Group Divisible Designs with Equal-Sized Holes[1]

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Abstract. In this paper we consider group divisible designs with equal-sized holes (HGDD) which is a generalization of modified group divisible designs [1] and HMOLS. We prove that the obvious necessary conditions for the existence of the HGDD is sufficient when the block size is three, which generalizes the result of Assaf [1].

1. Introduction

A *group divisible design* (GDD) denoted by GDD$[K, \lambda, M; v]$ is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a v-set, $\mathcal{G}$ and $\mathcal{A}$ are collections of some subsets of X (called groups and blocks respectively) such that

(1) $|G| \in M$ for every $G \in \mathcal{G}$;
(2) $|B| \in K$ for every $B \in \mathcal{A}$;
(3) $|G \cap B| \leq 1$ for every $G \in \mathcal{G}$ and every $B \in \mathcal{A}$; and
(4) every pairset $\{x, y\}$, where x and y belong to distinct groups, is contained in exactly λ blocks of $\mathcal{A}$.

The *group type* of a GDD$(X, \mathcal{G}, \mathcal{A})$ is the multiset $\{|G|: G \in \mathcal{G}\}$ and denoted by $1^i 2^j 3^k \ldots$, which means that in the multiset there are i occurrences of 1, j occurrences of 2, etc. A set of blocks is called a *parallel class* if the blocks partition X. For ease of notation we sometimes write a GDD$[K, \lambda, M; v]$ as a GDD$[K, \lambda]$ together with its group type or just as a GDD$[K, \lambda]$.

A *sub*-GDD$(Y, \mathcal{G}', \mathcal{A}')$ of a GDD$(X, \mathcal{G}, \mathcal{A})$ is a GDD whose points and blocks are respectively points and blocks of the GDD$(X, \mathcal{G}, \mathcal{A})$ and whose every group is contained in some group of the latter. If the sub-GDD is missing, then it is called an *incomplete* GDD, or say that the GDD has a *hole*. In fact the missing sub-GDD need not exist.

If a GDD has several equal-sized holes which partition the point set, we call it a *holey* GDD, or HGDD. We give a formal definition as following.

Let X be a v-set, where $v = tmn$. Let $\mathcal{G} = \{G_{ij}: 1 \leq i \leq n, 1 \leq j \leq t\}$ such that $\mathcal{G}$ is a partition of X and $|G_{ij}| = m$ for any $G_{ij} \in \mathcal{G}$. Let $\mathcal{A}$ be a collection of some k-subsets of X (called *blocks*) such that for any two points x and y of X from G_{ij} and G_{sh} respectively, there are λ blocks of $\mathcal{A}$ containing both x and y when $i \neq s$ and $j \neq h$, while when $i = s$ or $j = h$ no block $A \in \mathcal{A}$ can

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contain both x and y. Then we call the design $(X, \mathcal{G}, \mathcal{A})$ a (v, k, λ)-HGDD of type (n, m^t). We call $\cup_{j=1}^{t} G_{ij}$ a group and $\cup_{j=1}^{n} G_{ij}$ a *hole* of the HGDD.

The HGDD is a special case of DGDD (double group divisible design), however, for simplicity, we shall not state the definition of the latter here. The interested readers are referred to [14].

The existence of a $(v, k, 1)$-HGDD of type (k, m^t) is equivalent to the existence of $k-2$ HMOLS (holey mutually orthogonal Latin squares) of order tm which has been studied in [4], [9], [10], [11], [12] and [13].

A (v, k, λ)-HGDD of type $(n, 1^t)$ is called a modified group divisible design in [1] which was motivated by the problem of resolvable group divisible designs with $k = 3, \lambda = 2$ and other constructions of designs.

For any positive integers k, n, m, t and λ, the necessary conditions for the existence of an HGDD are the following.

