, c_2, c_3, c_4\}$ there exists i such that $c_i < c < c_{i+1}$.

If $b_i < a$ then b_ic_i and b_ic_{i+1} have to be placed on page 3, and, hence, ac must be assigned to page 1 and ce is on page 2. In this case there is no space for c_id_i. Therefore, $a < b_i$. By symmetry, we also have $d_i < e$. Thus, we have $a < b_i < d_i < e$. Edges b_ic_{i+1} and c_id_i must be assigned to different pages and, hence, ac and ce have to be drawn on one page, which could be only page 3. Hence, it follows also that we can have only $c' < c_i$ or $c_{i+1} < c'$ in L for the element which left $c' \in \{c_1, c_2, c_3, c_4\} \setminus \{c_i, c_{i+1}, c\}$. We show that there is no space to insert c' in L.

Let us assume that $c' < c_i$ (see Figure 2). Then $c'e$ and c_ie are on page 2, ac_{i+1} and b_ic_{i+1} are on page 1, ac_i is on page 3. If there exists edge $c'v$ such that $c_{i+1} < v$ in L then $c'v$ must be drawn on page 4. If there exist edges uc and $c'v$ such that

$u < c'$ and $c < v < c_{i+1}$ in L then either uc intersects $c'v$ on page 1 or uc is on page 3, hence, $u < c$ and edge uc'', where $c'' = c'$ or $c'' = c_i$, must be drawn on page 4. Therefore, $c' = c_{i-1}$. Edges $b_{i-1}c_i$ and $c_{i-1}d_{i-1}$ have to be placed on distinct pages, then ac_{i-1} must be on page 1. In this case $c_{i-1}d_{i-1}$ intersects b_ic_{i+1} on page 1. Therefore, there is no element $c' \in \{c_1, c_2, c_3, c_4\} \setminus \{c_i, c_{i+1}, c\}$ such that $c' < c_i$ in L. In a similar way we can show that $c_{i+1} < c'$ for no such c'. This completes the proof that no c lies between c_i and c_{i+1} in L.

By symmetric arguments, the relations $c_{i+1} < c < c_i$ are also impossible in L. ∎

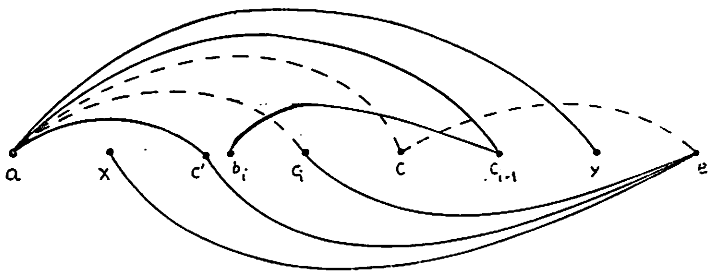


Figure 2

Lemma 1 implies the following:

Corollary. *In every linear extension L of P such that $pn(P, L) \leq 3$ we have either $a < x < c_1 < c_2 < c_3 < c_4 < y < e$ or $a < x < c_4 < c_3 < c_2 < c_1 < y < e$.*

We now assume that $a < x < c_1 < c_2 < c_3 < c_4 < y < e$ and show

Lemma 2. *For every $v \in P, v \neq a, e$ we have $a < v < e$ in L.*

Proof of Lemma 2: It is sufficient to show that $a < b_i$ in L for every $i = 1, 2, 3$ and $d_i < e$ will follow by symmetry. Let us assume that $b_i < a$ for some $i \in \{1, 2, 3\}$. Edges b_ic_i and b_ic_{i+1} have to be on page 3, ac_{i+1} on page 1, c_ie on page 2, c_id_i on page 2, hence, $c_{i+1}e$ must be on page 3. If $i < 3$ (see Figure 3) then $b_{i+1}c_{i+2}$ have to be on page 2, ac_{i+2} on page 1, $c_{i+2}e$ on page 3, so there is no space for d_{i+1}. Therefore, $i = 3$, $a < b_1$ and $a < b_2$ (see Figure 4). Edges c_1e, c_2e, and c_3e, must be assigned to page 2. Edge c_2d_2 have to be placed on page 1, then b_2c_3 must be on page 3, therefore, ac_2 is on page 1, c_1d_1 must be on page 3 and ac_1 is on page 1. In this case there is no page for adding edge b_1c_2. ∎

