# ALMOST RESOLVABLE BIBDS WITH BLOCK-SIZE 5 OR 6

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**Abstract.** An obvious necessary condition for the existence of an almost resolvable $B(k, k-1; v)$ is $v \equiv 1 \pmod{k}$. We show in this paper that the necessary condition is also sufficient for $k = 5$ or 6 possibly excepting 8 values of $v$ when $k = 5$ and 3 values of $v$ when $k = 6$.

## 1. Introduction.

A *balanced imcomplete block design* (BIBD) with parameters $v$, $k$ and $\lambda$ (simply $B(k, \lambda; v)$) is a pair $(X, \mathcal{A})$, where $X$ is a $v$-set of points and $\mathcal{A}$ is a set of $k$-subsets of $X$ called *blocks* such that any 2-subset of $X$ is contained in exactly $\lambda$ blocks. A $B(k, \lambda, v)$ is *resolvable*, denoted by $RB(k, \lambda, v)$, if there exists a partition of its set of blocks into subsets called *parallel classes* each of which in turn partitions the set $X$. A $B(k, k-1; v)$ is said to be *almost resolvable*, denoted by $AR(k, v)$, if its set of blocks can be partitioned into some families called *almost parallel classes* such that each family forms a partition of $X - \{x\}$ for some $x \in X$,where $x$ is called a *singleton*. An obvious necessary condition for the existence of a $AR(k, v)$ is $v \equiv 1 \pmod{k}$.

It has been shown in [2] that for each positive integer $v \equiv 1 \pmod{5}$ there exists an $AR(5, v)$ with at most 26 possible exceptions. That is

**Lemma 1.1.** *Let $v$ be a positive integer. If $v \equiv 1 \pmod{5}$ and $v \notin E$, then an $AR(5, v)$ exists, where* $E = \{46, 51, 86, 116, 141, 161, 196, 201, 226, 236, 261, 266, 291, 296, 326, 351, 376, 411, 471, 476, 501, 591, 596, 711, 766, 986\}$.

In this paper, we shall give a new technique used in the construction of almost resolvable BIBDs. As an appliction, we improve the above result, and reduce this number of possible exceptions to 8. It is also shown that an $AR(6, v)$ exists for each $v \equiv 1 \pmod{6}$ with at most 3 possible exceptions.

For the concepts not defined in this paper, the reader is referred to [1]. In what follows, we shall adopt the following notations:

$$B(K, \lambda) = \{v: \text{a } PBD\ B(K, \lambda; v) \text{ exists}\},$$
$$AR(k) = \{v: \text{an } AR(k, v) \text{ exists}\}.$$

## 2. Preliminaries.

In this section we shall define several types of incomplete designs and state some preliminary results which will be used later.

Let $(X, \mathcal{A})$ be a BIBD. If a set of points $Y \subseteq X$ has the property that, for each $A \in \mathcal{A}$, either $|Y \cap A| \leq 1$ or $A \subseteq Y$, then we say that $Y$ is a *subdesign* or *flat* of the BIBD. An $AR(k, w)$ is a *subsystem* of an $AR(k, v)$ if the almost parallel classes of the $AR(k, w)$ are induced by the almost parallel classes of the $AR(k, v)$. Note that the almost parallel class $P$ of the $AR(k, w)$, which is induced by the almost parallel class $P(x)$ with singleton $x$ of the $AR(k, v)$, has $x$ as its singleton. If we remove the subsystem $AR(k, w)$ from $AR(k, v)$, leaving a hole, we obtain an *incomplete system* $IAR(k, v; w)$.

An *incomplete group divisible design* (IGDD) is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

(1) $X$ is a set of *points*, and $Y \subseteq X$;
(2) $\mathcal{G}$ is a partition of $X$ into *groups*;
(3) $\mathcal{A}$ is a set of *blocks*, each of which intersects each group in at most one point;
(4) no block contains two members of $Y$;
(5) every pair of points $\{x, y\}$ from distinct groups, such that at least one of $x, y$ is in $X - Y$, occurs in precisely $\lambda$ blocks of $\mathcal{A}$.

We say that an IGDD $(X, Y, \mathcal{G}, \mathcal{A})$ is a $(k, \lambda)$-IGDD if $|A| = k$ for every block $A \in \mathcal{A}$. The *type* of the IGDD is defined to be the multiset of ordered pairs $\{(|G|, |G \cap Y|): G \in \mathcal{G}\}$. We sometimes use the "exponential" notation for its description. Note that when $Y = \emptyset$, a $(k, \lambda)$-IGDD is just a $(\{k\}, \lambda)$-GDD. By $(K, \lambda)$-GDD we mean a GDD with block sizes in $K$ and index $\lambda$.