Lemma 1.1. *The necessary conditions for the existence of a* (v, k, λ)-HGDD *of type* (n, m^t) *are that* $v = tmn, t \geq k, n \geq k, \lambda(t-1)(n-1)m \equiv 0 \pmod{k-1}$ *and* $\lambda v(t-1)(n-1)m \equiv 0 \pmod{k(k-1)}$.

Proof: $\lambda(t-1)(n-1)m/(k-1)$ is the number of blocks containing each point of the design and $\lambda v(t-1)(n-1)m/(k(k-1))$ is the total number of the blocks. The other conditions follow from the definition of HGDD directly.

The above necessary conditions are not sufficient for the existence of an HGDD. For example, there exist no $(24, 4, 1)$-HGDD of type $(4, 1^6)$ because there exist no GDD$[\{4\}, 1, \{6\}, 24]$ (see [7]). But in this paper we shall prove that the conditions of Lemma 1.1 are sufficient for the existence of HGDD when $k = 3$. In [1] Assaf has proved the following.

Theorem 1.2. *Let t, n and λ be positive integers. The necessary and sufficient conditions for the existence of an* $(nt, 3, \lambda)$-HGDD *of type* $(n, 1^t)$ *are that* $n \geq 3, t \geq 3, \lambda(t-1)(n-1) \equiv 0 \pmod 2$ *and* $\lambda tn(t-1)(n-1) \equiv 0 \pmod 6$.

We shall generalize Assaf's result and obtain the main Theorem of this paper as following.

Theorem 1.3. *The necessary and sufficient conditions for the existence of a* $(v, 3, \lambda)$-HGDD *of type* (n, m^t) *are that* $v = tmn, t \geq 3, n \geq 3, \lambda(t-1)(n-1)m \equiv 0 \pmod 2$ *and* $\lambda v(t-1)(n-1)m \equiv 0 \pmod 6$.

2. Constructions and related designs

In this section we shall give the main recursive constructions of this paper. The designs of PBDs and TDs will be used. So we first state the definitions of these designs below.

A *pairwise balanced design* (or PBD) of index λ, denoted by $B[K, \lambda; v]$, is a pair $(X, \mathcal{A})$ where X is a set of v elements (called points) and $\mathcal{A}$ is a collection of subsets (called blocks) of X, such that every unordered pair of points is contained in exactly λ blocks of $\mathcal{A}$ and every block in $\mathcal{A}$ has its size in K.

A *transversal design* TD(k, v) is a GDD$[\{k\}, 1, \{v\}; kv]$. It is well-known that the existence of a TD(k, v) is equivalent to the existence of $k-2$ mutually orthogonal Latin squares (MOLS) of order v. Furthermore, if there exists $k-1$ MOLS of order v, then there exists a TD(k, v) which has v/k parallel classes. For the existence of MOLS the reader is referred to [3] and [8].

In [1] the following lemma has been proved.

Lemma 2.1. *For every positive $t \neq 2$, there exists a* TD$(3, t)$ *which has at least one parallel class of the blocks.*

For the existence of GDD$[\{3\}, \lambda]$, the following result is given in [5].

Lemma 2.2. *There exists a* GDD$[3, \lambda, m, um]$ *if and only if* $u \geq 3$, $\lambda(u-1)m \equiv 0 \pmod 2$ *and* $\lambda u(u-1)m^2 \equiv 0 \pmod 6$.

Let $B(K) = \{v$: there exists a $B\{K, 1; v]\}$. If $B(A) = A$ for some set A, then we say that the set A is PBD-*closed.* If A_0 is a finite subset of A such that $B(A_0) = B(A)$, then A_0 is called a *basis* of A. Let $\mathrm{N}_a = \{v \in \mathrm{N} : v \geq a\}$. The following results are obtained by Hanani and Wilson (see [2]).