Finally,

Lemma 3. *For a linear extension L, edges ac_i, ac_{i+1}, cannot be drawn on one page (1 or 3) and edges c_ie, $c_{i+1}e$ on another one (2 or 3, or 2, respectively).*

Proof of Lemma 3: Let us assume that edges ac_i, ac_{i+1} are drawn on one page and c_ie, $c_{i+1}e$ are on another one. By Lemma 2 we have $a < b_i$ and $d_i < e$, hence, b_i

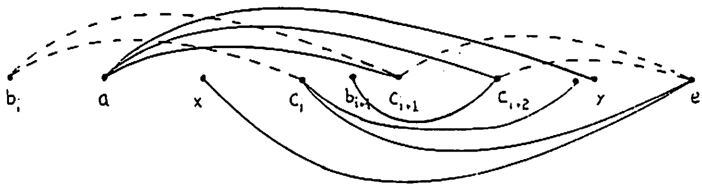


Figure 3

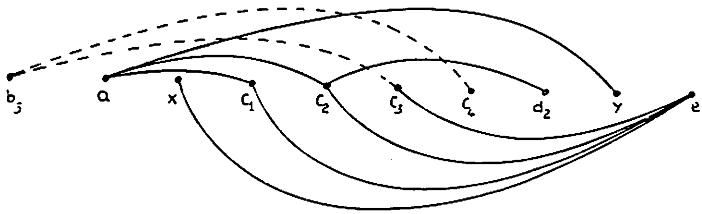


Figure 4

c_{i+1} intersects ac_i and c_ie, $c)id_i$ intersects ac_{i+1} and $c_{i+1}e$. Therefore, b_ic_{i+1} and c_id_i must be drawn on the third page, but there b_ic_{i+1} intersects c_id_i. ∎

To complete the proof of the Theorem, we distinguish several cases:

1. If both ac_1 and ac_2 are on page 3 (see Figure 5) then c_1e must be assigned to page 2, c_2e is on page 3, ac_3 and ac_4 are on page 1.
   a. If c_3e is assigned to page 3 then, by Lemma 3, c_4e is on page 2, b_3c_4 is on page 2, c_2d_2 on page 2, so c_1e_1 intersects b_1c_2 on page 1.
   b. If c_3e is assigned to page 2 then, by Lemma 3, c_4e is on page 3, b_3c_4 is on page 3, so c_2d_2 intersects b_3c_4 or c_4e on page 3.

In a similar way one can show that both c_3e and c_4e cannot be placed on page 3.

2. If both ac_2 and ac_3 are assigned to page 3 (see Figure 6) then ac_1 is on page 1, c_1e and c_2e are on page 2, c_3e on page 3 (by Lemma 3), ac_4 is on page 1, c_1d_1 and b_2c_3 are on page 1, c_2d_2 and b_3c_4 are on page 2, so c_4e is on page 3. Hence, there is no space for c_3d_3.

In a similar way on can show that both c_2e and c_3e cannot be placed on page 3.

3. If both ac_3 and ac_4 are assigned to page 3 (see Figure 7) then c_1e, c_2e, c_3e, are on page 2, ac_2 is on page 1 (by case 2), ac_1 and c_4e are on page 3 (by Lemma 3), so b_2c_3 intersects ac_1 or c_1d_1 on page 3.

In a similar way one can show that both c_3e and c_4e cannot be placed on page 3.

