Cyclic and Rotational Oriented Triple Systems[1]

Biagio Micale and Mario Pennisi

Dipartimento di Matematica
Università di Catania
Viale A. Doria 5
95125 Catania
Italy

Abstract. An *oriented* (or *ordered*) triple means either a Mendelsohn or a transitive triple. An *oriented* (or *ordered*) *triple system* of order v, briefly OTS(v), is a pair (V, B), where V is a v-set and B is a collection of oriented triples of elements of V, such that every ordered pair of distinct elements of V belongs to exactly one member of B. It is known that an OTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$. An OTS(v) is *cyclic* if it admits an automorphism consisting of a single cycle of length v; an OTS(v) is *rotational* if it admits an automorphism consisting of a single fixed point and one cycle of length $v-1$. In this note we give some constructions of OTS(v)'s which allow to determine the spectrum of cyclic and of rotational OTS(v)'s.

1. Introduction.

Let V be a finite set. In what follows an *ordered pair* of elements belonging to B will always be an ordered pair (x, y) where $x \neq y$.

A *Mendelsohn triple* on V is a set of three ordered pairs of the form

$$[a, b, c] = \{(a, b), (b, c), (c, a)\};$$

observe that [a,b,c] = [b,c,a] = [c,a,b].

A *transitive triple* on V is a set of three ordered pairs of the form

$$\langle a, b, c\rangle = \{(a, b), (b, c), (a, c)\}.$$

An *oriented triple* on V means either a Mendelsohn or a transitive triple on V.

In [8] such a triple is called *ordered triple*. But we like "oriented triple" better than "ordered triple" to eliminate ambiguity with the usual notion of ordered triple.

An *oriented* (or *ordered*) *triple system* of order v, briefly OTS(v), is a pair (V, B), where V is a v-set and B is a collection of oriented triples on V, called *blocks*, such that every ordered pair of distinct elements of B belongs to exactly one member of B.

In particular, if the triples in B are all Mendelsohn or all transitive then (V, B) is a *Mendelsohn triple system*, MTS(v), or a *transitive triple system*, TTS(v), respectively.

If (V, B) is an OTS(v) then

$$|B| = \frac{v(v-1)}{3}$$

therefore a necessary condition for the existence of OTS(v)'s is $v \equiv 0, 1 \pmod 3$. It is known ([7], [8], [9]) that the spectrum for OTS(v)'s and TTS(v)'s is the set of all $v \equiv 0, 1 \pmod 3$, and for MTS(v)'s is the set of all $v \equiv 0, 1 \pmod 3$, $v \neq 6$.

An OTS(v) is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length v.

In [3] and [5] it is proved that:

Proposition 1. *A cyclic MTS(v) exists if and only if* $v \equiv 1, 3 (mod 6)$, $v \neq 9$; *a cyclic TTS(v) exists if and only if* $v \equiv 1, 4, 7 (mod 12)$.

A *twofold triple system* of order v ([1]) is a pair (V, B), where V is a v-set and B is a collection of 3-subsets of V, with the property that every 2-subset of V appears in exactly two triples.

In [3] it is observed that if one omits the direction in the blocks of an OTS(v) then obtain a twofold triple system of order v and in [4] it is proved that a cyclic twofold triple system of order v exists only if $v \equiv 0,1,3,4,7,9 \pmod{12}$, $v \neq 9$.

From this it follows that if a cyclic OTS(v) exists then $v \equiv 0,1,3,4,7,9 \pmod{12}$, $v \neq 9$. Further from Proposition 1 it follows that for $v \equiv 1,3,4,7,9 \pmod{12}$, $v \neq 9$, a cyclic OTS(v) exists.

In this note we prove that also for $v \equiv 0 \pmod{12}$ a cyclic OTS(v) can be constructed, therefore *the spectrum of cyclic* OTS(v)'s *is the set of all* $v \equiv 0,1,3,4,7,9 \pmod{12}$, $v \neq 9$.

An OTS(v) is said to be *rotational* if it admits an automorphism consisting of a single fixed point and one cycle of length $v-1$.