Lemma 2.3. *Let* $A_1 = \mathrm{N}_3$, $A_2 = \{v \in \mathrm{N}_3 : v \equiv 0$ *or* $1 \pmod 3)\}$ *and* $A_3 = \{v \in \mathrm{N}_3 : v \equiv 1 \pmod 2)$, *then* A_1, A_2 *and* A_3 *are* PBD-*closed and their bases are* $B_1 = \{3, 4, 5, 6, 8\}$, $B_2 = \{3, 4, 6\}$ *and* $B_3 = \{3, 5\}$ *respectively.*

The proof of the following lemma is simple. It just comes from the definition of HGDD directly. Nevertheless, as we shall see, this lemma is very useful in our proofs.

Lemma 2.4. *There exists an* HGDD *of type* (n, m^t) *if and only if there exists an* HGDD *of type* (t, m^n).

In order to use the above Lemma, we need the concept of double PBD-closed set. A *double* PBD-*closed set* is a cartesian product of two PBD-closed sets A and B. If A and B has their bases A_0 and B_0 respectively, we call $A_0 \times B_0$ the *basis* of the double PBD-closed set $A \times B$.

To give a point x weight $t(x)$ means that the point x is replaced by the set $\{x\} \times I_{t(x)}$ where $I_{t(x)} = \{1, 2, \ldots, t(x)\}$. The following four lemmas are the main recursive constructions in this paper.

Lemma 2.5. *If there exist a* TD(n, t) *which has one parallel class and a* GDD$[\{k\}, \lambda]$ *of type* m^n, *then there exists a* (tmn, k, λ)-HGDD *of type* (n, m^t).

Proof: Give weight m to every point of a TD(n, t) which has one parallel class. Then input GDD$[\{k\}, \lambda]$ of type m^n to every block of this TD except those in the parallel class.

Lemma 2.6. *If there exist a* $B[K,1;v]$ *and a* (ktm,k',λ)-HGDD *of type* (k,m^t) *for every* $k \in K$*, then there exists a* (vtm,k',λ)-HGDD *of type* (v,m^t).

Proof: Give weight mt to every point of the $B[K,1;v]$. For every block of size k, input an HGDD of type (k,m^t).

Lemma 2.7. *If there exist a* (v,k,λ_1)-HGDD *of type* (n,m^t) *and a* GDD$[\{k'\},\lambda_2]$ *of type* r^k*, then there exists a* $(rv,k',\lambda_1,\lambda_2)$-HGDD *of type* $(n,(mr)^t)$.

Proof: Give every point of the HGDD weight r and input GDDs of type r^k to every block of this HGDD.

Lemma 2.8. *Let* $A\times B$ *be a double* PBD*-closed set and* $A_0\times B_0$ *be its basis. If for every* $(n,t)\in A_0\times B_0$ *there exists an* (mnt,k,λ)-HGDD *of type* (n,m^t)*, then for any* $(u,v)\in(A\times B)\cup(B\times A)$ *there exists an* (muv,k,λ)-HGDD *of type* (u,m^v).

Proof: Conclusion follows from Lemma 2.4, Lemma 2.5 and Lemma 2.6 immediately.

3. The existence of HGDDs of $\lambda = 1$

In this section we prove the existence of $(v,3,1)$-HGDD. From Theorem 1.2 we have the following lemma.

Lemma 3.1. *The necessary and sufficient conditions for the existence of an* $(nt,3,1)$-HGDD *of type* $(n,1^t)$ *are that* $n\geq 3, t\geq 3, (t-1)(n-1)\equiv 0 \pmod 2$ *and* $tn(t-1)(n-1)\equiv 0 \pmod 6$.

Now we consider the case with holes of size $m>1$. From the necessary condition of Lemma 1.1 it is not difficult to know that to treat the cases $m=2,3$ and 6 is the main task of this section.

Lemma 3.2. *There exists a* $(v,3,1)$-HGDD *of type* $(3,2^t)$ *for every positive integer* $t\geq 3$.