4. If both ac_1 and ac_2 are assigned to page 1 then
   a. If c_1e is on page 3 (see Figure 8) then ac_3, ac_4 are on page 1. By Lemma 3, c_2e is placed on page 2, c_3e is on page 3, c_4e is on page 2, so c_2d_2 intersects b_3c_4 on page 2.

b. If $c_1 e$ is assigned to page 2 (see Figure 9) then $c_2 e$ is on page 3 (by Lemma 3), therefore, ac_3 and ac_4 are on page 1. Again, by Lemma 3, $c_3 e$ is on page 2, $c_4 e$ is on page 3, so $c_2 d_2$ intersects $b_3 c_4$ on page 3.

In a similar way one can show that both $c_3 e$ and $c_4 e$ cannot be placed on page 1.

5. If both ac_2 and ac_3 are assigned to page 1 then ac_1 is placed on page 3.

a. If $c_2 e$ is assigned to page 3 (see Figure 10) then, by Lemma 3, $c_1 e$ is on page 2, $c_3 e$ is on page 2, $c_4 e$ is on page 3, ac_4 is on page 1, so $c_2 d_2$ intersects b_3 c_4 on page 3.

b. If $c_2 e$ is assigned to page 2 (see Figure 11) then, by Lemma 3, $c_3 e$ is on page 3, ac_4 is on page 1, $c_4 e$ is on page 2, so $c_1 d_1$ intersects $b_2 c_3$ on page 3.

In a similar way one can show that both $c_2 e$ and $c_3 e$ cannot be placed on page 2.

6. If both ac_3 and ac_4 are assigned to page 1 then ac_2 is on page 3 (by case 5), ac_1 is on page 1 (by case 1), $c_1 e$ is on page 2.

a. If $c_2 e$ is assigned to page 3 (see Figure12) then $c_3 e$ is on page 2 (by case 2). By Lemma 3, $c_4 e$ is on page 3, so $c_2 d_2$ intersects $b_3 c_4$ on page 3.

b. If $c_2 e$ is assigned to page 2 (see Figure 13) then $c_3 e$ is on page 3 (by case 2). By Lemma 3, $c_4 e$ is placed on page 2, so $c_2 d_2$ intersects $b_3 c_4$ on page 2.

In a similar way one can show that both $c_1 e$ and $c_2 e$ cannot be placed on page 2.

7. Then, we have only one possibility left ac_4 is on page 1 (if ac_4 is assigned to page 3 then $c_1 e$ and $c_2 e$ must be on page 2) ac_3 is on page 3, ac_2 is on page 1 and ac_1 is on page 3. In a similar way, we have to place $c_1 e$ on page 2, $c_2 e$ on page 3. Therefore, ac_3 intersects $c_2 e$ on page 3. ∎

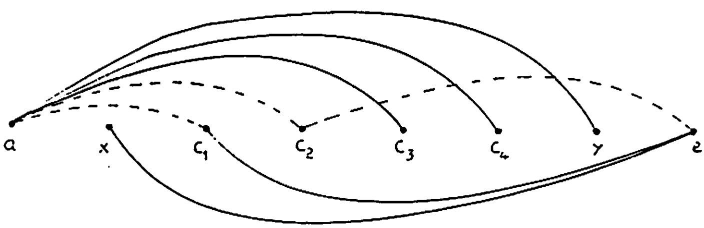


Figure 5

Now we we the poset P in Figure 1 to construct a planar poset Q (see Figure 14) which needs 4 pages.

