Packing and covering of the complete graph, IV: the trees of order seven

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Abstract. It is shown that the maximal number of pairwise edge disjoint trees of order seven in the complete graph K_n, and the minimum number of trees of order seven, whose union is K_n are $\left\lfloor \frac{n(n-1)}{12} \right\rfloor$ and $\left\lceil \frac{n(n-1)}{12} \right\rceil$, $n \geq 11$, respectively. ($\lfloor x \rfloor$ denotes the largest integer not exceeding x and $\lceil x \rceil$ the least integer not less than x).

1. Introduction.

Graphs in our context are undirected, finite, and have no multiple edges or loops. We refer to [H] for the basic definitions.

We denote by $P(n, H)$, the *packing number*, namely, the maximal number of pair-wise edge disjoint graphs H, in the complete graph K_n, and by $C(n, H)$, the *covering number*, namely, the minimum number of graphs H whose union is K_n.

As usual $\lfloor x \rfloor$ will denote the largest integer not exceeding x and $\lceil x \rceil$ the least integer not less than x.

In [R1], [R2], and [R3], it was proved that:

(1) $P(n,T) = \left\lfloor \frac{n(n-1)}{2e(T)} \right\rfloor$ and

(2) $C(n,T) = \left\lceil \frac{n(n-1)}{2e(T)} \right\rceil$, for $n \geq n_0$,

where T was any tree of order less than equal six, $e(T)$ is the number of edges of T and n_0 was a constant determined in the various cases.

It was asked in [R3] if (1) and (2) are true for all trees.

Our purpose in this paper is to answer that question in the affirmative for all trees of order seven.

Definition: A graph H is said to have a *G-decomposition* if it is the union of edge disjoint subgraphs each isomorphic to G. Wc denote this fact by $G \mid H$. ∎

The *G-decomposition* problem, for $H = K_n$, is to determine the set of naturals $N(G)$, such that K_n has a *G-decomposition* if and only if $n \in N(G)$.

Note that *G-decomposition* is actually an exact packing and covering.

In the proof of our problems of packing and covering, we make a great use of the results obtained by Huang and Rosa [HR], for the *G-decomposition* problem in cases when G is a tree of order seven.

We denote $H = \cup_{i=1}^{t} G_i$ when the graph H is the union of t edge disjoint graphs G_i, $i = 1, 2, \ldots, t$.

The results of these problems which will be discussed in details in the remainder of the paper, can be summarized in:

Main Theorem (Packing and Covering).

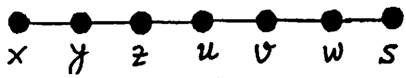
(a) $P(n,T) = \left\lfloor \frac{n(n-1)}{12} \right\rfloor$, $n \geq 11$ *and T any tree of order seven.*

(b) $C(n,T) = \left\lceil \frac{n(n-1)}{12} \right\rceil$, $n \geq 11$ *and T any tree of order seven.* ∎

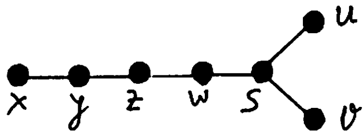
Remark: In the case $7 \leq n < 11$ we shall give the exact values of the packing and covering numbers for each tree in consideration.

The relevant trees to our problems are:

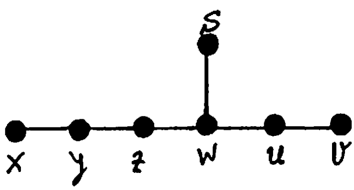
(i) T_1, the path of length six, which is denoted (x, y, z, u, v, w, s).



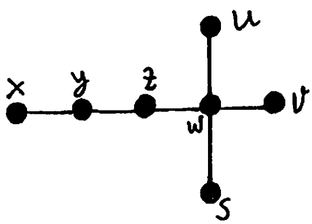
(ii) T_2, is denoted $(x, y, z, w, s; u, v)$.



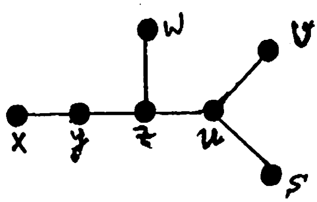
(iii) T_3, is denoted $(x, y, z, w(s), u, v)$.



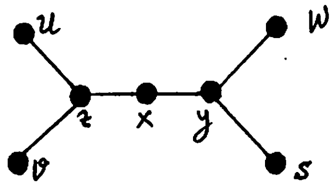
(iv) T_4, is denoted $(w; u, v, s, z - y, x)$.



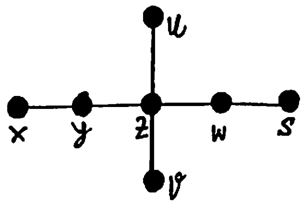
(v) T_5, is denoted $(x, y, z(w), u; v, s)$.



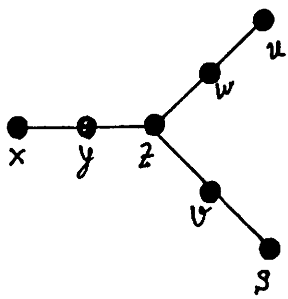
(vi) T_6, is denoted $(u, v; z, x, y; w, s)$.



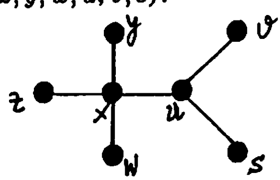
(vii) T_7, is denoted $(x, y, z; u, v, w - s)$.



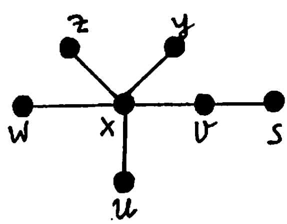
(viii) T_8, is denoted $(x, y, z; w(u), v(s))$.



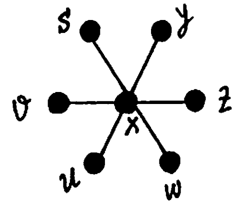
(ix) T_9, is denoted $(x; z, y, w, u; v, s)$.



(x) T_{10}, is denoted $(x; y, z, w, u, v - s)$.



(xi) T_{11}, is denoted $(x; y, z, w, u, v, s)$.



2. Preliminary results.

Notation: The vertex set of K_n is defined to be $\boldsymbol{Z}_n$, and addition of vertex labels are done mod n.

Lemma 2.1. $T_i \mid K_{2,6}, K_{3,6}$, $i = 4, 6, 10$.

Proof: Let $V(K_{2,6}) = A \cup B$ where $A = \{a, b\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. Let $V(K_{3,6}) = A' \cup B'$, where $A' = A \cup \{c\}$ and $B' = B$. The T_i-decomposition of $K_{2,6}$ and $K_{3,6}$ $i = 4, 6, 10$ is found in Table 1.

Table 1
The decomposition

	$K_{2,6}$	$K_{3,6}$
T_4	$(a; 0, 1, 2, 3-b, 4)$ $(b; 0, 1, 2, 5-a, 4)$	$(a; 0, 1, 2, 3-b, 4)$ $(b; 0, 1, 2, 5-c, 3)$ $(c; 0, 1, 2, 4-a, 5)$
T_6	$(0, 1; a, 2, b; 3, 4)$ $(0, 1; b, 5, a; 3, 4)$	$(0, 1; a, 2, b; 3, 4)$ $(0, 2; c, 5, a; 3, 4)$ $(0, 5; b, 1, c; 3, 4)$
T_{10}	$(a; 0, 1, 2, 3, 4-b)$ $(b; 0, 1, 2, 3, 5-a)$	$(a; 0, 1, 2, 3, 4-b)$ $(b; 0, 1, 2, 5, 4-c)$ $(c; 0, 1, 2, 3, 5-a)$

Corollary 1. *$T_i \mid K_{2\alpha+3\beta,6t}$ for $i = 4, 6, 10$, α, β non-negative integers at least one is positive and t a positive integer.* ▮

Lemma 2.2. $T_1, T_7 \mid K_{t,6}$ *for* $t = 3, 4, 5$.

Proof: Let $V(K_{3,6}) = A \cup B$ where $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3, 4, 5\}$. Let $V(K_{4,6}) = A_1 \cup B_1$ where $A_1 = A \cup \{d\}$ and $B_1 = B$, and $V(K_{5,6}) = A_2 \cup B_2$ where $A_2 = A \cup \{d, e\}$ and $B_2 = B$. The T_1, T_7-decompositions of $K_{t,6}, t = 3, 4, 5$ are found in Table 2.