Proof: Make use of Lemma 2.5 by letting $n=3$ and $m=2$. The required TD and GDD are given in Lemmas 2.1 and 2.2.

Lemma 3.3. *There exists a* $(v,3,1)$-HGDD *of type* $(4,2^t)$ *for any positive integer* $t\in B_1$.

Proof: For $t=4,5$ or 8, there exists a TD$(4,t)$ which has a parallel class (for there exist 3 MOLS of order t). So we can use Lemma 2.5 to obtain a $(8t,3,1)$-HGDD of type $(t,2^4)$ by inputting a GDD$[\{3\},1]$ of type 2^4. An HGDD of type $(3,2^4)$ is given in Lemma 3.2. An HGDD of type $(4,2^6)$ is given below.

points: $(\{a,b\}\cup Z_{10})\times Z_4$
groups: $(\{a,b\}\cup Z_{10})\times\{i\}, i=0,1,2,3$
holes: $\{a,b\}\times Z_4, \{i,i+5\}, i=0,1,2,3,4$
blocks: $\{0_1 1_2 3_0\}$ $\{0_1 9_2 6_0\}$ $\{0_1 8_2 4_0\}$ mod(10,4)
develop the first coordinates of the elements in the following blocks $i\to i+2$ modulo 10
$\{a_0 0_1 2_2\}$ $\{b_0 1_1 3_2\}$ $\{a_0 1_1 2_3\}$ $\{b_0 2_1 3_3\}$
$\{a_0 1_2 3_3\}$ $\{b_0 2_2 4_3\}$ $\{a_1 0_0 1_2\}$ $\{b_1 1_0 2_2\}$
$\{a_1 0_2 3_3\}$ $\{b_1 1_2 4_3\}$ $\{a_1 0_3 3_0\}$ $\{b_1 1_3 4_0\}$
$\{a_2 2_1 1_3\}$ $\{b_2 1_1 0_3\}$ $\{a_2 2_0 0_3\}$ $\{b_2 3_0 1_3\}$
$\{a_2 3_1 1_0\}$ $\{b_2 4_1 2_0\}$ $\{a_3 1_0 0_2\}$ $\{b_3 2_0 1_2\}$
$\{a_3 0_1 3_2\}$ $\{b_3 1_1 4_2\}$ $\{a_3 3_1 0_0\}$ $\{b_3 4_1 1_0\}$

The proof is complete.

Lemma 3.4. *There exists a* $(v,3,1)$-HGDD *of type* $(6,2^t)$ *for any positive integer* $t\in B_1$.

Proof: The $(v,3,1)$-HGDDs of type $(3,2^6)$ and $(4,2^6)$ are given in Lemmas 3.2 and 3.3 respectively. So we have the HGDDs of type $(6,2^3)$ and $(6,2^4)$. The HGDD of type $(5,2^6)$ comes from Lemma 2.7 by letting $(n,m,t,r,k,k')=(5,1,6,2,3,3)$. The required HGDD and GDD come from Lemma 3.1 and 2.2. From 5 MOLS of order 8 we obtain a TD$(6,8)$ which has a parallel class. Using Lemma 2.5 by taking $(n,m,t,k)=(6,2,8,3)$, a HGDD of type $(6,2^8)$ is obtained. An HGDD of type $(6,2^6)$ is displayed below.

points: $(\{a,b\})\cup Z_{10})\times Z_6$
groups: $(\{a,b\})\cup Z_{10}\times\{i\}, i=0,1,2,3,4,5$
holes: $\{a,b\}\times Z_6, \{i,i+5\}\times Z_6, i=0,1,2,3,4$
blocks: $\{a_1 1_5 0_0\}$ $\{b_1 0_5 2_0\}$ $\{0_1 9_4 3_0\}$ $\{0_1 2_4 9_0\}$
$\{0_1 3_4 2_0\}$ $\{0_1 4_4 6_0\}$ $\{0_1 3_3 4_5\}$ mod(10,6)
develop the first coordinates of the elements of the following blocks $i\to i+2$ modulo 10 and the second coordinates (written as subscripts) modulo 6
$\{a_1 3_2 1_4\}$ $\{b_1 4_2 2_4\}$ $\{a_1 0_2 3_3\}$ $\{b_1 1_2 4_3\}$
$\{a_1 4_3 0_4\}$ $\{b_1 5_3 1_4\}$

The proof is completed.