Theorem 2. *For the poset Q we have $pn(Q) = 4$.*

Proof: Let L be a linear extension of Q, let $x \in \{c_1, \ldots, c_{16}\}$ be such that for every $c \in \{c_1, \ldots, c_{16}\}$, $c \neq x$ we have $x < c$ in L. Then there exists i such that $x \notin \{c_i, \ldots, c_{i+7}\}$. Let $y \in \{c_i, \ldots, c_{i+7}\}$ be such that for every

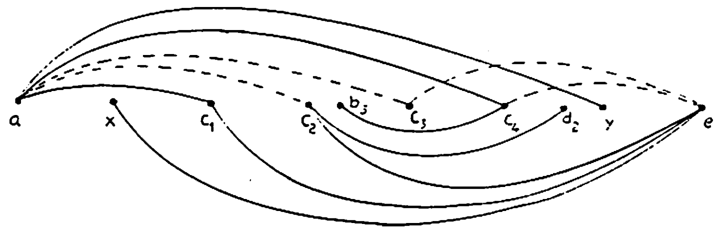


Figure 6

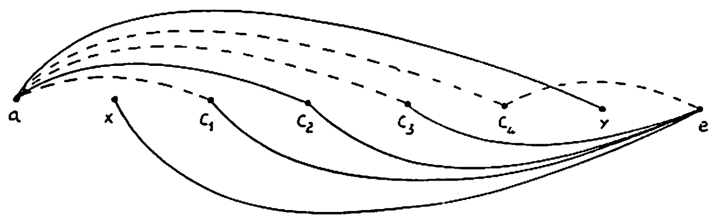


Figure 7

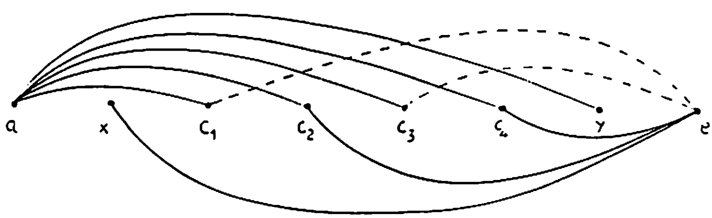


Figure 8

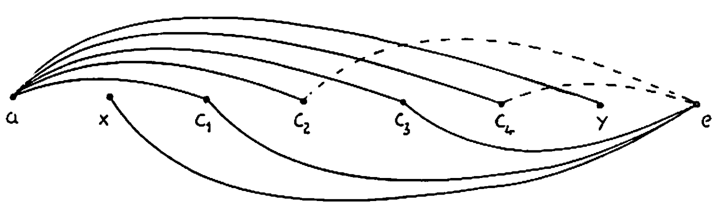


Figure 9

$v \in \{c_i, \ldots, c_{i+7}\}$ we have $v \leq y$. Then there exists j such that $i \leq j \leq i + 7$ and $y \notin \{c_j, \ldots, c_{j+3}\}$. By Theorem 1, the subposet of Q on the set $\{1, 2, x, y, b_j, \ldots, b_{j+2}, c_j, \ldots, c_{j+3}, d_j, \ldots, d_{j+2}\}$ is isomorphic to the poset P defined in Theorem 1, hence, $pn(P, L) > 3$. Thus, $pn(Q) > 3$.