Table 2
The decomposition

	T_7	T_1
$K_{3,6}$	$(b,2,a;0,1,3-c)$	$(0,a,1,b,2,c,3)$
	$(a,5,b;3,4,0-c)$	$(1,c,4,b,5,a,2)$
	$(a,4,c;2,5,1-5)$	$(4,a,3,b,0,c,5)$
$K_{4,6}$	$(b,2,a;0,1,3-c)$	$(0,a,1,b,2,c,3)$
	$(a,5,c;0,1,2-d)$	$(0,b,5,c,4,d,3)$
	$(a,4,b;0,5,3-d)$	$(b,3,a,2,d,1,c)$
	$(b,1,d;0,5,4-c)$	$(c,0,d,5,a,4,b)$
$K_{5,6}$	$(b,2,a;0,1,3-c)$	$(0,a,1,b,2,c,3)$
	$(a,4,c;0,1,2-e)$	$(0,c,4,d,5,e,3)$
	$(a,5,b;3,4,1-d)$	$(0,d,1,c,5,a,4)$
	$(b,0,d;2,5,3-e)$	$(1,e,4,b,3,a,2)$
	$(c,5,e;0,1,4-d)$	$(3,d,2,e,0,b,5)$

Corollary 2. $T_1, T_7 \mid K_{3\alpha+4\beta+5\gamma,6t}$ *where* α,β,γ *are non-negative integers at least one positive and* t *a positive integer.* ∎

Lemma 2.3. $T_2 \mid K_{t,6}$ *for* $t = 4,5,6$.

Proof: Let $V(K_{t,6}) = A \cup B$ where $A = \{a,b,c,d,e,f\}$ (according to the value of t), and $B = \mathbf{Z}_5$. The T_2-decomposition is found in Table 3.

Table 3
The decomposition

$K_{4,6}$	$K_{5,6}$	$K_{6,6}$
$(0,a,1,b,2;c,d)$	$(a,0,b,2,c;1,3)$	$(a,0,b,2;1,5)$
$(2,a,4,b,0;c,d)$	$(c,4,d,5,e;2,3)$	$(a,2,d,1,b;4,5)$
$(3,d,1,c,5;a,b)$	$(d,2,a,3,b;1,4)$	$(c,4,d,5,e;1,3)$
$(5,d,4,c,3;a,b)$	$(e,4,a,1,d;0,3)$	$(e,4,a,5,f;0,2)$
	$(1,e,0,c,5;a,b)$	$(2,e,1,a,3;b,d)$
		$(4,f,3,c,0;a,e)$

Corollary 3. $T_2 \mid K_{4\alpha+5\beta+6\gamma,6t}$ *where* α,β,γ *are non-negative integers at least one positive and* t *a positive integer.* ∎

Lemma 2.4. $T_3, T_9 \mid K_{t,6}$ *where* $t = 3,4,5$

Proof: Let $V(K_{3,6}) = A \cup B$ where, $A = \{a, b, c\}$ and $B = \mathbb{Z}_5$. Let $V(K_{4,6}) = A' \cup B$ where $A' = A \cup \{d\}$. Let $V(K_{5,6}) = C \cup B$ where $C = A' \cup \{e\}$. The decomposition is in Table 4.

Table 4
The decomposition

	T_3	T_9
$K_{3,6}$	$(2, b, 5, c(0), 4, a)$	$(0, 1, 2; a, 3; b, c)$
	$(3, c, 2, a(1), 0, b)$	$(0, 1, 2; b, 5; a, c)$
	$(5, a, 3, b(4), 1, c)$	$(0, 1, 2; c, 4; a, b)$
$K_{4,6}$	$(1, b, 2, d(4), 3, a)$	$(0, 1, 2; a, 3; b, c)$
	$(1, d, 5, b(3), 4, c)$	$(1, 2, 3; d, 4; a, c)$
	$(3, c, 2, a(1), 0, b)$	$(1, 2, 5; c, 0; b, d)$
	$(4, a, 5, c(1), 0, d)$	$(1, 2, 4; b, 5; a, d)$
$K_{5,6}$	$(0, a, 1, b(2), 3, c)$	$(0, 1, 4; b, 5; a, c)$
	$(0, b, 5, c(2), 1, e)$	$(1, 2, 4; a, 3; b, c)$
	$(1, d, 5, a(4), 3, e)$	$(3, 4, 5; d, 2; b, e)$
	$(5, e, 4, d(3), 2, a)$	$(3, 4, 5; e, 1; c, d)$
	$(b, 4, c, 0(d), e, 2)$	$(a, d, e; 0, c; 2, 4)$

Corollary 4. $T_3, T_9 \mid K_{3\alpha+4\beta+5\gamma,6t}$ *where* α, β, γ *are non-negative integers at least one positive and* t *a positive integer.* ∎

Lemma 2.5. $T_5, T_8 \mid K_{t,6}$ *where* $t = 3, 4, 5$.

Proof: Let $V(K_{3,6}) = A \cup B$ where, $A = \{a, b, c\}$ and $B = \mathbb{Z}_5$. Let $V(K_{4,6}) = A' \cup B$ where $A' = A \cup \{d\}$. Let $V(K_{5,6}) = C \cup B$ where $C = A' \cup \{e\}$. The decomposition is in Table 5.

Table 5
The decomposition

	T_5	T_8
$K_{3,6}$	$(0, a, 1(b), c; 2, 3)$	$(0, a, 1; b(2), c(3))$
	$(0, c, 4(a), b; 2, 3)$	$(0, b, 5; a(3), c(2))$
	$(0, b, 5(c), a; 2, 3)$	$(0, c, 4; a(2), b(3))$
$K_{4,6}$	$(0, a, 1(b), c; 2, 3)$	$(0, a, 1; b(2), c(3))$
	$(0, b, 2(a), d; 2, 3)$	$(0, b, 4; c(2), d(1))$
	$(0, c, 5(d), a; 3, 4)$	$(0, d, 3; a(2), b(5))$
	$(0, d, 4(c), b; 3, 4)$	$(0, c, 5; a(4), d(2))$

Table 5 continued

	T_5	T_8
$K_{5,6}$	$(0,a,1(b),c;2,3)$	$(0,a,1;b(2),c(3))$
	$(0,b,2(d),e;1,3)$	$(0,c,4;d(3),e(5))$
	$(0,c,4(e),d;1,5)$	$(1,d,5;a(4),b(3))$
	$(0,d,3(a),b;4,5)$	$(1,e,2;a(3),c(5))$
	$(0,e,5(c),a;2,4)$	$(2,d,0;b(4),e(3))$

Corollary 5. $T_5, T_8 \mid K_{3\alpha+4\beta+5\gamma,6t}$ *where* α,β,γ *are non-negative integers at least one positive and* t *a positive integer.* ∎

3. Proof of Main Theorem in the various cases.

Huang and Rosa [HR] proved the following:

Theorem 3.1. $N(T_i) = \{n \mid n \equiv 0,1,4,9 \pmod{12}, i = 1,2,\ldots,10\}$ *and* $N(T_{11}) = \{n \mid n \equiv 0,1,4,9 \pmod{12}, n \geq 12\}$. ∎

As a result of Theorem 3.1 we have to prove the Main Theorem for the listed trees only in the cases:

(1) $n = 12m + k, k = 2,3,5,6,7,8,10,11.$

Theorem 3.2. *The Main Theorem is valid for* T_1 *for all* $n \geq 7$.

Proof: The proof will take case of several cases according to the various values of n. We use the well-known decomposition (see [H p. 89]) of K_n, $n-$ odd, into $\frac{n-1}{2}$ spanning cycles and for $n-$ even into $\frac{n}{2}$ Hamilton paths.

For $n-$ odd the proof of the theorem follows immediately by cutting the P_7-paths from the suitable Eulerian tour, created by union of the above mentioned spanning cycles, leaving a path of length less then seven, unpacked. We demonstrate it with $n = 11$.

$$K_{11} = \bigcup_{i=0}^{4}(0,1+i,2+i,10+i,9+i,4+i,8+i,5+i,7+i,6+i).$$

When n is even, put $n-1 \equiv q \pmod 6$. Let, $K_n = \cup_{i=0}^{\frac{n}{2}-1} P_i$, where, $P_i = \left(1+i, 2+i, n+i, 3+i, n-1+i, \ldots, \frac{n+4}{2}+i, \frac{n+2}{2}+i\right)$, and addition is done modulo n. First we deal with $n = 8$.

We give the construction for $q = 3,5$. For $q = 1$ the construction is similar.

Let $q = 3$. From each of the paths P_i delete the edges $(1+i, 2+i)$ and $\left(\frac{n+4}{2}+i, \frac{n+2}{2}+i\right)$, so that the remaining paths are P_7 decomposable. The union of the rest of the edges results in a spanning cycle, a fact which completes the proof in this case.