Lemma 3.5. *There exist* $(v,3,1)$-HGDDs *of type* $(n,3^t)$ *for any* $(n,t)\in B_1\times B_3$.

Proof: This proof is similar to the proof of Lemma 3.2. Here we let $m=n=3$ in Lemma 2.5 to show that HGDDs of type $(3,3^t)$ exist for any $t\in B_1$. For

$t \in \{5,8\}$ there exists TD$(5,t)$ which has a parallel class (there exist 4 MOLS of order t). So the HGDD of type $(5,3^t)$ can be constructed by again using Lemma 2.5 where the required GDD$[\{3\},1]$ of type 3^5 comes from Lemma 2.2. A $(v,3,1)$-HGDD of type $(t,1^5)$ exists from Lemma 3.1 where $t \in \{4,6\}$. So the HGDD of type $(t,3^5)$ exists for $t \in \{4,6\}$ by Lemma 2.7. This completes the proof.

Lemma 3.6. *There exists a* $(v,3,1)$-HGDD *of type* $(n,6^t)$ *for any* $(n,t) \in B_1 \times B_1$.

Proof: From 7 MOLS of order 8 we obtain a TD$(8,8)$ which has a parallel class. Using Lemma 2.5 by inputting a GDD$[\{3\},1]$ of type 6^8, we obtain a HGDD of type $(8,6^8)$. By using Lemma 2.7 we can construct HGDDs of type $(3,6^t)$, $(4,6^t)$, $(5,6^t)$ and $(6,6^t)$ from HGDDs of type $(3,2^t)$, $(4,2^t)$, $(5,3^t)$ and $(6,2^t)$ respectively, where $t \in B_1$. The required GDDs come from Lemma 2.2. This completes the proof.

Theorem 3.7. *The necessary and sufficient conditions for the existence of a* $(v,3,1)$-HGDD *of type* (n,m^t) *are that* $v = tmn, t \geq 3, n \geq 3, (t-1)(n-1)m \equiv 0 \pmod 2$ *and* $v(t-1)(n-1)m \equiv 0 \pmod 6$.

Proof: To prove this Theorem we distinguish four cases. By virtue of Lemma 2.8, we need only consider the pairs (n,t) which are from the basis of certain double PBD-closed set.

Case 1: $m \not\equiv 0 \pmod 2$ and $m \not\equiv 0 \pmod 3$. In this case t and n should satisfy $(n-1)(t-1) \equiv 0 \pmod 2$ and $v(n-1)(t-1) \equiv 0 \pmod 6$. So we can use Lemma 2.7 to construct an HGDD of type (n,m^t) from an HGDD of type $(n,1^t)$ which comes from Lemma 3.1.

Case 2: $m \equiv 0 \pmod 2$ and $m \not\equiv 0 \pmod 3$. In this case t and n should satisfy $tn(n-1)(t-1) \equiv 0 \pmod 3$. Without loss of generality, we assume that $(n,t) \in A_2 \times A_1$. For $m = 2$, Lemmas 3.2, 3.3 and 3.4 tell us that for any $(n,t) \in B_2 \times B_1$, there exists a $(v,3,1)$-HGDD of type $(n,2^t)$. For $m > 2$, the HGDD of type (t,m^n) can be obtained from an HGDD of type $(t,2^n)$ by using Lemma 2.7. The required GDD of type $(m/2)^3$ comes from Lemma 2.2.