We also use *incomplete frame*. An incomplete $(k, \lambda)$-frame is a $(k, \lambda)$-IGDD $(X, Y, \mathcal{G}, \mathcal{A})$ in which the set of blocks $\mathcal{A}$ can be partitioned into *holey parallel classes*, each of which is a partition of $X - H$ for some $H \in \mathcal{G}$, or a partition of $X - (H \cup Y)$ for some $H \in \mathcal{G}$. Simple calculation shows that, for each group $H$, there are exactly $\lambda|H \cap Y|/(k - 1)$ holey parallel classes which partition $X - (H \cup Y)$, and $\lambda|H - Y|/(k - 1)$ holey parallel classes which partition $X - H$. An incomplete $(k, \lambda)$-frame with $Y \neq \emptyset$ is called a $(k, \lambda)$*-frame*. Remember that the type of an incomplete $(k, \lambda)$-frame is the type of the underlying $(k, \lambda)$-IGDD.

The following lemmas are slight generalizations of the constructions in [4] and [5].

**Lemma 2.1.** *Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD with index utility and let a non-negative integral weight $W_x$ be assigned to each point $x \in X$. For every block $A \in \mathcal{A}$, suppose that there exists a $(k, \lambda)$-frame of type $\{W_x: x \in A\}$. Then there exists an incomplete $(k, \lambda)$-frame of type $\{(\sum_{x \in G} W_x, \sum_{x \in G \cap Y} W_x): G \in \mathcal{G}\}$.*

**Lemma 2.2.** *Suppose that the following designs exist:*

(1) *a incomplete* $(k,\lambda)$*-frame of type* $\{(t_1,u_1),(t_2,u_2),\ldots,(t_n,u_n)\}$;
(2) *an* $IAR(k,t_i+a;u_i+a)$ *for* $1\le i\le n$; *and*
(3) *an* $AR(k,u+a)$.

*Then there exists an* $AR(k,t+a)$ *where* $t=\Sigma t_i$, $u=\Sigma u_i$ *and* $a\ge 0$.

**Lemma 2.3.** *Suppose there is a* $(k,\lambda)$*-frame of type* $\{t_1,t_2,\ldots,t_n\}$, *and let* $\varepsilon\ge 0$. *For* $1\le i\le n$, *suppose there is a* $(k,\lambda)$*-frame of type* $T_i\cup\{\varepsilon\}$, *where* $\sum_{t\in T_i} t=t_i$. *Then there is a* $(k,\lambda)$*-frame of type* $\{\varepsilon\}\cup\left(\cup_{1\le i\le n}T_i\right)$.

We shall also make use of the following lemma.

**Lemma 2.4.** *Let* $v=p^r$ *be any prime power and* $k>2$ *be such that* $k$ *is a divisor of* $v-1$. *Then there exists an* $AR(k,v)$ *which can be generated by a collection of base blocks under the additive group of* $GF(p^r)$.

Proof: Let $X=GF(v)$, and let $w$ be a primitive element of $X$. The base blocks of required design are $\{w^i,w^{i+d},\ldots,w^{i+(k-1)d}\}$, where $i=0,1,\ldots,d-1$ and $d=(v-1)/k$. ∎

## 3. Resolvable TD determined by a Latin rectangle.

We recall that an $r\times n$ matrix $A=(a_{ij})_{r\times n}$ is an $r\times n$ $(r\le n)$ *latin rectangle* on set $S$ of cardinality $n$ if the elements in each row are all different and so are the elements on each column.

It is well known that a *transversal design* (TD) $TD(m,n)$ is a $(\{m\},1)$-GDD of type $n^m$. A TD $(X,\mathcal{G},\mathcal{A})$ is said to be *resolvable* if its set of blocks $\mathcal{A}$ can be partitioned into some families called *parallel classes* such that each family forms a partition of $X$. Here we introduce the idea of a *resolvable TD determined by some Latin rectangle*. Informally, such a design is a resolvable TD with the property that the set of blocks can be produced from one distinguished parallel class by means of some latin rectangle.