Table 6

n	*Packing*	*Remains* for *Covering*
8	(1,7,6,5,4,3,2)	
	(2,1,4,7,5,0,6)	$(2,7,3,5)\cup(0,1)$
	(3,1,6,4,2,0,7)	
	(4,0,3,6,2,5,1)	
9	T_1-decomposition (Theorem 3.1)	
$12m, 12m+1$		
$12m+4, 12m+9$	T_1-decomposition (Theorem 3.1)	

Let $q = 5$. Delete from each path P_i, $(1+i, 2+i, n+i)$ and $\left(\frac{n+4}{2}+i, \frac{n+2}{2}+i\right)$. The remaining paths are P_7-decomposable, and so is the union of the deleted edges, excluding a path of length three. We demonstrate it for $n = 18$. After the decomposition of $P_i \setminus \{(1+i, 2+i, n+i) \cup \left(\frac{n+4}{2}+i, \frac{n+2}{2}+i\right)\}$, we add the following three P_7-paths: (2,3,4,5,6,7,8,9), (10,11,12,13,14,15,16,17), (17,18,2,1,3,5,7,9). The non-packed path is (2,4,6,8). This completes the proof of the theorem. ∎

Theorem 3.3. *The Main Theorem is valid for T_2 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 7

n	*Packing*	*Remains* for *Covering*
7	$(0+i, 6+i, 1+i, 5+i, 2+i, 3+i, 4+i), i=0,1,2$	$(0,5,6)\cup(1,4)$
8	$(0+i, 7+i, 1+i, 6+i, 2+i, 4+i, 5+i), i=0,1,2$	
	(2,3,4,1,5;0,7)	(4,5,6;0,7)
9	T_2-decomposition (Theorem 3.1)	
10	$(0+i, 6+i, 1+i, 5+i, 2+i, 3+i, 4+i), i=0,1,2$	
	(1,4,8,9,7;5,6)	
	(3,7,8,6,9;1,2)	
	(7,4,9,5,8;0,1)	(9,0,5,6)
	(9,3,8,2,7;0,1)	
11	$(0+i, 1+i, 10+i, 2+i, 9+i; 5+i, 6+i), i=0,\dots,3$	
	$(4+i, 5+i, 3+i, 6+i, 2+i; 7+i, 8+i), i=0,1,2$	
	(0,6,1,7,9;8,10)	(0,10)
	(9,0,5,10,8;6,7)	
12,13	T_2-decomposition (Theorem 3.1)	

Table 7 continued

n	*Packing*	*Remains* for *Covering*
14	$(0+i, 1+i, 13+i, 2+i, 12+i; 6+i, 7+i), i=0,\dots,6$	
	$(9+i, 10+i, 8+i, 11+i, 7+i; 0+i, 1+i), i=0,\dots,4$	(5,13)
	$(2+i, 7+i, 8+i, 6+i, 9+i; 4+i, 5+i), i=0,1$	
	$(12,5,0,13,6;1,11)$	
15	$(2+i, 14+i, 3+i, 13+i, 4+i; 12+i, 0), i=0,\dots,6$	
	$(9+i, 7+i, 10+i, 6+i, 11+i; 5+i, 0), i=0,\dots,6$	
	$(1,2,3,4,5;6,12)$	$(1,8) \cup (2,9) \cup (6,13)$
	$(3,11,12,13,14;1,7)$	
	$(6,7,8,9,10;11,3)$	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let

(2) $K_{12m+2} = K_{12(m-1)} \cup K_{14,12(m-1)} \cup K_{14}, \ m \geq 2$.

Then, using Theorem 3.1, Corollary 3, and Table 7, for $n = 14$ we have the packing and the covering as well.

k = 3.

Let

(3) $K_{12m+3} = K_{12(m-1)} \cup K_{15,12(m-1)} \cup K_{15}, \ m \geq 2$.

Then, using Theorem 3.1, Corollary 3, and Table 7, for $n = 15$ we have the packing and the covering as well.

k = 5.

Let

(4) $K_{12m+5} = K_{12m} \cup K_{5,12m} \cup K_5$.

We have by Theorem 3.1, and Corollary 3, that $T_2 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m + j\}, j = 0, 1, \dots, 4$.

Take a tree from the decomposition of $K_{5,12m}$, say, $(12m, 0, 12m+1, 1, 12m+2; 2, 3)$ and with the non-packed K_5 create the following trees:

$$(12m, 0, 12m+1, 1, 12m+2; 12m+3, 12m+4) \text{ and}$$
$$(12m+3, 12m+4, 12m, 12m+1, 12m+2; 2, 3).$$

Hence, the packing is completed and we are left with the non-packed path $(12m+2, 12m, 12m+3, 12m+1, 12m+4)$.

k = 6.

Let

(5) $K_{12m+6} = K_{12m} \cup K_{6,12m} \cup K_6$.

Using Theorem 3.1, and Corollary 3, $T_2 \mid K_{12m} \cup K_{6,12m}$.

Denote the vertices of K_6 by, $\{12m + j\}, j = 0, 1, \dots, 5$.

Take some tree from the T_2-decomposition of $K_{6,12m}$, say, $(12m, 0, 12m + 1, 1, 12m + 2; 2, 3)$.

Using that tree and the unpacked graph K_6 we create the following trees T_2:

$(0, 12m, 12m + 1, 12m + 2, 12m + 3; 12m + 4, 12m + 5)$;
$(0, 12m + 1, 1, 12m + 2, 12m + 4; 12m, 12m + 5)$; and
$(12m + 4, 12m + 1, 12m + 3, 12m, 12m + 2; 2, 3)$.
We are left with the non-packed star:
$(12m + 5; 12m, 12m + 1, 12m + 2)$ for the covering.

k = 7, 8, 10, 11.

Let

(6) $K_{12m+k} = K_{12m} \cup K_{k,12m} \cup K_k$.

From Theorem 3.1, and Corollary 3, we have that $T_2 \mid K_{12m} \cup K_{k,12m}$. In Table 7 we find the packing and covering of K_k. Hence, the proof is completed. ∎

Theorem 3.4. *The Main Theorem is valid for T_3 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 8

n	*Packing*	*Remains* for *Covering*
7	(0,2,6,4(5),1,3)	
	(0,1,2,3(4),5,6)	(0,6,3) ∪ (2,5)
	(6,1,5,0(3),4,2)	
8	(0,2,6,4(5),1,3)	
	(0,1,2,3(4),5,6)	(1,6,7;0,5)
	(0,6,3,7(4),2,5)	
	(7,1,5,0(3),4,2)	
9	T_3-decomposition (Theorem 3.1)	
10	(0,6,3,7(4),2,5)	
	(1,3,8,5(4),9,6)	
	(4,8,9,1(0),2,3)	
	(7,1,5,0(3),4,2)	(1,4,9) ∪ (0,8)
	(8,1,6,7(5),0,9)	

Table 8 continued

n	*Packing*	*Remains* for *Covering*
	(8,2,9,3(4),5,6)	
	(9,7,8,6(4),2,1)	
11	(0,6,3,10(1),7,2)	
	(1,3,8,5(4),9,6)	
	(1,4,9,10(2),8,0)	
	(3,7,4,10(0),6,5)	
	(4,8,9,1(0),2,3)	
	(7,1,5,0(3),4,2)	(2,5)
	(8,1,6,7(5),0,9)	
	(8,2,9,3(4),5,10)	
	(9,7,8,6(4),2,1)	
12,13	T_3-decomposition (Theorem 3.1)	
14	(0,6,3,10(1),7,2)	
	(1,3,8,5(4),9,6)	
	(1,4,9,10(2),8,0)	
	(2,5,11,4(12),13,3)	
	(3,7,4,10(0),6,5)	
	(4,8,9,1(0),2,3)	
	(5,12,3,11(10),13,2)	
	(6,13,7,12(0),8,11)	
	(7,1,5,0(3),4,2)	(11,12)
	(7,11,6,12(10),13,5)	
	(8,1,6,7(5),0,9)	
	(8,2,9,3(4),5,10)	
	(9,7,8,6(4),2,1)	
	(9,12,1,13(5),0,11)	
	(10,13,9,11(1),2,12)	
$12m$, $12m+1$		
$12m+4$, $12m+9$	T_3-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (2).

Then, using Theorem 3.1, Corollary 4, and Table 8, for $n = 14$ we have the packing and the covering as well.

k = 3.

Let

(7) $K_{12m+3} = K_{12m} \cup K_{3,12m} \cup K_3$.

Using Corollary 4 Theorem 3.1 we have that $T_3 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m, 12m+1, 12m+2$. Take some T_3 from the decomposition of $K_{3,12m}$, say, $(2, 12m+1, 5, 12m+2(0), 4, 12m)$, and together with the non-packed triangle create the tree, $(2, 12m+1, 5, 12m+2(0), 12m, 4)$, leaving the path $(4, 12m+2, 12m+1, 12m)$ non-packed, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

We have by Theorem 3.1, and Corollary 4, that $T_3 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0, 1, \ldots, 4$.

Take a tree from the decomposition of $K_{5,12m}$, say, $(0, 12m, 1, 12m+1(2), 3, 12m+2)$ and with the non-packed K_5 create the following trees:

$$(0, 12m, 1, 12m+1(12m+3), 12m+4, 12m+2) \text{ and}$$
$$(2, 12m+1, 3, 12m+2(12m+3), 12m, 12m+4).$$

Hence, the packing is completed and we are left with $(12m+2, 12m+1, 12m, 12m+3, 12m+4)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1, and Corollary 4, $T_3 \mid K_{12m} \cup K_{6,12m}$.