In Figure 15 we show a four-page embedding of Q. ∎

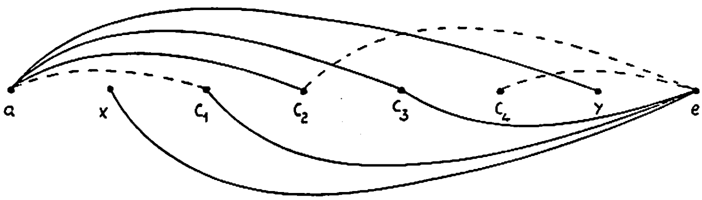


Figure 10

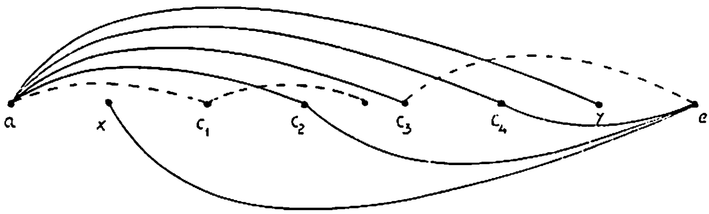


Figure 11

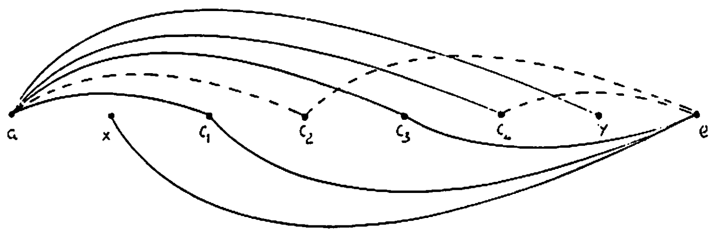


Figure 12

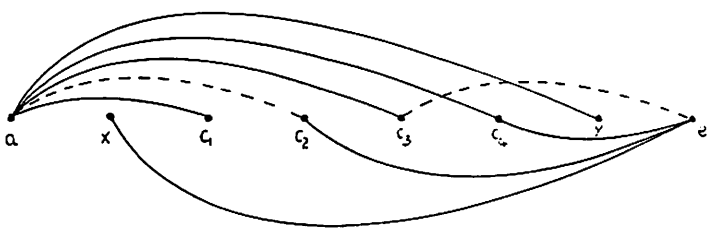


Figure 13

3. Six-page embeddings.

The purpose of this section is to disprove a conjecture of Czyzowicz [4] that the page number is unbounded on a class of nested iterations of the poset Q of Figure 14.

A *simple diamond* is the poset with the diagram shown in Figure 16, and Figure 15 shows the diagram of a *diamond of degree* 0 $Q^0 = Q$. Simple diamonds