**Definition 3.1:** Let $I_m=\{0,1\ \ldots,m-1\}$ and $G=\{g_0=0,g_1,\ldots,g_{n-1}\}$ be an Abelian group of order $n$. Let $L=(a_{ij})_{m\times(n-1)}$ be a Latin rectangle defined on $G-\{0\}$. Suppose that $\mathcal{A}$ is the union of the following families of $m$-subsets of $I_m\times G$:

$$\mathcal{A}_0=\{A_{g0}:A_{g0}=\{(0,g),(1,g),\ldots,(m-1,g)\},g\in G\},$$
$$\mathcal{A}_t=\{A_{gt}:A_{gt}=\{(0,g+a_{0t}),(1,g+a_{1t}),\ldots,(m-1,g+a_{(m-1)t})\},g\in G\},$$

where $1\le t\le n-1$ and $(a_{0t},a_{1t},\ldots,a_{(m-1)t})^T$ is the $t$-th column of $L$. If $(I_m\times G,\{\{s\}\times G: s\in I_m\},\mathcal{A})$ is a resolvable TD in which $\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_{n-1}$ forms its mutually disjoint parallel classes, then we say that it is a resolvable TD determined by $L$ and denote it by $RT^L(m,n)$.

**Remark:** Careful inspection of Definition 3.1 gives rise to the fact that $\mathcal{B}_g = \{A_{gj}\colon A_{gj} \in \mathcal{A}_j \text{ and } 0 \le j \le n-1\}$ $(g \in G)$ are also $n$ mutually disjoint parallel classes of the $RT^L(m,n)$.

We now present the following existence theorem for $RT^L(m,n)$.

**Theorem 3.2.** *Let $R_n$ be a commutative ring of order n with unity, and $U$ be a set of elements with multiplicative inverse in $R_n$. If $|U| \ge m$, and $u_1 - u_2 \in U$ for each pair $\{u_1, u_2\} \subset U$, then there exists an $RT^L(m,n)$ defined on $I_m \times R_n$.*

Proof: Let $R_n = \{r_0 = 0, r_1, \dots, r_{n-1}\}$, and $\{u_0, u_1, \dots, u_{m-1}\} \subset U$. We take

$$L = \begin{pmatrix} u_0 r_1 & u_0 r_2 & \cdots & u_0 r_{n-1} \\ u_1 r_1 & u_1 r_2 & \cdots & u_1 r_{n-1} \\ \vdots & \vdots & & \vdots \\ u_{m-1} r_1 & u_{m-1} r_2 & \cdots & u_{m-1} r_{n-1} \end{pmatrix}.$$

Then it is clear that $L$ is a latin rectangle on $R_n - \{0\}$. As in Definition 3.1, we write $\mathcal{A}_0 = \{A_{r0}\colon A_{r0} = \{(0,r),(1,r),\dots,(m-1,r)\} \text{ and } r \in R_n\}$, and $\mathcal{A}_t = \{A_{rt} : A_{rt} = \{(0, r + u_0 r_t), (1, r + u_1 r_t), \dots, (m-1, r + u_{m-1} r_t)\} \text{ and } r \in R_n\}$, where $t = 1, 2, \dots, n-1$. We need to prove that $(I_m \times R_n, \{\{s\} \times R_n,\ s \in I_m\}, \mathcal{A})$ is a resolvable TD in which $\mathcal{A} = \cup_{0 \le j \le n-1} \mathcal{A}_j$. Since $L$ is a Latin rectangle, it is easy to show that each $\mathcal{A}_j$ $(0 \le j \le n-1)$ forms a partition of $I_m \times R_n$, and hence is a parallel class. Note that each block of $\mathcal{A}$ meets each group $\{s\} \times R_n$ $(s \in I_m)$ in precisely one point. Therefore, we need only to prove that no pair of points from distinct groups occurs in two distinct blocks. In fact, assume there are two blocks $A_{rj_1} \in \mathcal{A}_{j_1}$ and $A_{r'j_2} \in \mathcal{A}_{j_2}$ $(0 \le j_1 \ne j_2 \le n-1, r, r' \in R_n)$ which contain the same pair of points of $I_m \times R_n$. Without loss of generality, we assume that one of the two points comes from the $s_1$-th group and the other from the $s_2$-th group. By definition, we have

$$\big((s_1, r + u_{s_1} r_{j_1}), (s_2, r + u_{s_2} r_{j_1})\big) = \big((s_1, r' + u_{s_1} r_{j_2}), (s_2, r' + u_{s_2} r_{j_2})\big),$$

namely,

$$\begin{cases} r + u_{s_1} r_{j_1} = r' + u_{s_1} r_{j_2}, \\ r + u_{s_2} r_{j_1} = r' + u_{s_2} r_{j_2} \end{cases}.$$