Denote the vertices of K_6 by, $\{12m+j\}, j = 0, 1, \ldots, 5$.

Take some tree from the T_3-decomposition of $K_{6,12m}$, say, $(0, 12m, 1, 12m+1(2), 3, 12m+2)$.

Using that tree and the unpacked graph K_6 we create the following trees T_3:

$(2, 12m+1, 12m+4, 12m(0), 12m+2, 12m+5)$;
$(12m, 12m+5, 12m+3, 12m+2(12m+4), 3, 12m+1)$; and
$(12m+5, 12m+4, 12m+3, 12m+1(12m+2), 1, 12m)$.
We are left with the non-packed path:
$(12m+3, 12m, 12m+1, 12m+5)$.

k = 7, 8, 10, 11.

Let K_{12m+k} be as in (6).

From Theorem 3.1, and Corollary 4, we have that $T_3 \mid K_{12m} \cup K_{k,12m}$. In Table 8 we find the packing and covering of K_k. Hence, the proof is completed. ∎

Lemma 3.5. $P(7,T_4) = 3, C(7,T_4) = 5$.

Proof: The proof is easy by straight forward verification. ∎

Theorem 3.6. *The Main Theorem is valid for T_4 for all $n \geq 8$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 9

n	*Packing*	*Remains* for *Covering*
8	$(0;4,5,6,1-2,3)$	
	$(3;0,4,5,6-1,7)$	
	$(4;1,5,6,7-1,0)$	$(5;2,6,1-3)$
	$(7;0,3,5,6-2,4)$	
9	T_4-decomposition (Theorem 3.1)	
10	$(0;4,5,6,1-2,3)$	
	$(3;0,4,5,6-1,7)$	
	$(4;1,5,6,7-1,0)$	
	$(7;0,3,5,6-2,4)$	$(8;0,1,9)$
	$(8;3,4,7,6-5,2)$	
	$(9;0,1,3,2-8,5)$	
	$(9;4,6,7,5-1,3)$	
11	$(0;4,5,6,1-2,3)$	
	$(3;0,4,5,6-1,7)$	
	$(4;1,5,6,7-1,0)$	
	$(7;0,3,5,6-2,4)$	
	$(8;3,4,7,6-5,2)$	$(0,8)$
	$(9;0,4,6,5-1,3)$	
	$(9;1,2,3,7-10,8)$	
	$(10;0,2,3,5-8,9)$	
	$(10;4,6,9,1-8,2)$	
$12m, 12m+1$		
$12m+4, 12m+9$	T_4-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let

(8) $K_{12m+2} = K_{12m} \cup K_{2,12m} \cup K_2$.

Then, using Corollary 1 and Theorem 3.1, we have the packing number leaving K_2 in (8) unpacked for the covering.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 1 and Theorem 3.1 we have that $T_4 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m$, $12m+1$, $12m+2$. Take some T_4 from the decomposition of $K_{3,12m}$, say, $(12m; 0, 1, 2, 3 - 12m+1, 4)$. Instead of the edge $(12m, 0)$ put $(12m, 12m+2)$, so that we are left with the path $(12m+2, 12m+1, 12m, 0)$ unpacked, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

Since $K_{5,12m} = 2m(K_{2,6} \cup K_{3,6})$ we have by Theorem 3.1 and Corollary 1 that $T_4 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0, 1, \ldots, 4$. Take a tree from the decomposition of $K_{5,12m}$, say, $(12m; 0, 1, 2, 3 - 12m+1, 4)$ and with the non-packed K_5 create the following trees:

$$(12m; 0, 1, 12m+1, 12m+2 - 12m+3, 12m+4) \text{ and}$$
$$(12m+1; 4, 12m+2, 12m+3, 3 - 12m, 2).$$

Hence, the packing is completed and we are left with $(12m+4; 12m+2, 12m+1, 12m - 12m+3)$ for the covering.

k = 6.

Let

(9) $K_{12m+6} = K_{12m+4} \cup K_{2,12m} \cup K_{2,4} \cup K_2$.

Using Theorem 3.1, and Corollary 1, $T_4 \mid K_{12m+4} \cup K_{2,12m}$.

Denote the vertices of $K_{2,4}$ by, $\{12m, 12m+1\}$, $\{12m+2, 12m+3, 12m+4, 12m+5\}$.

Take some tree from the T_4-decomposition of $K_{2,12m}$, say, $(12m; 0, 1, 2, 3 - 12m+1, 4)$.

Using that tree and the unpacked graph $K_{2,4} \cup K_2$ we create the following trees T_4:

$(12m; 0, 1, 12m+2, 12m+5 - 12m+1, 4)$; and
$(12m+1; 12m+2, 12m+3, 12m+4, 3 - 12m, 2)$.
Hence, the packing is completed and we are left with the non-packed star:
$(12m; 12m+1, 12m+3, 12m+4)$ for the covering.

k = 7.

Let

(10) $K_{12m+7} = K_{12m+4} \cup K_{3,12m} \cup K_{3,4} \cup K_3$.

Using Theorem 3.1 and Corollary 1, $T_4 \mid K_{12m+4} \cup K_{3,12m}$.
Denote the vertices of $K_{3,4}$ by $\{12m, 12m+1, 12m+2\}$ and $\{12m+3, 12m+4, 12m+5, 12m+6\}$.
Take two trees from the T_4-decomposition of $K_{3,12m}$, say, $(12m; 0,1,2,3-12m+1,4)$ and $(12m+1; 0,1,2,5-12m+2,3)$.
Using those trees and the unpacked graph $K_{3,4} \cup K_3$ we create the following trees:
$(12m; 0,1,2,12m+3-12m+1,4)$;
$(12m; 12m+4, 12m+5, 12m+6, 12m+1-5, 12m+2)$;
$(12m+2; 12m+3, 12m+4, 12m+5, 12m+1-3, 12m)$;
$(12m+1; 0,1,2,12m+6-12m+2,3)$. So the packing is completed.
We are left with the unpacked graph:
$(12m+1; 12m+4, 12m+5) \cup (12m, 12m+2)$ which proves the covering as well.

k = 8, 10, 11.

Let K_{12m+k} be as in (6). We have from Theorem 3.1, and Corollary 1, that $T_4 \mid K_{12m} \cup K_{12m,k}$. From Table 9 we have the packing and covering of K_k. Hence, the proof of Main Theorem for T_4 is completed. ∎

Theorem 3.7. *The Main Theorem is valid for T_5 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 10

n	*Packing*	*Remains* for *Covering*
7	$(0,1,2(3),4;5,6)$	
	$(1,4,3(0),6;2,5)$	$(0,2) \cup (3,5) \cup (1,6)$
	$(3,1,5(2),0;4,6)$	
8	$(0,1,2(3),4;5,6)$	
	$(0,7,3(4),1;5,6)$	
	$(1,4,7(2),6;0,5)$	$(0,3,6,2,5)$
	$(1,7,5(3),0;2,4)$	
9	T_5-decomposition (Theorem 3.1)	
10	$(0,1,2(3),4;5,6)$	
	$(0,7,3(4),1;5,6)$	
	$(1,4,7(2),6;0,5)$	$(0,8,9) \cup (2,5)$
	$(1,7,5(3),0;2,4)$	
	$(1,8,3(0),9;5,7)$	
	$(4,8,2(6),9;0,1)$	
	$(4,9,6(3),8;5,7)$	

Table 10 continued

n	*Packing*	*Remains* for *Covering*
11	(0,1,2(3),4;5,6)	
	(0,7,3(4),1;5,6)	
	(1,4,7(2),6;0,5)	(2,10)
	(1,7,5(3),0;2,4)	
	(2,5,10(4),3;8,9)	
	(3,0,10(1),9;5,8)	
	(4,8,2(6),9;0,1)	
	(4,9,6(3),8;5,7)	
	(9,7,10(6),8;0,1)	
12,13	T_5-decomposition (Theorem 3.1)	
14	(0,1,2(3),4;5,6)	
	(0,3,13(6),10;1,9)	
	(0,7,3(4),1;5,6)	
	(0,9,13(1),2;6,8)	
	(0,10,11(6),9;5,8)	
	(1,4,7(2),6;0,5)	(2,10)
	(1,7,5(3),0;2,4)	
	(2,5,10(4),3;8,9)	
	(4,9,6(3),8;5,7)	
	(8,0,12(5),10;6,7)	
	(9,7,11(5),8;1,10)	
	(10,2,11(3),12;6,7)	
	(11,1,12(2),13;4,5)	
	(12,4,11(0),13;7,8)	
$12m$, $12m+1$		
$12m+4$, $12m+9$	T_5-decomposition (Theorem 3.1)	

We have to prove the theorem in Ihe cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (2).

