The nonexistence of [51,5,33;3]-codes

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1. Introduction

Abstract — It is unknown whether or not there exists a [51,5,33;3]-code (meeting the Griesmer bound). The purpose of this paper is to show that there is no [51,5,33;3]-code.

Let $V(n;q)$ be an n-dimensional vector space consisting of row vectors over the Galois field $GF(q)$. If $\mathcal{C}$ is a k-dimensional subspace in $V(n;q)$ such that every nonzero vector in $\mathcal{C}$ has a Hamming weight (i. e., number of nonzero coordinates) of at least d, then $\mathcal{C}$ is denoted an $[n,k,d;q]$-code. It is well known [Griesmer, 1960, Solomon and Stiffler, 1965] that if there exists an $[n,k,d;q]$-code, then

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \tag{1.1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. The bound (1.1) is called the Griesmer bound. It is unknown whether or not there exists a [51,5,33;3]-code; which would meet the Griesmer bound. The purpose of this paper is to prove the following theorem.

Theorem 1.1. There is no [51,5,33;3]-code.

Remark 1.2. It is known that there exists a [52,5,33;3]–code. Hence Theorem 1.1 shows that $n_3(5,33) = 52$, where $n_q(k,d)$ denotes the smallest value of n for which there exists an $[n,k,d;q]$-code.

2. Preliminary results

Let $S_{k,q}$ be the set of all column vectors c, $\mathrm{c} = (c_0, c_1, \cdots, c_{k-1})^T$, in $W(k,q)$ which satisfy the following condition:

$$\exists i : 0 \leq i \leq k-1 : \begin{cases} c_i = 1, & \\ c_j = 0, & i < j \leq k-1 \end{cases} \tag{2.1}$$

where $W(k,q)$ denotes a k-dimensional vector space consisting of column vectors over $GF(q)$. Then $S_{k,q}$ consists of $(q^k - 1)/(q-1)$ nonzero vectors in $W(k,q)$. For any nonzero vector $\mathbf{y}$ in $W(k,q)$, there exists a unique vector $\mathbf{x}$ in $S_{k,q}$ and a unique element σ in $GF(q)$ such that $\mathbf{y} = \sigma\mathbf{x}$, and there is no element σ in $GF(q)$ such that $\mathbf{x}_2 = \sigma\mathbf{x}_1$ for any two vectors $\mathbf{x}_2$ and $\mathbf{x}_1$ in $S_{k,q}$. Hence the $(q^k-1)/(q-1)$ vectors in $S_{k,q}$ can be regarded as $(q^k-1)/(q-1)$ points in a finite projective geometry $PG(k-1,q)$ where $q \geq 3$.

Let F be a set of f points in $PG(t,q)$. If $|F \cap H| \geq m$ for any $(t-1)$-flat (i. e., hyperplane) in $PG(t,q)$ and $|F \cap H| = m$ for some $(t-1)$-flat in $PG(t,q)$, then F is called a $\{f, m; t, q\}$-minihyper, where $|A|$ denotes the number of points in the set A.

Proposition 2.1. [Hamada, 1987]. Let F be a set of f points in $S_{k,q}$, and let C be the subspace of $V(n;q)$ generated by a $k \times n$ matrix (denoted by $\mathbf{G}$) whose column vectors are all the vectors in $S_{k,q} \setminus F$, where $n = v_k - f$, $1 \leq f < v_k - 1$, and $v_k = (q^k-1)/(q-1)$.
(1) Let $H_z = \{\mathbf{y} \in S_{k,q} \mid \mathbf{z}\cdot\mathbf{y} = 0$ over $GF(q)\}$ for any nonzero vector $\mathbf{z}$ in $S_{k,q}$. Then H_z is a hyperplane in $PG(k-1,q)$, and the weight of the code vector $\mathbf{z}^{\mathrm{T}}\mathbf{G}$ is equal to $|F \cap H_z| + q^{k-1} - f$, where $\mathbf{z}^{\mathrm{T}}$ denotes the transpose of the vector $\mathbf{z}$.

(2) In the case $k \geq 3$ and $1 \leq d < q^{k-1}$, C is an $[n,k,d;q]$-code meeting the Griesmer bound if and only if F is a $\{v_k - n, v_{k-1} - n + d; k-1, q\}$-minihyper.