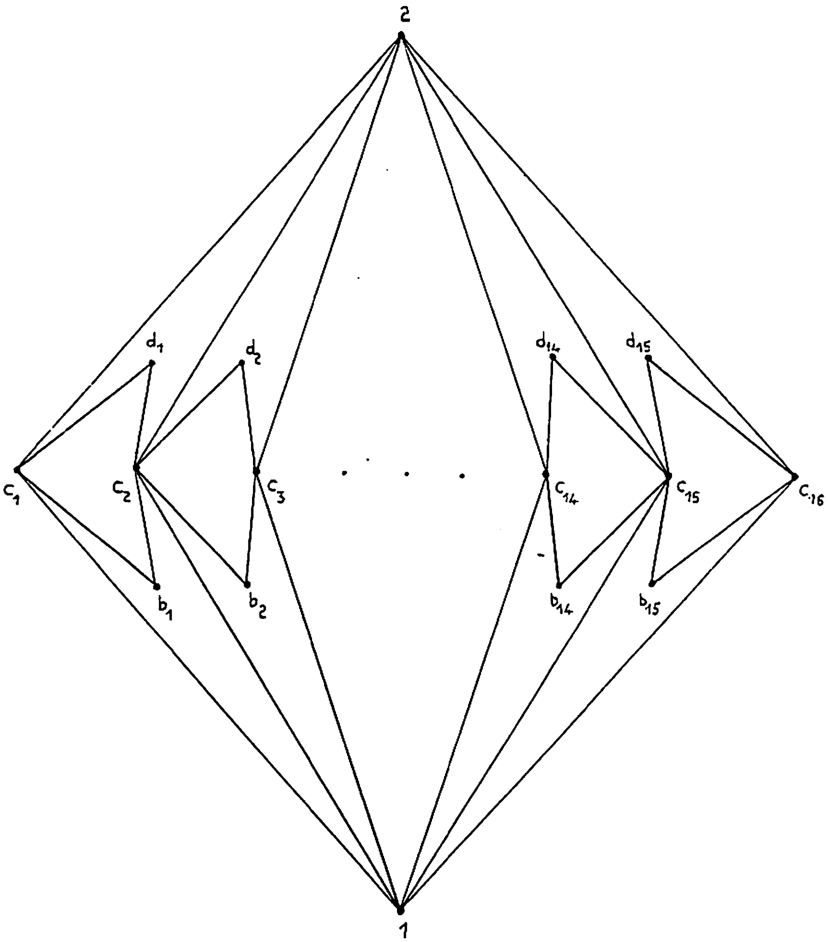
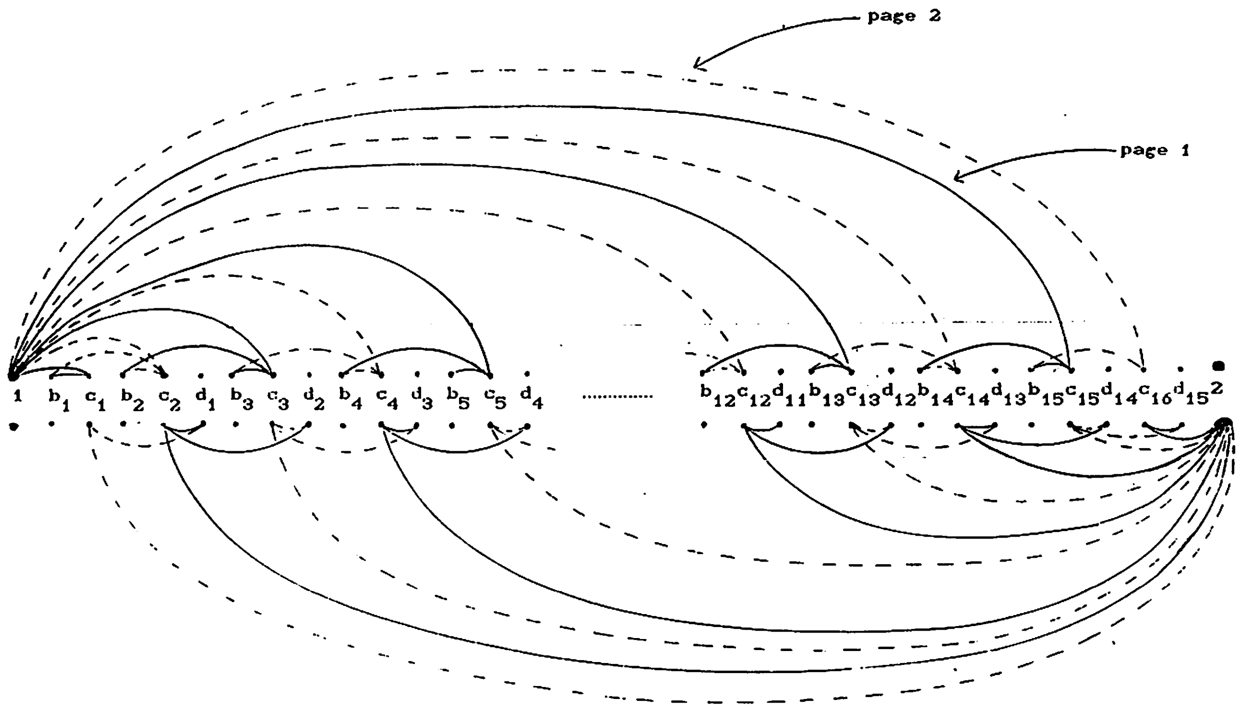


Figure 14

(b_i, c_i, c_{i+1}, d_i) for $i = 1, 2, \ldots, 15$ are distinguished in Q and called *holes*. A *diamond of degree* i Q^i for $i \geq 1$ is constructed from the diamond of degree $i-1$ by replacing all its 15^i holes by the poset Q. In what follows, for the diagram of Q^i we take the one which results from the above construction of Q^i with Q taken as shown in Figure 14. One can see that each diamond is a planar poset. We will show that $pn(Q^i) \leq 6$ for every $i \geq 1$.