Then, using Theorem 3.1, Corollary 5, and Table 10, for n = 14 we have the packing and the covering as well.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 5 and Theorem 3.1 we have that $T_5 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m$, $12m+1$, $12m+2$. Take some T_5 from the decomposition of $K_{3,12m}$, say, $(0, 12m, 1(12m+1), 12m+2; 2, 3)$. Then together with the non-packed triangle we consider the tree: $(12m+1, 1, 12m(0), 12m+2; 2, 3)$, leaving the path $(1, 12m+2, 12m+1, 12m)$ unpacked, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

We have by Theorem 3.1 and Corollary 5 that $T_5 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0, 1, \dots, 4$.

Take a tree from the decomposition of $K_{5,12m}$, say, $(0, 12m, 1(12m+1), 12m+2; 2, 3)$, and with the non-packed K_5 create the following trees:

$(0, 12m, 1(12m+1), 12m+2; 12m+3, 12m+4)$ and
$(12m+4, 12m+3, 12m+1(12m), 12m+2; 2, 3)$.
Hence, the packing is completed and we are left with
$(12m+1, 12m+4, 12m; 12m+2, 12m+3)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1 and Corollary 5, $T_5 \mid K_{12m} \cup K_{6,12m}$.

Denote the vertices of K_6 by, $\{12m+j\}$, $j = 0, 1, \dots, 5$.

Take some tree from the T_5-decomposition of $K_{6,12m}$, say, $(0, 12m, 1(12m+1), 12m+2; 2, 3)$.

Using that tree and the unpacked graph K_6 we create the following trees T_5:

$(12m+4, 12m, 1(12m+1), 12m+2; 12m+3, 12m+5)$;
$(12m+5, 12m, 12m+1(12m+4), 12m+2; 2, 3)$; and
$(12m+5; 12m+4, 12m+3(12m+1), 12m; 0, 12m+2)$.
We are left with the non-packed graph:
$(12m+1, 12m+5, 12m+3) \cup (12m+2, 12m+4)$.

k = 7, 8, 10, 11.

Let K_{12m+k} be as in (6).

From Theorem 3.1 and Corollary 5 we have that $T_5 \mid K_{12m} \cup K_{k,12m}$. In Table 10 we have the packing and covering of K_k. Hence, the proof is completed. ∎

Lemma 3.8. $P(7, T_6) = 3, C(7, T_6) = 5$.

Proof: The proof is easy by straight forward verification. ∎

Theorem 3.9. *The Main Theorem is valid for T_6 for all $n \geq 8$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 11

n	*Packing*	*Remains* for *Covering*
8	$(1,2;0,3,4;5,6)$	
	$(2,4;1,3,6;0,7)$	
	$(3,4;7,1,1;2,5)$	$(2,5,3) \cup (4,0,7)$
	$(0,1;5,7,2;3,4)$	
9	T_6-decomposition (Theorem 3.1)	
10	$(1,2;0,3,4;5,6)$	
	$(2,4;1,3,6;0,7)$	
	$(3,4;7,1,1;2,5)$	$(9;0,5,8)$
	$(0,1;5,7,2;3,4)$	
	$(1,2;8,3,9;4,5)$	
	$(1,2;9,7,8;4,5)$	
	$(2,3;5,8,0;4,7)$	
11	$(1,2;0,3,4;5,6)$	
	$(2,4;1,3,6;0,7)$	
	$(3,4;7,1,1;2,5)$	$(0,10)$
	$(0,1;5,7,2;3,4)$	
	$(1,2;8,3,9;4,5)$	
	$(1,2;9,7,8;4,5)$	
	$(1,2;10,9,8;0,5)$	
	$(2,9;5,3,10;6,7)$	
	$(5,8;10,4,0;7,9)$	
$12m, 12m+1$		
$12m+4, 12m+9$	T_6-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (8). Then, using Corollary 1 and Theorem 3.1, we have the packing number leaving K_2 in (8) unpacked for the covering.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 1 and Theorem 3.1 we have that $T_6 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m, 12m+1, 12m+2$. Take some T_6 from the decomposition of $K_{3,12m}$, say, $(0,1; 12m, 2, 12m+1; 3, 4)$. Replace the edge $(12m, 0)$ by $(12m, 12m+2)$, so that we are left with the path $(12m+2, 12m+1, 12m, 0)$ unpacked, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

Since $K_{5,12m} = 2m(K_{2,6} \cup K_{3,6})$ we have by Theorem 3.1 and Corollary 1 that $T_6 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0, 1, \dots, 4$. Take a tree from the decomposition of $K_{5,12m}$, say, $(0,1; 12m, 2, 12m+1; 3, 4)$, with the non-packed K_5 create the following trees:

$$(0,1; 12m, 2, 12m+1; 12m+2, 12m+3) \text{ and}$$
$$(3,4; 12m+1, 12m, 12m+4; 12m+2, 12m+3).$$

Now take the tree $(0,1; 12m+2, 2, 12m+3; 3, 4)$ and replace the edge $(0, 12m+2)$ by the edge $(12m, 12m+2)$. Hence, the packing is completed and we are left with the non-packed graph $(12m+4, 12m+1) \cup (0, 12m+2, 12m+3, 12m)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1 and Corollary 1, $T_6 \mid K_{12m} \cup K_{6,12m}$. Denote the vertices of K_6 by, $\{12m+j\}, j = 1, 2, \dots, 5$.

Take some trees from the T_6-decomposition of $K_{6,12m}$, say, $(0,1; 12m, 2, 12m+1; 3, 4), (0,1; 12m+1, 5, 12m+2; 3, 4)$ and together with the non-packed K_6 create the trees:

$(0,1; 12m, 2, 12m+1; 12m+3, 12m+4)$;
$(0,1; 12m, 12m+2, 12m+1; 3, 4)$; and
$(0,1; 12m+1, 5, 12m+2; 12m+3, 12m+4)$.
We are left with the path:
$(12m, 12m+1, 12m+5, 12m+4)$ for the covering.

k = 7.

Let K_{12m+7} be as in (6) for $k = 7$.

Using Theorem 3.1 and Corollary 1, $T_6 \mid K_{12m} \cup K_{7,12m}$.

Take as in the case $k = 6$. Then the unpacked path $(12m, 12m+1, 12m+5, 12m+4)$ with the star $(12m+6; 12m, 12m+1, 12m+2, 12m+3, 12m+4, 12m+5)$ and the tree $(0,1; 12m+5, 2, 12m+3; 4, 5)$ create the following trees:

$(0,1; 12m+5, 12m+1, 12m+6; 12m, 12m+2)$; and
$(4,5; 12m+3, 2, 12m+5; 12m+4, 12m+6)$.
We are left with the non-packed graph:
$(12m+3, 12m+6, 12m+4) \cup (12m, 12m+1)$ for the covering.

k = 8, 10, 11.

Let K_{12m+k} be as in (6). We have from Theorem 3.1, and Corollary 1, that $T_6 \mid K_{12m} \cup K_{12m,k}$. From Table 11 we have the packing and covering of K_k. Hence, the proof of Main Theorem for T_6 is completed. ∎

Theorem 3.10. *The Main Theorem is valid for T_7 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 12

n	*Packing*	*Remains* for *Covering*
7	(5,1,0;2,3,4−6)	
	(5,4,2;3,1,6−0)	(4;1,3,6)
	(1,6,5;0,2,3−4)	
8	(5,1,0;2,3,4−6)	
	(5,4,2;3,1,6−0)	
	(1,6,5;0,2,3−4)	(7;0,4,6) ∪ (1,3)
	(4,1,7;2,5,3−6)	
9	T_7-decomposition (Theorem 3.1)	
10	(5,1,0;2,3,4−6)	
	(5,4,2;3,1,6−0)	
	(1,6,5;0,2,3−4)	
	(4,1,7;2,5,3−6)	(8;0,4) ∪ (9,2)
	(0,7,8;5,6,3−1)	
	(4,7,9;0,5,8−2)	
	(7,6,9;3,4,1−8)	
11	(5,1,0;2,3,4−6)	
	(5,4,2;3,1,6−0)	
	(1,6,5;0,2,3−4)	
	(4,1,7;2,5,3−6)	
	(0,7,8;5,6,3−10)	(2,10)
	(4,7,9;0,5,8−2)	
	(7,6,9;3,4,1−8)	
	(0,8,10;6,7,9−2)	
	(3,1,10;0,5,4−8)	
12,13	T_7-decomposition (Theorem 3.1)	
14	(5,1,0;2,3,4−6)	
	(5,4,2;3,1,6−0)	
	(1,6,5;0,2,3−4)	
	(4,1,7;2,5,3−6)	

Table 12 continued

n	*Packing*	*Remains* for *Covering*
	$(0,7,8;5,6,3-10)$	
	$(4,7,9;0,5,8-2)$	
	$(7,6,9;3,4,1-8)$	
	$(0,8,10;6,7,9-2)$	$(5,12)$
	$(3,1,10;0,5,4-8)$	
	$(11,8,12;9,10,13-3)$	
	$(12,3,11;0,1,4-13)$	
	$(12,1,13;5,7,10-11)$	
	$(10,2,13;8,9,11-12)$	
	$(13,0,12;3,7,2-11)$	
	$(12,5,11;7,9,6-13)$	
$12m, 12m+1$		
$12m+4, 12m+9$	T_7-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (2).