Definition 2.2. Two $[n,k,d;q]$-codes C_1 and C_2 are said to be equivalent if there exist generator matrices $\mathbf{G}_i$ for C_i, $i = 1, 2$, such that $\mathbf{G}_2 = \mathbf{G}_1\mathbf{D}\mathbf{P}$ (or $\mathbf{G}_2 = \mathbf{G}_1\mathbf{P}\mathbf{D}$) for some permutation matrix $\mathbf{P}$ and some nonsingular diagonal matrix $\mathbf{D}$ with entries from $GF(q)$.

Remark 2.3. Proposition 2.1 shows that in the case $k \geq 3$ and $1 \leq d < q^{k-1}$ there is a one-to-one correspondence between the set of all nonequivalent $[n,k,d;q]$-codes meeting the Griesmer bound and the set of all $\{v_k - n, v_{k-1} - n + d; k-1, q\}$-minihypers.

Since there is a one-to-one correspondence between the set of all nonequivalent $[51,5,33;3]$-codes meeting the Griesmer bound and the set of all $\{70,22;4,3\}$-minihypers, it is sufficient to prove the following theorem in order to prove Theorem 1.1.

Theorem 2.4. There is no $\{70,22;4,3\}$-minihyper.

Remark 2.5. Refer references [Hamada, 1991, Hamada and Deza, 1991, Hamada and Helleseth, 1990, Hamada *et al.*, 1991] with respect to a characterization of $[n,k,d;q]$-codes meeting the Griesmer bound using minihypers in $PG(k-1,q)$.

3. The Proof of Theorem 2.4

In order to prove Theorem 2.4, we prepare the following three lemmas whose proofs will be given in Sections 4, 5, and 6, respectively.

Lemma 3.1. Suppose there exists a $\{70,22;4,3\}$*-minihyper. Then* $|F \cap H| = 22$, 25, *or* 31 *for any 3–flat H in* $PG(4,3)$, *and the following properties hold:*
(1) If $|F \cap H| = 25$, *then* $F \cap H$ *is a* $\{25,7;4,3\}$*-minihyper in* H.
(2) If $|F \cap H| = 31$, *then* $F \cap H$ *is a* $\{31,9;4,3\}$*-minihyper in* H.
(3) There exists at least one 3–flat H in $PG(4,3)$ *such that* $|F \cap H| = 31$.

Lemma 3.2. Any $\{31,9;3,3\}$*-minihyper must contain a 2–flat in* $PG(3,3)$.

Lemma 3.3. There is no $\{25,7;3,3\}$*-minihyper which contains a 2–flat in* $PG(3,3)$.

Proof of Theorem 2.4. Suppose there exists a $\{70,22;4,3\}$-minihyper F. Then it follows from Lemma 3.1 that there exists a 3–flat H in $PG(4,3)$ such that $F \cap H$ is a $\{31,9;4,3\}$-minihyper in H. Since H is a 3-flat, it follows from Lemma 3.2 that $F \cap H$ contains a 2–flat (denoted by V) in H.
Let H_i $(i = 1,2,3)$ be three distinct 3–flats in $PG(4,3)$, different from H, which contain V, where $|F \cap H_1| \le |F \cap H_2| \le |F \cap H_3|$. Since $|F| = 70$, $|F \cap H| = 31$, $|V| = 13$, and $|F \cap H_i| = 22$, 25, or 31 for $i = 1,2,3$, it follows that $\sum_{i=1}^{3} |F \cap (H_i \setminus V)| = |F| - |F \cap H| = 39$, and $|F \cap (H_i \setminus V)| = |F \cap H_i| - |V| = 9$, 12, or 18 for $i = 1,2,3$. Hence we have $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|) = (22,25,31)$.
Since $|F \cap H_2| = 25$, it follows from Lemma 3.1 that $F \cap H_2$ is a $\{25,7;4,3\}$-minihyper which contains the 2–flat V in H_2. Since H_2 is a 3–flat, this implies that there exists a $\{25,7;3,3\}$-minihyper which contains a 2–flat in $PG(3,3)$. This is contradictory to Lemma 3.3. Hence there is no $\{70,22;4,3\}$-minihyper. □

4. The proof of Lemma 3.1

Let $E = \{$ (1,2,1), (2,2,1), (3,2,1), (0,3,1), (0,0,2), (1,0,2), (2,0,2), (3,0,2), (0,1,2), (1,1,2), (2,1,2), (3,1,2), (0,2,2), (1,2,2), (2,2,2), (3,2,2), (0,3,2), (0,0,3) $\}$. Then for any integer m such that $22 \le m < 40$, there exists a unique ordered set $(m_1,m_2,m_3) \in E$ such that $m = m_1v_1 + m_2v_2 + m_3v_3$, where $v_1 = 1$, $v_2 = 4$, and $v_3 = 13$. In what follows, let $v_l = (3^l - 1)/(3-1)$ for any integer $l \ge 0$.