This means that $(u_{s_1} - u_{s_2})\, r_{j_1} = (u_{s_1} - u_{s_2}) r_{j_2}$. Since $u_{s_1} - u_{s_2} \in U$, we get $r_{j_1} = r_{j_2}$, and hence $j_1 = j_2$, which is a contradiction. The conclusion then follows. ∎

**Corollary 3.3.** *If $n = p_1^{e_1} \cdot p_2^{e_2} \dots p_r^{e_r}$ is the factorization of the integer $n$ into powers of the distinct primes $p_1, p_2, \dots, p_r$, and $m = \min\{p_i^{e_i} - 1; 1 \le i \le r\}$ then there exists an $RT^L(m,n)$. on $I_m \times (GF(p_1^{e_1}) \times \cdots \times GF(p_r^{e_r}))$.*

Proof: Take $R_n = GF(p_1^{e_1}) \times \ldots \times GF(p_r^{e_r})$. Then the conclusion follows from Theorem 3.2. ∎

## 4. A new construction.

We defined $RT^L(m,n)$ in Section 3. Our main application of these designs involves using them to establish the following new construction for almost resolvable BIBDs.

**Construction 4.1.** *Let $m,n,w$ be positive integers such that $n \equiv w \equiv 1 \pmod{k}$, $m \equiv 0 \pmod{k}$. Suppose that the following designs exist:*

(1) *an $RT^L(m,n)$ on $I_m \times G$, where $G$ is an Abelian group;*
(2) *an $AR(k,n)$ on $G$ which can be generated by a collection of base blocks under the action of $G$;*
(3) *an $RB(k,k-1;m)$ and an $AR(k,m+1)$; and*
(4) *an $AR(k,w)$, where $1 \le w \le n$.*

*Then there exists an $AR(k,mn+w)$.*

Proof: Let $G = \{g_0 = 0, g_1, \ldots, g_{n-1}\}$ be an Abelian group. Let $L = (a_{ij})_{m\times(n-1)}$ be a Latin rectangle defined on $G-\{0\}$. Suppose $(I_m \times G, \{\{s\}\times G\colon s \in I_m\}, \mathcal{A})$ is an $RT^L(m,n)$, which is determined by $L$. By Definition 3.1, we have $n$ parallel classes $\mathcal{A}_j$ $(j = 0,1,\ldots,n-1)$ which form a partition of $\mathcal{A}$, where

$$\begin{aligned}\mathcal{A}_0 &= \{A_{g0}\colon A_{g0} = \{(0,g),(1,g),\ldots,(m-1,g)\}, g \in G\},\\ \mathcal{A}_t &= \{A_{gt}\colon A_{gt} = \{(0,g+a_{0t}),(1,g+a_{1t}),\\ &\qquad \ldots,(m-1,g+a_{(m-1)t})\}, g \in G\},\end{aligned} \tag{4.1}$$

$(t = 1,2,\ldots,n-1)$. For each $1 \le t \le n-1$, $(a_{0t},a_{1t},\ldots,a_{(m-1)t})^T$ is the $t$-th column of $L$.

As already mentioed in the remark of Definition 3.1, we also have $n$ other parallel classes $\mathcal{B}_g (g \in G)$ which form another partition of $\mathcal{A}$, where

$$\mathcal{B}_g = \{A_{gj}\colon A_{gj} \in \mathcal{A}_j \text{ and } 0 \le j \le n-1\} \quad (g \in G). \tag{4.2}$$

Take $F = \{\infty_0,\infty_1,\ldots,\infty_{w-1}\}$ and $T = \{g_0,g_1,\ldots,g_{w-1}\} \subset G$.

At this point, we will construct an $AR(k,mn+w)$ on $(I_m \times G)\cup F$ as follows.