Then, using Theorem 3.1, Corollary 2, and Table 12 for $n = 14$, we have the packing and the covering as well.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 2 and Theorem 3.1 we have that $T_7 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m$, $12m+1$, $12m+2$. Take some T_7 from the decomposition of $K_{3,12m}$, say, $(12m+1, 2, 12m; 0, 1, 3-12m+2)$. Instead of the edge $(12m, 3)$ put $(12m, 12m+2)$, so that we are left with the path $(12m+2, 12m+1, 12m, 3)$ unpacked, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

We have by Theorem 3.1 and Corollary 2 that $T_7 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0, 1, \ldots, 4$.

Take a tree from the decomposition of $K_{5,12m}$, say, $(12m+1, 2, 12m; 0, 1, 3 - 12m+2)$ and with the non-packed K_5 create the following trees:

$$(2, 12m+1, 12m; 0, 1, 12m+3 - 12m+2) \text{ and}$$

$$(2, 12m, 12m+4; 12m+1, 12m+3, 12m+2 - 3).$$

Hence, the packing is completed and we are left with $(3, 12m, 12m+2, 12m+1, 12m+3)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1 and Corollary 2, $T_7 \mid K_{12m} \cup K_{6,12m}$. Denote the vertices of K_6 by, $\{12m+j\}, j = 0, 1, \ldots, 5$.

Take some tree from the T_7-decomposition of $K_{6,12m}$, say, $(12m+1, 2, 12m; 0, 1, 3 - 12m+2)$.

Using that tree and the unpacked graph K_6 we create the following trees T_7:

$(12m+1, 2, 12m; 12m+3, 12m+4, 3 - 12m+2)$;
$(0, 12m, 12m+5; 12m+1, 12m+2, 12m+3 - 12m+4)$ and
$(1, 12m, 12m+2; 12m+1, 12m+3, 12m+4 - 12m+5)$.
We are left with the non-packed star:
$(12m+1; 12m, 12m+3, 12m+4)$.

k = 7, 8, 10, 11.

Let K_{12m+k} be as in (6).

From Theorem 3.1 and Corollary 2, we have that $T_7 \mid K_{12m} \cup K_{k,12m}$. In Table 12 we find the packing and covering of K_k. Hence, the proof is completed. ∎

Theorem 3.11. *The Main Theorem is valid for T_8 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 13

n	*Packing*	*Remains* for *Covering*
7	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(2,0,5,3)
	(4,0,6;1(3),5(2))	
8	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(2,0,5,6) ∪ (3,7)
	(3,5,7;4(0),6(1))	
	(6,0,7;1(3),5(2))	
9	T_8-decomposition (Theorem 3.1)	
10	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(1,3) ∪ (2,7) ∪ (6,8)
	(2,5,8;9(3),7(0))	
	(3,5,7;4(0),6(1))	
	(8,1,9;0(2),5(6))	
	(8,2,9;6(0),7(1))	
	(9,4,8;0(5),3(7))	

11	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(4,10)
	(2,5,8;9(3),7(0))	
	(3,5,7;4(0),6(1))	
	(7,2,10;3(8),6(9))	
	(8,0,10;1(3),9(4))	
	(8,1,9;0(2),5(6))	
	(8,2,9;6(0),7(1))	
12,13	T_8-decomposition (Theorem 3.1)	
14	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(9,12)
	(2,5,11;8(3),7(0))	
	(3,5,7;4(0),6(1))	
	(5,8,9;12(6),13(11))	
	(8,1,9;0(2),5(6))	
	(8,2,9;6(0),7(1))	
	(11,2,12;3(1),4(10))	
	(11,4,13;2(10),3(9))	
	(11,12,13;1(10),7(2))	
	(12,1,11;3(10),9(6))	
	(12,5,13;8(0),10(9))	
	(13,0,12;8(7),10(6))	
	(13,6,11;0(10),9(4))	
15	(0,1,2;3(6),4(5))	
	(0,3,4;1(5),6(2))	(8,13,10,14)
	(0,11,1;12(2),13(3))	
	(0,12,5;13(2),14(1))	
	(0,13,6;11(5),14(2))	
	(0,14,3;11(2),12(6))	
	(2,5,8;9(3),7(0))	
	(3,5,7;4(0),6(1))	
	(4,11,7;12(8),13(9))	
	(7,2,10;3(8),6(9))	
	(8,0,10;1(3),9(4))	
	(8,1,9;0(2),5(6))	
	(8,2,9;6(0),7(1))	
	(8,11,13;12(10),14(7))	
	(10,4,12;11(9),14(8))	
	(10,11,14;4(13),9(12))	
$12m$, $12m+1$		
$12m+4$, $12m+9$	T_8-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (2).

Then, using Theorem 3.1, Corollary 5, and Table 13 for $n = 14$, we have the packing and the covering as well.

k = 3.

Let K_{12m+3} be as in (3).

Using Corollary 5, Theorem 3.1, and Table 13 for $n = 15$, we have the packing and the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

We have by Theorem 3.1 and Corollary 5 that $T_8 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m + j\}, j = 0, 1, \ldots, 4$.

Take a tree from the decomposition of $K_{5,12m}$, say, $(0, 12m, 1; 12m+1(2), 12m+2(3))$, and with the non-packed K_5 create the following trees:

$$(0, 12m, 12m+2; 12m+1(1), 12m+3(12m+4)) \text{ and}$$
$$(1, 12m, 12m+4; 12m+1(12m+3), 12m+2(3)).$$

Hence, the packing is completed and we are left with $(2, 12m+1, 12m, 12m+3) \cup (1, 12m+2)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1 and Corollary 5, $T_8 \mid K_{12m} \cup K_{6,12m}$. Denote the vertices of K_6 by, $\{12m + j\}, j = 0, 1, \ldots, 5$.

We do the same procedure as for $k = 5$ with the additional tree $(1, 12m+2, 12m+5; 12m+1(2), 12m+3(12m))$. We are left with the non-packed path $(12m+1, 12m, 12m+5, 12m+4)$.

k = 7, 8, 10, 11.

Let K_{12m+k} be as in (6).

From Theorem 3.1 and Corollary 5, we have that $T_8 \mid K_{12m} \cup K_{k,12m}$. In Table 13 we find the packing and covering of K_k. Hence, the proof is completed. ∎

Theorem 3.12. *The Main Theorem is valid for T_9 for all $n \geq 7$.*

Proof: The proof will take case of several cases according to the various values of n:

Table 14

n	*Packing*	*Remains* for *Covering*
7	(0,2,3; 1,4; 5,6), (2,3,4; 0,5; 1,6)	
	(3,4,5; 2,6; 0,1)	(3; 4,5,6)
8	(0,2,3; 1,4; 5,6), (2,3,4; 0,5; 1,6)	
	(3,4,5; 2,6; 0,1), (4,5,6; 3,7; 0,1)	(7; 2,4,5,6)
9	T_9-decomposition (Theorem 3.1)	
10	(0,2,3; 1,4; 5,6), (2,3,4; 0,5; 1,6)	
	(3,4,5; 2,6; 0,1), (4,5,6; 3,7; 0,1)	(5,6,9) ∪ (1,8)
	(2,3,4; 9,7; 5,6), (0,5,6; 8,7; 2,4)	
	(2,3,4; 8,9; 0,1)	
11	(0,2,3; 1,4; 5,6), (2,3,4; 0,5; 6,10)	
	(3,4,5; 2,6; 0,1), (4,5,6; 3,7; 0,1)	(8,10)
	(2,3,4; 9,7; 5,6), (0,5,6; 8,7; 2,4)	
	(2,3,4; 8,9; 0,1), (2,3,4; 10,9; 5,6)	
	(0,6,7; 10,1; 5,8)	
12,13	T_9-decomposition (Theorem 3.1)	
14	(0,2,3; 1,4; 5,6), (2,3,4; 0,5; 6,10)	
	(3,4,5; 2,6; 0,1), (4,5,6; 3,7; 0,1)	(9,12)
	(2,3,4; 9,7; 5,6), (0,5,6; 8,7; 2,4)	
	(2,3,4; 8,9; 0,1), (2,3,4; 10,9; 5,6)	
	(0,6,7; 10,1; 5,8), (0,1,2; 11,3; 12,13)	
	(4,5,6; 11,12; 1,2), (4,5,6; 13,12; 0,7)	
	(4,5,6; 12,8; 11,13), (7,9,10; 11,13; 0,1)	
	(2,7,9; 13,10; 8,12)	
$12m, 12m+1$		
$12m+4, 12m+9$	T_9-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (2).