Lemma 4.1. Suppose there exists a $\{v_2 + 2v_3 + v_4 (= 70), v_1 + 2v_2 + v_3 (= 22); 4,3\}$*-minihyper F.*
(1) If H is a 3–flat in $PG(4,3)$ *such that* $|F \cap H| = m_1v_1 + m_2v_2 + m_3v_3$ *for some ordered set* $(m_1,m_2,m_3) \in E$, *then* $F \cap H$ *is a* $\{m_1v_1 + m_2v_2 + m_3v_3, m_1v_0 + m_2v_1 + m_3v_2; 4,3\}$*-minihyper in the 3–flat H.*
(2) There is no 3–flat H in $PG(4,3)$ *such that* $|F \cap H| = m_1v_1 + m_2v_2 + m_3v_3$ *for any ordered set* $(m_1,m_2,m_3) \in E$ *unless* $m_1 + m_2 + m_3 = 4$.

Proof. (1) Let H be a 3-flat in $PG(4,3)$ such that $|F \cap H| = m_1v_1 + m_2v_2 + m_3v_3$ for some ordered set $(m_1, m_2, m_3) \in E$. Suppose there exists a 2–flat Δ in H such that $|F \cap \Delta| \leq -1 + m_1v_0 + m_2v_1 + m_3v_2$. Let H_i $(i = 1,2,3)$ be three distinct 3–flats in $PG(4,3)$, different from H, that contain Δ. Since $|F| = v_2 + 2v_3 + v_4 = 70$ and $|F \cap H_i| \geq v_1 + 2v_2 + v_3 = 22$ for $i = 1,2,3$, it follows that $|F| = |F \cap H| + \sum_{i=1}^{3}(|F \cap H_i| - |F \cap \Delta|) \geq 69 + m_1 + m_2 + m_3 \geq 71 > |F|$, a contradiction. Hence $|F \cap \Delta| \geq m_1v_0 + m_2v_1 + m_3v_2$ for any 2–flat Δ in H.

If $|F \cap \Delta| > m_1v_0 + m_2v_1 + m_3v_2$ for any 2–flat Δ in H, it follows that $|F \cap H| > m_1v_1 + m_2v_2 + m_3v_3$, a contradiction. Hence there exists a 2–flat Δ in H such that $|F \cap \Delta| = m_1v_0 + m_2v_1 + m_3v_2$. This implies that $F \cap H$ is a $\{m_1v_1 + m_2v_2 + m_3v_3, m_1v_0 + m_2v_1 + m_3v_2; 4, 3\}$-minihyper (cf. Theorem 2.2 in [Hamada, 1991]).

(2) Suppose there exists a 3–flat H in $PG(4,3)$ such that $|F \cap H| = m_1v_1 + m_2v_2 + m_3v_3$ for any ordered set $(m_1, m_2, m_3) \in E$ such that $m_1 + m_2 + m_3 > 4$. Then it follows from (1) that there exists a 2–flat Δ in H such that $|F \cap \Delta| = m_1v_0 + m_2v_1 + m_3v_2$. Let H_i $(i = 1,2,3)$ be three distinct 3–flats in $PG(4,3)$, different from H, that contain Δ. Since $|F| = |F \cap H| + \sum_{i=1}^{3}(|F \cap H_i| - |F \cap \Delta|) \geq 66 + m_1 + m_2 + m_3 > 70 = |F|$, we have a contradiction.

Suppose there exists a 3–flat H in $PG(4,3)$ such that $|F \cap H| = m_1v_1 + m_2v_2 + m_3v_3$ for any ordered set $(m_1, m_2, m_3) \in E$ such that $m_1 + m_2 + m_3 < 4$. Then $|F \cap H| = 2v_3$, $v_1 + 2v_3$, $v_2 + 2v_3$, or $3v_3$.

Case I: $|F \cap H| = 2v_3$. It follows from (1) that there exists a 2–flat Δ in H such that $|F \cap \Delta| = 2v_2 = 8$. Let H_i $(i = 1,2,3)$ be three distinct 3–flats in $PG(4,3)$, different from H, that contain Δ. Since $\sum_{i=1}^{3}|F \cap (H_i \setminus \Delta)| = |F| - |F \cap H| = 44$ and $|F \cap (H_i \setminus \Delta)| = |F \cap H_i| - |F \cap \Delta| \geq 14$ for $i = 1,2,3$, there exists a 3–flat Π in $\{H_1, H_2, H_3\}$ such that $|F \cap \Pi| = 23$ or 24. Since $2v_1 + 2v_2 + v_3 = 23$ and $3v_1 + 2v_2 + v_3 = 24$, this is a contradiction.