First,in view of hypothesis (2), we construct an $AR(k,n)$ on $G$. Without loss of generality, we assume that

$$\begin{aligned}&D_1 = \{g_1,g_2,\ldots,g_k\},\ D_2 = \{g_{k+1},g_{k+2},\ldots,g_{2k}\},\ldots,\\ &D_{\overline{n}} = \{g_{(\overline{n}-1)k+1},g_{(\overline{n}-1)k+2},\ldots,g_{n-1}\} \qquad (\overline{n} = (n-1)/k)\end{aligned}$$

are the collection of base blocks of the $AR(k,n)$. It follows that, for each $s \in I_m$,

$$\begin{aligned}D_{s1} &= \{(s,g_1),(s,g_2),\ldots,(s,g_k)\},\\ D_{s2} &= \{(s,g_{k+1}),(s,g_{k+2}),\ldots,(s,g_{2k})\},\\ &\ \ldots\\ D_{s\overline{n}} &= \{(s,g_{(\overline{n}-1)k+1}),(s,g_{(\overline{n}-1)k+2}),\ldots,(s,g_{n-1})\}\end{aligned} \tag{4.3}$$

are the collection of base blocks of an $AR(k,n)$ on $\{s\} \times G$ (as an isomorphic image of $G$), and they form an almost parallel class with singleton $(s,0)$. Note that we have $T - \{g_0\} = \{g_1, \ldots, g_{w-1}\} = \cup_{1 \le h \le \overline{w}} D_h \subset G$ where $\overline{w} = (w-1)/k$.

Second, we adjoin $w$ infinite points in $F$ to $w$ parallel classes $\mathcal{B}_{g_f}$ ($g_f \in T \subset G$) of the $RT^L(m,n)$, where $\infty_f$ is adjoined to exactly the parallel class $\mathcal{B}_{g_f}$ $(0 \le f \le w-1)$. This gives rise to a resolvable $(\{m, m+1\}, 1)$-GDD of type $n^m w^1$ with group set $\{\{s\} \times G \colon s \in I_m\} \cup \{F\}$. From (4.1) and (4.2), we know that this resolvable GDD has the following parallel classes as a partition of its set of blocks

$$\tilde{A}_j = \{A_{g_f j} \cup \{\infty_f\} \colon g_f \in T\} \cup \{A_{gj} \colon g \in G - T\},$$
$$j = 0, 1, \ldots, n-1.$$

Third, we use our hypotheses and the above resolvable GDD to construct a $B(k, k-1; mn+w)$ on $(I_m \times G) \cup F$ as follows: put an $AR(k,n)$ with the collection of base blocks (4.3) on each group $\{s\} \times G$ $(0 \le s \le m-1)$ of size $n$, an $RB(k, k-1; m)$ on each block of size $m$, an $AR(k, m+1)$ on each block of size $m+1$, and put an $AR(k,w)$ on $F$.

Finally, we prove that the $B(k, k-1; mn+w)$ obtained above is almost resolvable. For the convenience of notation, we make the following conventions:

(1) $(s,g) + g' = (s, g+g')$ for each $(s,g) \in I_m \times G$ and $g' \in G$;
(2) $T + g' = \{g + g' \colon g \in T\}$ for each $T \subset G$ and $g' \in G$;
(3) $D_{sh} + g' = \{(s, g+g') \colon (s,g) \in D_{sh}\}$ for each $g' \in G$ and $D_{sh}$ shown in (4.3)

It is well known that an $AR(k,v)$ has $v$ almost parallel classes, and an $RB(k, k-1; v)$ has $v-1$ parallel classes. We use the points of $A$ to label those almost parallel classes of the $AR(k, m+1)$ on $A$,for each block $A$ of size $m+1$. Write $\mathcal{D}_x(A)$ for the almost parallel class with singleton $x$ $(x \in A)$. Write also $\mathcal{F}_{\infty_0}, \mathcal{F}_{\infty_1}, \ldots, \mathcal{F}_{\infty_{w-1}}$ for the almost parallel classes of the $AR(k,w)$ on $F$. For each block $A$ of size $m$, we simply denote those parallel classes of the $RB(k,m)$ on $A$ by $\varepsilon_1(A), \varepsilon_2(A), \ldots, \varepsilon_{m-1}(A)$.

Take

$$\delta_{i0} = \left[\bigcup_{g_f \in T} \mathcal{D}_{(i,g_f)}(A_{g_f0} \cup \{\infty_f\})\right] \cup \left[\bigcup_{g \in G-T} \mathcal{E}_i(A_{g0})\right]$$

$$\cup \left[\bigcup_{1 \le h \le \overline{w}} D_{ih}\right] \qquad (i = 1, 2, \ldots, m-1);$$

$$\delta_{00} = \left[\bigcup_{g_f \in T} \mathcal{D}_{(0,g_f)}(A_{g_f0} \cup \{\infty_f\})\right] \tag{4.4}$$

$$\cup \left[\bigcup_{0 \le s \le m-1} \bigcup_{\overline{w}+1 \le h \le \overline{n}} D_{sh}\right] \cup \left[\bigcup_{1 \le h \le \overline{w}} D_{0h}\right] \text{; and}$$

$$\psi_0 = \bigcup_{g_f \in T} \mathcal{D}_{\infty_f}(A_{g_f0} \cup \{\infty_f\}).$$