Then, using Theorem 3.1, Corollary 4, and Table 14 for $n = 14$, we have the packing and the covering as well.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 4 and Theorem 3.1 we have that $T_9 \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (3) are labelled $12m$, $12m+1$, $12m+2$. Take some T_9 from the decomposition of $K_{3,12m}$, say, $(0,1,2;\ 12m,3;\ 12m+1,12m+2)$ and $(0,1,2;\ 12m+2,4;\ 12m,12m+1)$ and together with the non-packed triangle create the trees $(1,2,4;\ 12m,3;\ 12m+1,12m+2)$ and $(0,1,2;\ 12m+2,12m+1;\ 12m,4)$ leaving the path $(4,12m+2,12m,0)$ non-packed, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

We have by Theorem 3.1 and Corollary 4 that $T_9 \mid K_{12m} \cup K_{5,12m}$. Denote the vertices of K_5 by: $\{12m+j\}, j = 0,1,\ldots,4$.

Take two trees from the decomposition of $K_{5,12m}$, say, $(1,2,4;\ 12m+2,0;\ 12m+2,12m+4)$ and $(1,2,3;\ 12m+3,4;\ 12m+1,12m+3)$ and with the non-packed K_5 create the following trees:

$$(1,2,12m+2;\ 12m+3,12m;\ 12m+1,12m+4);$$
$$(12m+1,12m+3,12m+4;\ 12m+3,4;\ 12m,12m+2);\ \text{and}$$
$$(2,4,12m+4;\ 12m+2,0;\ 12m+2,12m+4).$$

Hence, the packing is completed and we are left with $(1,12m;\ 12m+2,12m+1;\ 12m+4)$ for the covering.

k = 6.

Let K_{12m+6} be as in (5). Using Theorem 3.1 and Corollary 4, $T_9 \mid K_{12m} \cup K_{6,12m}$. Denote the vertices of K_6 by, $\{12m+j\}, j = 0,1,\ldots,5$.

We proceed the case $k = 5$ so that with the non-packed graph left together with the star $(12m+5;\ 12m,12m+1,12m+2,12m+3,12m+4)$ we get the additional tree $(12m,12m+3,12m+4;\ 12m+5,12m+2;\ 1,12m+1)$.

We are left with the non-packed graph $(12m+4,12m+1,12m+5) \cup (12m,12m+2)$.

k = 7, 8, 10, 11.

Let K_{12m+k} be as in (6).

From Theorem 3.1 and Corollary 4, we have that $T_9 \mid K_{12m} \cup K_{k,12m}$. In Table 14 we find the packing and covering of K_k. Hence, the proof is completed. ∎

Lemma 3.13.

(i) $P(7,T_{10}) = 2, C(7,T_{10}) = 4$.
(ii) $P(8,T_{10}) = 3, C(8,T_{10}) = 5$.

Proof: Since $\Delta(T_{10}) = 5$ one can easily check that $P(7,T_{10}) = 2$, and $P(8,T_{10}) = 3$. The covering, hence, is obvious. ∎

Theorem 3.14. *The Main Theorem is valid for* T_{10} *for all* $n \geq 9$.

Proof: The proof will take case of several cases according to the various values of n:

Table 15

n	*Packing*	*Remains for Covering*
9	T_{10}-decomposition (Theorem 3.1)	
10	(0;4,5,6,7,1−2)	
	(2;4,5,6,7,3−1)	
	(3;4,5,6,7,0−2)	
	(1;4,5,8,9,6−7)	(1;2;3) ∪ (0,9)
	(7;1,4,5,9,8−0)	
	(8;2,3,4,9,6−5)	
	(9;2,3,4,6,5−8)	
11	(0;4,5,6,7,1−2)	
	(2;4,5,6,7,3−1)	
	(3;4,5,6,7,0−2)	
	(1;4,5,8,9,6−7)	
	(7;1,4,5,9,8−0)	(1,3)
	(8;2,3,4,9,6−5)	
	(9;2,3,4,6,5−8)	
	(10;3,4,5,6,0−9)	
	(10;2,7,8,9,1−3)	
$12m$, $12m+1$		
$12m+4$, $12m+9$	T_{10}-decomposition (Theorem 3.1)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (8).

Using Corollary 1 and Theorem 3.1, we have the packing number leaving K_2 in (8) unpacked for the covering.

k = 3.

Let K_{12m+3} be as in (7).

Using Corollary 1 and Theorem 3.1 we have that $T_{10} \mid K_{12m} \cup K_{3,12m}$.

Suppose the vertices of K_3 in (7) are labelled $12m$, $12m+1$, $12m+2$. Take some T_{10} from the decomposition of $K_{3,12m}$, say, $(12m; 0,1,2,3,4-12m+1)$. Instead of the edge $(12m,0)$ put $(12m, 12m+2)$, so that we are left with the path $(12m+2, 12m+1, 12m, 0)$ unpacked, a fact which proves the covering as well.

k = 5.

Let K_{12m+5} be as in (4).

Since $K_{5,12m} = 2m(K_{2,6} \cup K_{3,6})$ we have by Theorem 3.1 and Corollary 1 that $T_{10} \mid K_{12m} \cup K_{5,12m}$.

Denote the vertices of K_5 by: $\{12m + j\}, j = 0, 1, \dots, 4$.

Take one $K_{2,6}$ from $2m$, say, with vertex sets $\{12m, 12m+1\}$ and $\{0, 1, 2, 3, 4, 5\}$. We have two trees T_{10} in the decomposition:

$$(12m; 0, 1, 2, 3, 4 - 12m + 1) \text{ and } (12m + 1; 0, 1, 2, 3, 5 - 12m).$$

Using these trees and the unpacked K_5 we create three new trees T_{10}:

$(12m; 0, 1, 2, 12m + 4, 12m + 2 - 12m + 3)$;
$(12m; 3, 4, 5, 12m + 3, 12m + 1 - 1)$; and
$(12m + 1; 2, 3, 5, 12m + 2, 12m + 4 - 12m + 3)$.
We are left with the unpacked graph,
$(12m + 1; 0, 4, 12m + 3) \cup (12m + 2, 12m + 4)$,
which can be covered by one tree T_{10}.

Hence, the covering is also proved.

k = 6.

Let K_{12m+6} be as in (9). Using Theorem 3.1 and Corollary 1, $T_{10} \mid K_{12m+4} \cup K_{2,12m}$.

Denote the vertices of $K_{2,4}$ by, $\{12m, 12m+1\}$ and $\{12m+2, 12m+3, 12m+4, 12m+5\}$.

Take some tree from the T_{10}-decomposition of $K_{2,12m}$, say, $(12m; 0, 1, 2, 3, 4 - 12m + 1)$.

Using that tree and the unpacked graph $K_{2,4} \cup K_2$ we create the following trees T_{10}:

$(12m; 0, 1, 2, 3, 12m + 1 - 12m + 2)$;
$(12m; 12m + 2, 12m + 3, 12m + 4, 12m + 5, 4 - 12m + 1)$.
We are left with the star: $(12m + 1; 12m + 3, 12m + 4, 12m + 5)$ unpacked.

Hence, the covering is proved as well.

k = 7.

Let K_{12m+7} be as in (10).

Using Theorem 3.1 and Corollary 1, $T_{10} \mid K_{12m+4} \cup K_{3,12m}$.

Denote the vertices of $K_{3,4}$ by $\{12m, 12m+1, 12m+2\}$ and $\{12m+3, 12m+4, 12m+5, 12m+6\}$. Take two trees from the T_{10}-decomposition of $K_{3,12m}$, say, $(12m; 0, 1, 2, 3, 4 - 12m + 1)$ and $(12m + 2; 0, 1, 2, 3, 5 - 12m + 1)$.

Using those trees and the unpacked graph $K_{3,4} \cup K_3$, we create the following trees:

$(12m; 0, 1, 2, 3, 12m+1 - 12m+3)$;
$(12m; 12m+3, 12m+4, 12m+5, 12m+6, 4 - 12m+1)$;
$(12m+2; 0, 1, 2, 3, 12m+1 - 12m+4)$;
$(12m+2; 12m+3, 12m+4, 12m+5, 12m+6, 5 - 12m+1)$.
So the packing is completed. We are left with the unpacked graph:
$(12m+1; 12m+5, 12m+6) \cup (12m, 12m+2)$,
which proves the covering as well.

k = 8, 10, 11.

Let K_{12m+k} be as in (6).

We have from Theorem 3.1 and Corollary 1 that $T_{10} \mid K_{12m} \cup K_{12m,k}$. From Table 15 we have the packing and covering of K_k. Hence, the proof of Main Theorem for T_{10} is completed. ∎

Lemma 3.15.

(i) $P(7, T_{11}) = 1, C(7, T_{11}) = 6$,
(ii) $P(8, T_{11}) = 3, C(8, T_{11}) = 7$,
(iii) $P(9, T_{11}) = 5, C(9, T_{11}) = 8$,
(iv) $P(10, T_{11}) = 7, C(10, T_{11}) = 9$.

Proof: The proof of (i) - (iii) is easy and follows immediately. It is easy to see that $P(10, T_{11}) = 7$. In order to see that $C(10, T_{11}) = 9$ observe that the seven stars T_{11} in the packing of K_{10} leave three vertices, say, x, y, z, which are not the centers of any star. So that the triangle (x, y, z) is left non-packed. Hence, $C(10, T_{11}) = 9$. ∎

The following is obvious:

Lemma 3.16. $T_{11} \mid K_{s,12t}$, *for all* s, t *positive integers.* ∎

Theorem 3.17. *The Main Theorem is valid for* T_{11} *for all* $n \geq 11$.