Case II: $|F \cap H| = v_1 + 2v_3$, $v_2 + 2v_3$, or $3v_3$. Using a method similar to Case I, it can be shown that there exists a 3–flat Π in $PG(4,3)$ such that $|F \cap \Pi| = 2v_2 + 2v_2 + v_3 = 23$, a contradiction. This completes the proof.

□

Lemma 4.2. *(1) There is no* $\{28, 8; 3, 3\}$*-minihyper.*
(2) There is no $\{34, 10; 3, 3\}$*-minihyper.*

Proof. (1) Suppose there exists a $\{28, 8; 3, 3\}$-minihyper. Since $v_3 = 13$ and $v_4 = 40$, it follows from Remark 2.3 that there exists a [12,4,7;3]-code. Since there is no [12,4,7;3]-code, this is a contradiction.

(2) Suppose there exists a $\{34, 10, 3, 3\}$-minihyper. Then it follows from Remark 2.3 that there exists a [6,4,3;3]-code; a contradiction.

□

Proof of Lemma 3.1. It follows from Lemma 4.1 that $|F \cap H| = v_1 + 2v_2 + v_3 (= 22)$, $3v_2 + v_3 (= 25)$, $2v_1 + 2v_3 (= 28)$, $v_1 + v_2 + 2v_3 (= 31)$, $2v_2 + 2v_3 (= 34)$, or $v_4 (= 40)$ for any 3–flat H in $PG(4,3)$.

Case I: $|F \cap H| = 2v_1 + 2v_3 (= 28)$. It follows from Lemma 4.1 that $F \cap H$ is a $\{2v_1 + 2v_3, 2v_0 + 2v_2; 4, 3\}$-minihyper in H. Since H is a 3–flat, this implies that there exists a $\{28, 8; 3, 3\}$-minihyper, which is contradictory to (1) in Lemma 4.2. Hence there is no 3–flat H in $PG(4,3)$ such that $|F \cap H| = 28$.

Case II: $|F \cap H| = 2v_2 + 2v_3 (= 34)$. Using a method similar to Case I, we have a contradiction from Lemmas 4.1 and 4.2. Hence there is no 3–flat H in $PG(4,3)$ such that $|F \cap H| = 34$.

Case III: $|F \cap H| = v_4 (= 40)$. This implies that F contains the 3–flat H. Let V be any 2–flat in H and let H_i $(i = 1, 2, 3)$ be three distinct 3–flats in $PG(4,3)$, different from H, that contain V where $|F \cap H_1| \leq |F \cap H_2| \leq |F \cap H_3|$. Since $|F| = 70$, $|V| = 13$ and $|F \cap H_i| = 22, 25, 31$, or 40 for $i = 1, 2, 3$, it follows that $\sum_{i=1}^{3} |F \cap (H_i \setminus V)| = |F| - |F \cap H| = 30$ and $|F \cap (H_i \setminus V)| = |F \cap H_i| - |V|$ =9, 12, 18, 27, for $i = 1, 2, 3$. Hence we have $(|F \cap H_1|, |F \cap H_2|, |F \cap H_3|) = (22, 22, 25)$.

Since $|F \cap H_3| = 3v_2 + v_3 = 25$, it follows from Lemma 4.1 that $F \cap H_3$ is a $\{3v_2 + v_3, 3v_1 + v_2; 4, 3\}$-minihyper in H_3. Since H_3 is a 3–flat, this implies that there exists a $\{25, 7; 3, 3\}$-minihyper which contains a 2–flat in $PG(3,3)$. Hence we have a contradiction from Lemma 3.3.

From Cases I-III, it follows that $|F \cap H| = 22$, 25, or 28 for any 3–flat H in $PG(4,3)$.

(1)-(2). Since $3v_2 + v_3 = 25$ and $v_1 + v_2 + 2v_3 = 31$, it follows from Lemma 4.1 that (1) and (2) in Lemma 3.1 hold.