Similarly, for each $t$, $1 \le t \le n-1$, we take

$$\delta_{it} = \left[\bigcup_{g_f + a_{it} \in T + a_{it}} \mathcal{D}_{(i,g_f+a_{it})}(A_{g_ft} \cup \{\infty_f\})\right]$$

$$\cup \left[\bigcup_{g \in G-(T+a_{it})} \mathcal{E}_i(A_{gt})\right] \cup \left[\bigcup_{1 \le h \le \overline{w}} (D_{ih} + a_{it})\right],$$

$$(i = 1, 2, \ldots, m-1);$$

$$\delta_{0t} = \left[\bigcup_{g_f + a_{0t} \in T + a_{0t}} \mathcal{D}_{(0,g_f+a_{0t})}(A_{g_ft} \cup \{\infty_f\})\right] \tag{4.5}$$

$$\cup \left[\bigcup_{0 \le s \le m-1} \bigcup_{1+\overline{w} \le h \le \overline{n}} (D_{sh} + a_{st})\right] \cup \left[\bigcup_{1 \le h \le \overline{w}} (D_{0h} + a_{0t})\right] \text{; and}$$

$$\psi_t = \bigcup_{g_f \in T} \mathcal{D}_{\infty_f}(A_{g_ft} \cup \{\infty_f\}).$$

For each $1 \le t \le n-1$, $\{\delta_{st}\colon 0 \le s \le m-1\} \cup \{\psi_t\}$ is a partition of the blccks of the resolvable designs arising from $\bar{A}_t$, together with the blocks $D_{s1} + a_{st}, D_{s2} + a_{st}, \ldots, D_{s\overline{n}} + a_{st}$ $(s = 0, 1, \ldots, m-1)$. Also, $\{\delta_{s0}\colon 0 \le s \le m-1\} \cup \{\psi_0\}$ is a partition of the blocks of the resolvable designs arising from $\bar{A}_0$, together with the blocks shown in (4.3). Therefore, the families of blocks shown in (4.4) and (4.5) form a partition of all blocks the of $B(k, k-1; mn+w)$ on $(I_m \times G) \cup F$ except those blocks of the $AR(k, w)$ on $F$. Further, we know that $\delta_{sj}$ $(0 \leq s \leq m-1, 0 \leq j \leq n-1)$ is an almost parallel class of the $B(k, k-1; mn+w)$, with singleton $(s, a_{sj})$ when $j \neq 0$, or $(s, 0)$. Now, we need only partition the blocks contained in $\cup_{0 \leq j \leq n-1} \psi_j$ together with the blocks of the $AR(k, w)$ on $F$ into almost parallel classes. This can be done by taking

$$\delta_f = \mathcal{F}_{\infty_f} \cup \left[\cup_{0 \leq j} \mathcal{D}_{\infty_f}(A_{g_f j} \cup \{\infty_f\})\right], \quad f = 0, 1, \ldots, w-1.$$

This completes the proof. ∎

## 5. Results for $AR(k, v)$ with $k = 5$ or $6$.

In this section,we shall rely heavily on Construction 4.1 to obtain our results for almost resolvable BIBDs with block size 5 or 6.

Constructions using finite fields and elementary Abelian groups provide us with the following

**Lemma 5.1.** *Let $q$ be any prime power and $k > 2$ be such that $k$ is a divisor of $q-1$. Suppose that there is an $AR(k, v)$ on an Abelian group $G$ which is generated by a collection of $(v-1)/k$ base blocks. Then there exists an incomplete $(k, k-1)$-frame of type $(v, 1)^q$.*