Case 3: $m \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 3$. From the necessary condition described in Lemma 1.1, t and n should satisfying $(t-1)(n-1) \equiv 0$ (mod 2). We assume that $(n,t) \in A_1 \times A_3$. From Lemma 3.5 we know that for any $(n,t) \in B_1 \times B_3$ there exists an HGDD of type $(n,3^t)$. The HGDDs of type (t,m^n), where $m > 3$, can be constructed by using Lemma 2.7. The required GDD of type $(m/3)^3$ comes from Lemma 2.2.

Case 4: $m \equiv 0 \pmod 6$. For $m = 6$, the conclusion follows from Lemma 3.6. For $m > 6$, the HGDD of type (t,m^n) can be obtained by using Lemma 2.7 from the fact that there exists a GDD$[\{3\},1]$ of type $(m/6)^3$.

4. The existence of HGDDs of $\lambda > 1$

We now consider the case $\lambda > 1$ to complete the proof of our main result. By virtue of Theorem 1.2, we should treat the case $m > 1$.

Lemma 4.1. *There exists a* $(v, 3, 2)$-HGDD *of type* $(n, 3^t)$ *for any* $(n, t) \in B_1 \times B_1$.

Proof: As the existence of $(v, 3, 1)$-HGDD of type $(3, 3^t)$ and $(5, 3^t)$ has been proved in Theorem 3.7, we can repeat every block of these designs two times to obtain $(v, 3, 2)$-HGDDs of type $(3, 3^t)$ and $(5, 3^t)$. From Theorem 1.2 we know that a $(v, 3, 2)$-HGDD of type $(4, 1^t)$ or $(6, 1^t)$ exists. So a $(v, 3, 2)$-HGDD of type $(4, 3^t)$ or $(6, 3^t)$ can be constructed by using Lemma 2.7. Finally, from a TD$(8, 8)$ which has a parallel class (7 MOLS of order 8 exist) and a GDD$[\{3\}, 2]$ of type 3^8 we can obtain a $(v, 3, 2)$-HGDD of type $(8, 3^8)$. This completes the proof.

Theorem 4.2. *The necessary and sufficient conditions for the existence of a* $(v, 3, 2)$-HGDD *of type* (n, m^t) *are that* $v = tmn$, $t \geq 3$, $n \geq 3$ *and* $v(t-1)(n-1)m \equiv 0 \pmod 3$.

Proof: When $m \not\equiv 0 \pmod 3$, t and n should satisfy $tn(t-1)(n-1) \equiv 0 \pmod 3$. From Theorem 1.2 we know that there exists a $(tn, 3, 2)$-HGDD of type $(t, 1^n)$. So a $(tmn, 3, 2)$-HGDD of type (t, m^n) can be obtained by using Lemma 2.7. The required GDD$[\{3\}, 1]$ of type m^3 comes from Lemma 2.2.

When $m \equiv 0 \pmod 3$, there exists an HGDD of type $(t, 3^n)$ by Lemmas 4.1 and 2.8. Using Lemma 2.7 by inputting GDD$[\{3\}, 1]$ of type $(m/3)^3$ yields a $(v, 3, 2)$-HGDD of type (t, m^n) for $m > 3$. The conclusion follows.

Lemma 4.3. *There exists a* $(v, 3, 3)$-HGDD *of type* $(t, 2^n)$ *for any* $(n, t) \in B_1 \times B_1$.

Proof: As the existence of $(v, 3, 1)$-HGDDs of type $(4, 2^t)$ or $(6, 2^t)$ has been proved in Theorem 3.7, we can repeat every block of these designs three times to obtain $(v, 3, 3)$-HGDDs of type $(4, 2^t)$ or $(6, 2^t)$. From Theorem 1.2 we know that the $(v, 3, 3)$-HGDDs of type $(3, 1^n)$ and type $(5, 1^n)$ exist. So the $(v, 3, 3)$-HGDD of type $(3, 2^n)$ and type $(5, 2^n)$ can be constructed by using Lemma 2.7. Finally, from a TD$(8, 8)$ which has a parallel class (7 MOLS of order 8 exist) and a GDD$[\{3\}, 3]$ of type 2^8 we can obtain a $(v, 3, 3)$-HGDD of type $(8, 2^8)$. This completes the proof.