(3) Let n_i be the number of 3–flats H in $PG(4,3)$ such that $|F \cap H| = i$ for $i = 22, 25, 31$. Since (i) there are $v_5 (= 121)$ 3–flats in $PG(4,3)$ and (ii) there are $v_4 (= 40)$ 3–flats Π in $PG(4,3)$ such that $P \in \Pi$ for any point P in F, and (iii) there are v_3 3–flats Π in $PG(4,3)$ such that $Q_1 \in \Pi$ and $Q_2 \in \Pi$ for any two distinct points Q_1 and Q_2 in F, it follows that

$$n_{22} + n_{25} + n_{31} = v_5$$
$$22n_{22} + 25n_{25} + 31n_{31} = |F|v_4$$
$$\binom{22}{2} n_{22} + \binom{25}{2} n_{25} + \binom{31}{2} n_{31} = \binom{|F|}{2} v_3.$$

Hence we have $(n_{22}, n_{25}, n_{31}) = (95, 16, 10)$. This completes the proof. □

5. The proof of Lemma 3.2

Let C be an $[n, k, d; q]$-code and let A_i and B_i be the number of codewords of weight i in the code C and in its dual code $C^{\perp}$, respectively. The following lemma due to MacWilliams plays an important role in the proof of Lemma 3.2.

***Lemma 5.1** (The MacWilliams Identities).*

$$\sum_{j=0}^{n-t}\binom{n-j}{t}A_j = q^{k-t}\sum_{j=0}^{t}\binom{n-j}{n-t}B_j$$

for $t = 0, 1, \ldots, n$.

***Lemma 5.2.** If F is a $\{31, 9; 3, 3\}$-minihyper, then $|F \cap \Delta| = 9, 10, 12,$ or 13 for any 2–flat Δ in $PG(3,3)$ and $|F \cap L| \geq 2$ for any 1–flat L in $PG(3,3)$.*

Proof. Suppose there exists a 1–flat L in $PG(3,3)$ such that $|F \cap L| \leq 1$. Let $\Delta_i (i = 1, 2, 3, 4)$ be the four 2–flats in $PG(3,3)$ which contain L. Since $|F| = 31$ and $|F \cap \Delta_i| \geq 9$ for $i = 1, 2, 3, 4$, it follows that $|F| = \sum_{i=1}^{4}|F \cap \Delta_i| - 3|F \cap L| \geq 33 > |F|$, a contradiction. Hence $|F \cap L| \geq 2$ for any 1–flat L in $PG(3,3)$.
Suppose there exists a 2–flat Δ in $PG(3,3)$ such that $|F \cap \Delta| = 11$. Since $|\Delta| = v_3 = 13$, there exists a point Q in Δ such that $Q \notin F$. Let $L_i (i = 1, 2, 3, 4)$ be four 1–flats in the 2–flat Δ passing through the point Q. Since $\sum_{i=1}^{4}|F \cap (L_i \setminus \{Q\})| = |F \cap \Delta| = 11$ and $|(L_i \setminus \{Q\})| = 3$ for $i = 1, 2, 3, 4$, there exists a 1–flat L in $\{L_1, L_2, L_3, L_4\}$ such that $|F \cap L| = |F \cap (L \setminus \{Q\})| = 2$. Let Δ_i $(i = 1, 2, 3)$ be three distinct 2–flats in $PG(3,3)$, different from Δ, that contain L. Then $|F| = |F \cap \Delta| + \sum_{i=1}^{3}|F \cap \Delta_i| - 3|F \cap L| \geq 32 > |F|$, a contradiction. Hence there is no 2–flat Δ in $PG(3,3)$ such that $|F \cap \Delta| = 11$. Since $9 \leq |F \cap \Delta| \leq \Delta = 13$ for any 2–flat Δ in $PG(3,3)$, this completes the proof.

□

***Lemma 5.3.** Any $[9, 4, 5; 3]$-code has the unique weight enumerator $1 + 36z^5 + 24z^6 + 18z^8 + 2z^9$.*

Proof. Let G be a 4×9 generator matrix of a $[9, 4, 5; 3]$-code C. Without loss of generality, we can assume that any column vector of G belongs to the set $S_{4,3}$. Let $\tilde{G}$ denote the set of 9 column vectors in G and let F be the set $S_{4,3} \setminus \tilde{G}$. It follows from Proposition 2.1 that F is a $\{31, 9; 3, 3\}$-minihyper.
Since C is a $[9, 4, 5; 3]$-code, it follows from Proposition 2.1 and Lemma 5.2 that $A_0 = 1, A_1 = A_2 = A_3 = A_4 = 0 = A_7, B_0 = 1$, and $B_1 = B_2 = 0$. Since $|\tilde{G} \cap L| + |F \cap L| = |L|$ for any 1–flat L in $PG(3,3)$, it follows from Lemma