To simplify the description of an embedding of Q^i on 6 pages, we regard a linear extension of a poset as a mapping from its element set to distinct points on the real line and identify each element with its image. Thus, the interval between the images of two elements u and v is simply called an *interval* and denoted by $[u, v]$. We distinguish several classes of elements in posets Q^i. Let A_{i-1} (respectively,



page 2
page 1
1
b_1 c_1 b_2 c_2 d_1 b_3 c_3 d_2 b_4 c_4 d_3 b_5 c_5 d_4
....
b_{12} c_{12} d_{11} b_{13} c_{13} d_{12} b_{14} c_{14} d_{13} b_{15} c_{15} d_{14} c_{16} d_{15} 2

Figure 15

E_{i-1}) be the set of maximal (resp., minimal) elements of Q^{i-1}. We denote the remaining elements of Q^{i-1} from left to right in its diagram by $c_1, \ldots, c_n$. A hole of Q^{i-1} is denoted by (e_j, c_j, c_{j+1}, a_j), where $e_j \in E_{i-1}$, $a_j \in A_{i-1}$ for $1 \leq j < n$. Let $c_{j,1}, \ldots, c_{j,14}$ denote the middle elements from left to right added in hole (e_j, c_j, c_{j+1}, a_j) of Q^{i-1} while constructing Q^i.

We shall construct a 6-page embedding of a poset Q^{i-1} $(i \geq 1)$ with the following properties:

P1. $c_j > c_k$ for $j > k$

P2. All edges from c_j $(1 \leq j \leq n)$ to elements a_i (resp., to c_j from e_i) such that $a_i > c_j$ (resp., $e_i < c_j$) are drawn on one page, which is called a *front page* (resp., a *back page*) of c_j. An edge $c_j a_i$ (resp., $e_i c_j$) is called a *front edge* (resp., a *back edge*) of c_j.

P3. A hole (e_j, c_j, c_{j+1}, a_j) is embedded as follows (see Figure 17) intervals $[e_j, c_j]$ and $[c_{j+1}, a_j]$ contain no other elements, A' is a subset of A_{i-1}, E' is subset of E_{i-1}, $[[c_j, c_{j+1}]]$, which denotes the interval between A' and E', contains no other elements.

P4. Elements c_j and c_{j+1} have different back pages and different front pages.

We now use three back pages (namely, pages 1, 2, and 3), and three front pages (namely, pages 4, 5, and 6) to embed Q^i on 6 pages for $i \geq 0$). The embedding will have properties $P1 - P4$.

For $i = 0$, the embedding of Q shown in Figure 15 satisfies $P1 - P4$. In this case, the back pages of $c_1, c2$ are, respectively, 1, and 2, and their front pages are 4, and 5.

Suppose that given is an embedding of Q^{i-1} $(i \geq 1)$ on 6 pages which satisfies $P1 - P4$. For every hole (e_j, c_j, c_{j+1}, a_j) $(1 \leq j \leq n)$ of Q^{i-1} we proceed as follows:

1. Embed $c_{j,1}, \ldots, c_{j,14}$ in $[[c_j, c_{j+1}]]$ n the order $c_{j,1} < \ldots < c_{j,14}$.

2. Embed the elements of $A_i \setminus A_{i-1}$ and $E_i \setminus E_{i-1}$ in such a way that every hole of Q^i satisfies $P3$.

3. Elements $c_{j,1}, c_{j,3}, \ldots, c_{j,13}$ have the same back and front pages as c_{j+1}. Elements $c_{j,2}, c_{j,4}, \ldots, c_{j,14}$ have back and front pages different from back and front pages of c_j and c_{j+1}. .

We can easily see that such an embedding of elements is a linear extension of Q^i. Moreover, by the definition, this embedding has properties $P1 - P4$. We will show that no two edges intersect on page 1, and a similar conclusion holds true for the remaining pages.