Proof: Let $q = kn+1$, and $D_i (i = 1, 2, \ldots, (v-1)/k)$ be the collection of base blocks of an $AR(k, v)$ on $G$, where $D_i = \{d_{i0}, d_{i1}, \ldots, d_{i,k-1}\}$ for $1 \leq i \leq (v-1)/k$. Without loss of generality, assume $d_{ir} \neq 0$ $(1 \leq i \leq (v-1)/k,$ $0 \leq r \leq k-1)$. Let $x$ be a primitive element of $GF(q)$. Take $X = GF(q) \times G$. Let $\mathcal{B}$ consist of the following $nv$ blocks:

$$\{(x^j, d_{i0}), (x^{j+n}, d_{i1}), \ldots, (x^{j+(k-1)n}, d_{i,k-1})\}$$
$$\{(x^{j+n}, d_{i0}), (x^{j+2n}, d_{i1}), \ldots, (x^j, d_{i,k-1})\}$$
$$\cdots$$
$$\{(x^{j+(k-1)n}, d_{i0}), (x^j, d_{i1}), \ldots, (x^{j+(k-2)n}, d_{i,k-1})\}$$
$$\{(x^j, 0), (x^{j+n}, 0), \ldots, (x^{j+(k-1)n}, 0)\},$$

where $j = 0, 1, \ldots, n-1$ and $i = 1, 2, \ldots, (v-1)/k$.

It is readily checked that $\mathcal{B}$ is the collection of base blocks of an $(k, k-1)$-frame of type $v^q$ on the Abelian group $GF(q) \times G$, with group set $\{\{g\} \times G : g \in GF(q)\}$. The required incomplete $(k, k-1)$-frame then can be obtained by removing the following blocks:

$\{(x^j, 0), (x^{j+n}, 0), \ldots, (x^{j+(k-1)n}, 0)\}$ mod $(q, -)$, where $j = 0, 1, \ldots, n-1$.

This completes the proof. ∎

We also need the following obvious result.

**Lemma 5.2.** *An $AR(k, v)$ is equivalent to a $(k, k-1)$-frame of type $1^v$.*

*Let $P_k = \{n: n > 1$ is a prime power such that $n \equiv 1 \pmod{k}\}$.*

**Lemma 5.3.** $AR(k) \supset B(P_k, 1)$.

Proof: For each $v \in B(P_k)$, there is a $B(P_k, 1; v)$, namely, a $(P_k, 1)$-GDD of type $1^v$. Give weight 1 to every point of the GDD and apply Lemma 2.1 with $Y = \emptyset$. Since a $(k, k-1)$-frame of type $1^p$ exists for each $p \in P_k$ from Lemma 2.4 and Lemma 5.2, we obtain a $(k, k-1)$-frame of type $1^v$. Then $v \in AR(k)$. ∎

The following result was obtained in [3].

**Lemma 5.4.** *Let $v$ be a positive integer. If $v \equiv 1 \pmod{6}$ and $v \notin Q$, then $v \in B(P_6, 1)$, where $Q = \{55, 115, 145, 205, 235, 253, 265, 295, 319, 355, 391, 415, 445, 451, 493, 649, 655, 667, 685, 697, 745, 781, 799, 805, 1243, 1255, 1315, 1585, 1795, 1819, 1921\}$.*

As an immediate consequence of Lemma 5.3 and Lemma 5.4, we have

**Lemma 5.5.** *$AR(6)$ contains all positive integers $v \equiv 1 \pmod{6}$ possibly excepting those values of $v$ in $Q$.*

**Lemma 5.6.** *There exists a $(5, 4)$-frame of type $4^6$.*

Proof: Delete one point from a resolvable $B(5, 1; 25)$ to produce the required $(5, 4)$-frame of type $4^6$. ∎

**Lemma 5.7.** $\{261, 501\} \subset AR(5)$.

Proof: Give weight 4 to the $(\{6\}, 1)$-GDDs of type $5^{13}$ and $5^{25}$ which can be constructed respectively by deleting one point from a $B(6, 1; 66)$ and a $B(6, 1; 126)$ (see [6]). This guarantees that both $(5, 4)$-frames of type $20^{13}$ and $20^{25}$ exist by applying Lemma 2.1 with $Y = \emptyset$ and Lemma 5.6. We then use Lemma 2.3 with $\varepsilon = 1$ and the fact that $21 \in AR(5)$ to get the required result. ∎

**Lemma 5.8.** *Incomplete $(5, 4)$-frames of type $(26, 1)^{11}$ and $(32, 2)^6$ exist.*

Proof: It is knowm (see [2]) that an $AR(5, 26)$ exists on $Z_{26}$ which can be generated by a collection of base blocks. Therefore, the first incomplete frame exists from Lemma 5.1.