Proof: The proof will take case of several cases according to the various values of n:

Table 16

n	*Packing*	*Remains* for *Covering*
11	(0; 5, 6, 7, 8, 9, 10), (1; 0, 2, 3, 8, 9, 10)	
	(2; 0, 3, 7, 8, 9, 10), (3; 0, 4, 5, 8, 9, 10)	
	(4; 0, 1, 2, 5, 8, 9), (5; 1, 2, 6, 7, 8, 9)	(8, 9)
	(6; 1, 2, 3, 4, 8, 9), (7; 1, 3, 4, 6, 8, 9)	
	(10; 4, 5, 6, 7, 8, 9)	
17	(0; 5, 6, 7, 8, 9, 10), (1; 0, 2, 3, 8, 9, 10)	
	(2; 0, 3, 7, 8, 9, 10), (3; 0, 4, 5, 8, 9, 10)	
	(4; 0, 1, 2, 5, 8, 9), (5; 1, 2, 6, 7, 8, 9)	
	(6; 1, 2, 3, 4, 8, 9), (7; 1, 3, 4, 6, 8, 9)	
	(9; 8, 12, 13, 15, 16), (10; 4, 5, 6, 7, 8, 9)	
	(11; 0, 1, 2, 3, 4, 5), (11; 6, 7, 8, 10, 12, 13)	(11; 9, 14, 15, 16)
	(12; 0, 1, 2, 3, 4, 5), (12; 6, 7, 8, 10, 13, 14)	
	(13; 0, 1, 2, 3, 4, 5), (13; 6, 7, 8, 10, 14, 15)	
	(14; 0, 1, 2, 3, 4, 5), (14; 6, 7, 8, 10, 15, 16)	
	(15; 0, 1, 2, 3, 4, 5), (15; 6, 7, 8, 10, 12, 16)	
	(16; 0, 1, 2, 3, 4, 5), (15; 6, 7, 8, 10, 12, 13)	

We have to prove the theorem in the cases of (1) for $m \geq 1$.

k = 2.

Let K_{12m+2} be as in (8). Then, using Theorem 3.1 and Lemma 3.16 we have the packing, leaving the edge K_2 non-packed, for the covering.

k = 3.

Let K_{12m+3} be as in (7). Then by Theorem 3.1 and Lemma 3.16 we have that $T_{11} \mid K_{12m} \cup K_{3,12m}$. Now take three stars T_{11} of that decomposition, say, $(12m; 0,1,2,3,4,5), (12m+1; 0,1,2,3,4,5), (12m+2; 0,1,2,3,4,5)$. In the first star we replace the edge $(12m, 0)$ by $(12m, 12m+1)$, in the second star we replace the edge $(12m+1, 0)$ by $(12m+1, 12m+2)$, and in the third star we replace the edge $(12m+2, 0)$ by the edge $(12m+2, 12m)$. Hence, the packing was not changed and we are left with the non-packed star $(0; 12m, 12m+1, 12m+2)$ for the covering.

k = 5.

Let $K_{12m+5} = K_{12(m-1)} \cup K_{17,12(m-1)} \cup K_{17}$, $m \geq 2$.

From Theorem 3.1, Lemma 3.11, and Table 16 for $n = 17$, we have the packing and the covering, as well, in this case.

k = 6.

Let, K_{12m+6} be as in (5). Using Theorem 3.1 and Lemma 3.16 we have, $T_{11} \mid K_{12m} \cup K_{6,12m}$

Denote the vertices of K_6 by, $\{12m + j\}, j = 0, 1, \ldots, 5$.

We take some stars of that decomposition and change some of their edges creating new stars in a way that the packing is not changed but leaving the necessary non-packed star for the covering. The arrow denotes the new star obtained from the old one.

$$(12m; 0,1,2,3,4,5) \rightarrow (12m; 12m+1,1,2,3,4,5)$$
$$(12m+1; 0,1,2,3,4,5) \rightarrow (12m+1; 12m+2,1,2,3,4,5)$$
$$(12m+2; 0,1,2,3,4,5) \rightarrow (12m+2; 12m,1,2,3,4,5)$$
$$(12m+3; 0,1,2,3,4,5) \rightarrow (12m+3; 12m,1,2,3,4,5)$$
$$(12m+4; 0,1,2,3,4,5) \rightarrow (12m+4; 12m,1,2,3,4,5)$$
$$(12m+5; 0,1,2,3,4,5) \rightarrow (12m+5; 12m+2,1,2,3,4,5)$$
$$(12m; 6,7,8,9,10,11) \rightarrow (12m; 12m+5,7,8,9,10,11)$$
$$(12m+1; 6,7,8,9,10,11) \rightarrow (12m+1; 12m+3,7,8,9,10,11)$$
$$(12m+2; 6,7,8,9,10,11) \rightarrow (12m+2; 12m+3,7,8,9,10,11)$$
$$(12m+3; 6,7,8,9,10,11) \rightarrow (12m+3; 12m+5,7,8,9,10,11)$$
$$(12m+4; 6,7,8,9,10,11) \rightarrow (12m+4; 12m+1,7,8,9,10,11)$$
$$(12m+5; 6,7,8,9,10,11) \rightarrow (12m+5; 12m+1,7,8,9,10,11)$$
$$(0; 12m, 12m+1, 12m+2, 12m+3, 12m+4, 12m+5)$$
$$(6; 12m, 12m+1, 12m+2, 12m+3, 12m+4, 12m+5)$$

We are left with the non-packed star $(12m+4; 12m+2, 12m+3, 12m+5)$, for the covering.

k = 7.

Let K_{12m+7} be as in (6). From Theorem 3.1 and Lemma 3.16 we have $T_{11} \mid K_{12m} \cup K_{7,12m}$. Denote the vertices of K_7 by $\{12m+j\}, j = 0, 1, \ldots, 6$. Take the star $(12m; 12m+1, 12m+2, 12m+3, 12m+4, 12m+5, 12m+6)$, so that we are left with a non-packed K_6. From here we continue as in the case $k = 6$.

k = 8.

Let K_{12m+8} be as in (6). From theorem 3.1 and Lemma 3.16 we have $T_{11} \mid K_{12m} \cup K_{8,12m}$. Denote the vertices of K_8 by $\{12m+j\}, j = 0, 1, \ldots, 7$. Take the stars $(12m; 12m+1, 12m+2, 12m+3, 12m+4, 12m+5, 12m+6)$, $(12m+7; 12m, 12m+1, 12m+2, 12m+3, 12m+4, 12m+5)$, and $(12m+6; 12m+$

$7, 12m+1, 12m+2, 12m+3, 12m+4, 12m+5)$, so that we are left with a non-packed K_5. From here we continue as in the case $k = 5$, where for $m = 1$ we have K_{17} and its packing and covering as in Table 11.

k = 10.

Let K_{12m+10} be as in (6). From Theorem 3.1 and Lemma 3.16 we have that $T_{11} \mid K_{12m} \cup K_{10,12m}$. In addition we pack K_{10} with the stars:

$(12m; 12m+4, 12m+5, 12m+6, 12m+7, 12m+8, 12m+9)$
$(12m+1; 12m, 12m+2, 12m+3, 12m+7, 12m+8, 12m+9)$
$(12m+2; 12m, 12m+3, 12m+6, 12m+7, 12m+8, 12m+9)$
$(12m+3; 12m, 12m+5, 12m+6, 12m+7, 12m+8, 12m+9)$
$(12m+4; 12m+1, 12m+2, 12m+3, 12m+7, 12m+8, 12m+9)$
$(12m+5; 12m+1, 12m+2, 12m+4, 12m+7, 12m+8, 12m+9)$
$(12m+6; 12m+1, 12m+4, 12m+5, 12m+7, 12m+8, 12m+9)$

We are left with the non-packed triangle $(12m+7, 12m+8, 12m+9)$. We take three stars T_{11} from the above decomposition, say, $(12m+7; 0, 1, 2, 3, 4, 5)$, $(12m+8; 0, 1, 2, 3, 4, 5)$ and $(12m+9; 0, 1, 2, 3, 4, 5)$. We replace the edges $(12m+7, 0)$ by $(12m+7, 12m+8)$, $(12m+8, 0)$ by $(12m+8, 12m+9)$, $(12m+9, 0)$ by $(12m+9, 12m+7)$, so that we are left with the non-packed star $(0; 12m+7, 12m+8, 12m+9)$ for the covering.

k = 11.

Let K_{12m+11} be as in (6). From Theorem 3.1 and Lemma 3.16 we have that $T_{11} \mid K_{12m} \cup K_{11,12m}$. In Table 16 we find the packing and the covering of K_{11}. Hence, the proof is completed. ∎

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