5.2 and $|L| = 4$ that $|\tilde{G} \cap L| \leq 2$ for any 1–flat L in $PG(3,3)$. This implies that no three column vectors in G are linearly dependent and $B_3 = 0$. Hence it follows from Lemma 5.1 that

$$\begin{aligned} A_5 + A_6 + A_8 + A_9 &= 80 \\ 4A_5 + 3A_6 + A_8 &= 234 \\ \binom{4}{2} A_5 + \binom{3}{2} A_6 &= 288 \\ \binom{4}{3} A_5 + \binom{3}{3} A_6 &= 168. \end{aligned}$$

From the above equations, we have $A_5 = 36, A_6 = 24, A_8 = 18$, and $A_9 = 2$. This completes the proof.

□

Proof of Lemma 3.2. Let F be any $\{31, 9; 3, 3\}$-minihyper. It follows from Proposition 2.1 and Lemma 5.3 that there exists one 2–flat Δ in $PG(3,3)$ such that $|F \cap \Delta| = 13$. Since $|\Delta| = 13$, this implies that F contains the 2–flat Δ in $PG(3,3)$.

□

6. The proof of Lemma 3.3

Suppose there exists a $\{25, 7; 3, 3\}$-minihyper F which contains a 2–flat (denoted by V) in $PG(3,3)$. Without loss of generality, we can assume that $F \subset S_{4,3}$. Let G be a 4×15 matrix whose column vectors are all the vectors in $\tilde{G} := S_{4,3} \setminus F$ and let C be the subspace of $V(15,3)$ generated by the matrix G.

Let $H_z = \{\mathbf{y} \in S_{4,3} \mid \mathbf{z} \cdot \mathbf{y} = 0 \text{ over } GF(3)\}$ for any nonzero vector $\mathbf{z}$ in $W(4,3)$. Then H_z is a 2–flat in $PG(3,3)$. Since F is a $\{25, 7; 3, 3\}$-minihyper which contains a 2–flat V in $PG(3,3)$, there exist two vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ in $S_{4,3}$ such that $|F \cap H_{z_1}| = 7$ and $H_{z_2} = V$.

Since $w(\mathbf{z}^{\mathrm{T}}\mathrm{G}) = |F \cap H_z| + 2$ for any nonzero vector $\mathbf{z}$ in $W(4,3)$ and $\tilde{G} \subset S_{4,3}$, it follows from $|F \cap H_{z_2}| = |V| = 13$ that C is a $[15, 4, 9; 3]$-code such that $A_{15} \geq 2$ and $B_1 = B_2 = 0$. This is contradictory to Theorem 4.2 (and its proof) in [Hill and Newton, 1988]. Hence there is no $\{25, 7; 3, 3\}$-minihyper F which contains a 2–flat in $PG(3,3)$. □

References

J. H. Griesmer, "A bound for error-correcting codes", *IBM Journal of Res. and Dev.*, vol. 4(5) pp.532–542, 1960.

N. Hamada and M. Deza, "A characterization of $\{2v_{\alpha+1} + 2v_{\beta+1}, 2v_\alpha + 2v_\beta; t, q\}$-min·hypers in PG$(t, q)$ $(t \geq 3, q \geq 5$ and $0 \leq \alpha < \beta < t)$ and its applications to error-correcting codes", *Discrete Math.*, vol. 91 pp.xxx–xxx, 1991.

N. Hamada and T. Helleseth, "A characterization of some minihypers in a finite projective geometry $PG(t, 4)$", *European J. Combinatorics*, vol. 11 pp.541–548, 1990.

N. Hamada, T. Helleseth, and Ø. Ytrehus, "There are exactly two nonequivalent [20,5,12;3]-codes", *ARS Combinatoria*, vol. ??, 1991? To appear.

N. Hamada, "Characterization of min·hypers in a finite projective geometry and its applications to error-correcting codes", *Bulletin of Osaka Women's University*, vol. 24 pp.1–24, 1987.

N. Hamada. *Combinatorial Aspects of Design Experiments*, chapter 4: Error-Correcting Codes. To appear in Discrete Math., 1991.

R. Hill and D. E. Newton, "Some optimal ternary codes", *ARS Combinatoria*, vol. Twenty-Five A pp.61–72, 1988.

G. Solomon and J. J. Stiffler, "Algebraically punctured cyclic codes", *Inform. Contr.*, vol. 8 pp.170–179, 1965.

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