Remove one block from a $TD(6, 16)$, which exists since 16 is a prime power, we get a $(6, 1)$-IGDD of type $(16, 1)^6$. Give each point of the IGDD weight 2 and apply Lemma 2.1. This produces the second incomplete frame since a $(5, 2)$-frame of type $2^6$ exists (see [2]). ∎

**Lemma 5.9.** *There exist an $IAR(5, 31; 6)$ and an $IAR(5, 36; 6)$.*

Proof: Remove one block from a $B(6, 1; 31)$ (see [6]) to produce an $IAR(5, 31; 6)$. An $IAR(5, 36; 6)$ exists from [2]. ∎

**Corollary 5.10.** $\{196, 291\} \subset AR(5)$.

Proof: Apply Lemma 2.2, Lemma 5.8, and Lemma 5.9. ∎

Now we are able to give the main result of this paper.

**Theorem 5.11.** *Suppose that* $v \geq 6$ *and* $v \equiv 1 \pmod 5$. *Then* $v \in AR(5)$ *with at most 8 possible exceptions. The possible exceptions are* 46, 51, 116, 141, 201, 266, 296 *and* 351.

Proof: Combining Lemma 5.7 and Corollary 5.10 with Lemma 1.1, we need only to prove that $\{86, 161, 226, 236, 326, 376, 411, 471, 476, 591, 596, 711, 766, 986\} \subset AR(5)$. We give the proof using Construction 4.1 with parameters shown in Table 1. The required $RT^L(5, n)$ come from Corollary 3.3. The other conditions of Construction 4.1 are satisfied because of Lemma 1.1 and Lemma 2.4.

Table 1

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v=5n+w$ | 86 | 161 | 226 | 236 | 326 | 376 | 411 | 471 | 476 | 591 | 596 | 711 |
| $n$ | 16 | 31 | 41 | 41 | 61 | 71 | 71 | 81 | 81 | 101 | 101 | 121 |
| $w$ | 6 | 6 | 21 | 31 | 21 | 21 | 56 | 66 | 71 | 86 | 91 | 106 |
| $v=5n+w$ | 766 | 986 | | | | | | | | | | |
| $n$ | 131 | 181 | | | | | | | | | | |
| $w$ | 111 | 81 | | | | | | | | | | |

∎

**Theorem 5.12.** *Suppose that* $v \geq 7$ *and* $v \equiv 1 \pmod 6$. *Then* $v \in AR(6)$ *if* $v \neq 55, 145, 355$.

Proof: From Lemma 5.5, we need only to consider those values of $v$ in $Q$. We apply Construction 4.1 with parameters shown in Table 2. All conditions of Construction 4.1 are satisfied by Lemma 2.4, Lemma 5.5, and Corollary 3.3.

Table 2

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v=6n+w$ | 115 | 205 | 235 | 253 | 265 | 295 | 319 | 391 | 415 | 445 | 451 |
| $n$ | 19 | 31 | 37 | 37 | 43 | 49 | 49 | 61 | 61 | 73 | 73 |
| $w$ | 1 | 19 | 13 | 31 | 7 | 1 | 25 | 25 | 49 | 7 | 13 |
| $v=6n+w$ | 493 | 649 | 655 | 667 | 685 | 697 | 745 | 781 | 799 | 805 | 1243 |
| $n$ | 79 | 103 | 103 | 103 | 109 | 109 | 121 | 127 | 127 | 127 | 199 |
| $w$ | 19 | 31 | 37 | 49 | 31 | 43 | 19 | 19 | 37 | 43 | 49 |
| $v=6n+w$ | 1255 | 1315 | 1585 | 1795 | 1819 | 1921 | | | | | |
| $n$ | 199 | 199 | 229 | 283 | 283 | 283 | | | | | |
| $w$ | 61 | 121 | 211 | 97 | 121 | 223 | | | | | |

∎

Summarizing the results in Theorem 5.11 and Theorem 5.12, we have the following

**Theorem 5.13.** *The necessary condition for the existence of an* $AR(k,v)$*, namely,* $v \equiv 1 \pmod{k}$*, is also sufficient for* $k = 5$ *or 6 with the possible exceptions of* $(v,k) = (46,5), (51,5), (116,5), (141,5), (201,5), (266,5), (296,5), (351,5), (55,6), (145,6)$ *and* $(355,6)$.

## 6. Concluding remarks.

The constructions and results established in this paper may be used to discuss the existence of resolvable $B(k, k-1; v)$, which is now under investigation. An almost complete solution to the existence problem for $k = 5$ has been recently obtained, which will be reported in subsequent papers.

## Note added in Proof

Recently, S.C. Furino and the authors have constructed $AR(k,v)$s for $(v,k) = (46,5), (116,5), (266,5), (296,5), (351,5)$ and $(355,6)$. Therefore there are now 5 unsolved cases.

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