Theorem 4.4. *The necessary and sufficient conditions for the existence of a* $(v, 3, 3)$-HGDD *of type* (n, m^t) *are that* $v = tmn$, $t \geq 3$, $n \geq 3$ *and* $(t-1)(n-1)m \equiv 0 \pmod 2$.

Proof: When $m \not\equiv 0 \pmod 2$, t and n should satisfy $(t-1)(n-1) \equiv 0 \pmod 2$. From Theorem 1.2 we know that there exists a $(tn, 3, 3)$-HGDD of type $(t, 1^n)$.

So a $(tmn, 3, 3)$-HGDD of type (t, m^n) can be obtained by using Lemma 2.7. The required GDD[$\{3\}$, 1] of type m^3 comes from Lemma 2.2.

When $m \equiv 0 \pmod 2$, there exists an HGDD of type $(t, 2^n)$ by Lemma 4.3. Using Lemma 2.7 by inputting GDD[$\{3\}$, 1] of type $(m/2)^3$ yields a $(v, 3, 3)$-HGDD of type $(t, m)^n$ for $m > 2$. The conclusion follows.

Theorem 4.5. *The necessary and sufficient conditions for the existence of a* $(v, 3, 6)$-HGDD *of type* (n, m^t) *are that* $v = tmn, t \geq 3$ *and* $n \geq 3$.

Proof: From Theorem 1.2 we know that a $(v, 3, 6)$-HGDD of type $(n, 1^t)$ exists. So we can use Lemma 2.7 to obtain a $(v, 3, 6)$-HGDD of type (n, m^t) by inputting a GDD[$\{3\}$, 1] of type m^3.

We are now in a position to prove our main Theorem which is restated below for the reader's convenience.

Theorem 4.6. *The necessary and sufficient conditions for the existence of a* $(v, 3, \lambda)$-HGDD *of type* (n, m^t) *are that* $v = tmn, t \geq 3, n \geq 3, \lambda(t-1)(n-1)m \equiv 0 \pmod 2$ *and* $\lambda v(t-1)(n-1)m \equiv 0 \pmod 6$.

Proof: When $\lambda \not\equiv 0 \pmod 2$ and $\lambda \not\equiv 0 \pmod 3$, then t, n and m should satisfy $(t-1)(n-1)m \equiv 0 \pmod 2$ and $v(n-1)(t-1) \equiv 0 \pmod 6$, so there exists a $(v, 3, 1)$-HGDD of type (n, m^t) by Theorem 3.8. The required $(v, 3, \lambda)$-HGDD then can be obtained from this design by repeating every block λ times.

When $\lambda \equiv 0 \pmod 2$ and $\lambda \not\equiv 0 \pmod 3$, then $v(t-1)(n-1)m \equiv 0 \pmod 3$, so there exists a $(v, 3, 2)$-HGDD of type (n, m^t) by Theorem 4.2 and then the required $(v, 3, \lambda)$-HGDD of the same type.

When $\lambda \not\equiv 0 \pmod 2$ and $\lambda \equiv 0 \pmod 3$, then $(t-1)(n-1)m \equiv 0 \pmod 2$, so there exists a $(v, 3, 3)$-HGDD of type (n, m^t) by Theorem 4.4 and then the $(v, 3, \lambda)$-HGDD of the same type.

When $\lambda \equiv 0 \pmod 6$, there exists a $(v, 3, 6)$-HGDD of type (n, m^t) by Theorem 4.5 and then the required $(v, 3, \lambda)$-HGDD of the same type. The proof is completed.

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