By the inductive assumption, no two edges of Q^{i-1} intersect. Let ec be an edge in $E(Q^i) - E(Q^{i-1})$ and in the diamond obtained from hole (e_j, c_j, c_{j+1}, a_j) of

Q^{i-1}, that is, $e \in E_i$ and $c \in \{c_j, c_{j,1}, \dots, c_{j,14}, c_{j+1}\}$. We assume that ec is drawn on page 1. By 1 and 2, we have $e_j \leq e < c \leq c_{j+1}$.

If $c = c_{j+1}$ then $e, c_{j,14}, c$ are in the same hole of Q^1 (see Figure 18). Interval $[e, c]$ contains only $c_{j,14}$ and some elements of E_i. Element $c_{j,14}$ has a back page different from that of c (by 3), hence, its back edges are not drawn on page 1. Every element c' ($c' \neq c_{j,14}$), of which a back edge $e'\ c'$ ($e' \in E_i$) crosses ec must have $e' \in [e, c]$ and $c' > c$. of which a bacl edge e'c' - e'Ei Hence, $e'c'$ crosses $e_j c_{j+1}$ too. If $e'c' \in E(Q^{i-1})$ then $e'c'$ is not drawn on page 1 (because $e_j c_{j+1} \in E(Q^{i-1})$ an it is placed on page 1). Otherwise, by 1 and 2, $e'c'$ must be an edge of diamond in Q^i obtained from the hole $(e_{j+1}, c_{j+1}, c_{j+2}, a_{j+1})$, then c' has a back page different from that of c (by 3).

If $c = c_j$ then $e, c_j, c_{j,1}$ are in the same hole of Q^i, hence, $[e, c]$ does not contain any other element of Q^i (by 2).

If $c = c_{j,k}$ ($1 \leq k \leq 14$) then two cases are possible:

1. e and c are in the same hole of Q^i, then $[e, c]$ contains no elements (by 2).

2. $e = e_j$ (see Figure 19). Then, there are only two elements, c_j and $c_{j,k+1}$ (if $k = 14$ then c_{j+1}), of which back edges cross ec (by 1 and 2). But, by 3, both of them have back pages different from that of c.

Therefore, edge ec does not cross any other edge on page 1.

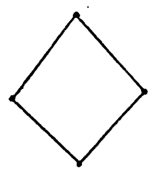


Figure 16. A simple diagram

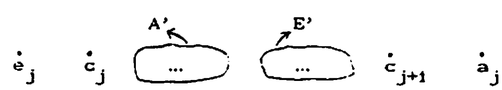


Figure 17. The embedding of a simple diamond of $D^{i-1}(Q)$

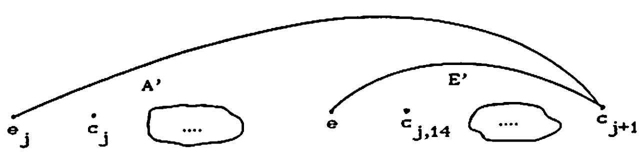


Figure 18

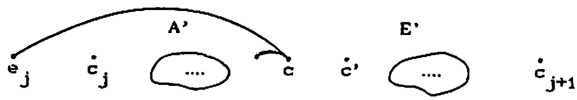


Figure 19

Acknowledgements.

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References

1. F. Bernhart, P.C. Kainen, *The book thickness of a graph*, J. Combin. Theory **B-27** (1979), 320–331.
2. D. Kelly, I. Rival, *Planar lattices*, Canad. J. Math. 27 (1975), 636–665.
3. R. Nowakowski, A. Parker, *Ordered set, pagenumbers and planarity*, Manuscript (1989.).
4. R. Nowakowski. A letter to M.M. Syslo, June 1989.
5. M.M. Syslo, *Bounds to the page Number of Partially Ordered Sets*, Proceeding of the WG (1989) (to appear).
6. M. Yannakakis, *Four pages are necessary and sufficient for planar graphs*, 18th STOC (1986), 104–108.