In [2] it is proved that:

Proposition 2. *A rotational MTS(v) exists if and only if* $v \equiv 1, 3, 4 (mod 6)$, $v \neq 10$.

It is easy to see that a rotational TTS(v) cannot exist. In the second part of this note we prove that for $v = 10$ and for each $v \equiv 0 \pmod 6$ a rotational OTS(v) can be constructed, for that *a rotational* OTS(v) *exists for every* $v \equiv 0, 1 \pmod 3$.

2. Cyclic OTS(v)'s.

First, we give some simple observations on the structure of a cyclic OTS(v).

Using standard representations of cyclic designs as sets of difference blocks, a cyclic OTS(v), (Z_v, B), is equivalent to a partitioning of the set $\{1, 2, \ldots, v-1\}$ into m-difference triples and t-difference triples; an m-difference triple is a triple

$(a,b,c) = (b,c,a) = (c,a,b)$, where $|\{a,b,c\}| = 1$ or 3 and $a+b+c=0$; a t-difference triple is a triple (a,b,c), with $a+b-c=0$. The m-difference triples give the Mendelsohn triples of $(\mathbf{Z}_v, B)$ and the t-difference triples give the transitive triples of $(\mathbf{Z}_v, B)$.

Theorem 1. *If $v \equiv 0 (mod 12)$ then a cyclic OTS(v) exists.*

Proof: First consider the case $v = 12$. Let M be the set of the two m-difference triples (4,4,4) and (8,8,8), and let T be the set of the three t-difference triples (1,10,11), (2,5,7) and (3,6,9). The set $M \cup T$ is a partitioning of $\{1,2,\ldots,11\}$ and, therefore, a cyclic OTS(12) exists.

Now, we study the case $v = 12h$, $h \geq 2$. Consider the following set of m-difference triples:

$$M = \{(4h,4h,4h),(8h,8h,8h)\}.$$

Further, consider the following t-difference triples:

$$d_i = (2i-1, 10h-i+1, 10h+i), \quad i = 1,2,\ldots,h,h+2,h+3,\ldots,2h-1$$
$$d_i' = (2i, 6h-i, 6h+i), \quad i = 1,2,..,2h-1;$$

let

$$T_1 = \{d_i : i = 1,2,..,h,h+2,h+3,..,2h-1\},$$
$$T_2 = \{d_i' : i = 1,2,\ldots,2h-1\}$$

and let

$$T = T_1 \cup T_2 \cup \{(2h+1,4h-1,6h),(9h,11h+1,8h+1)\}.$$

It is easy to verify that $M \cup T$ is a partitioning of the set $\{1,2,\ldots,v-1\}$ and, therefore, a cyclic OTS(v) exists for every $v \equiv 0 \pmod{12}$, $v \geq 24$. ∎

From the previous Theorem it follows that *a cyclic* OTS(v) *exists if and only if* $v \equiv 0,1,3,4,7,9 \pmod{12}$, $v \neq 9$.

3. Rotational OTS(v)'s.

We observe that a rotational OTS(v), $(\mathbf{Z}_{v-1} \cup \{x\}, B)$, where x is the fixed point, exists if and only if there is $z^* \in \mathbf{Z}_{v-1}$ and a partitioning of $\{1,2,\ldots,v-2\} - \{z^*\}$ into m-difference triples and t-difference triples; z^* give all the blocks (Mendelsohn triples) which contain the fixed point x.

For $v = 10$, the m-difference triple (3,3,3) and the two t-difference triples (1,4,5) and (2,6,8) give a partitioning of the set $\{1,2,\ldots,8\} - \{7\}$, hence a rotational OTS(10) exists.

Theorem 2. *If $v \equiv 0(mod 6)$ then a rotational OTS(v) exists.*

Proof: Let $v = 6h$, $h \geq 1$. We consider the following t-difference triples:

$$d_i = (2i-1, 3h-i, 3h+i-1), \qquad i = 1,2,\ldots,h$$
$$d'_i = (2i, 5h-i-1, 5h+i-1), \qquad i = 1,2,\ldots,h-1.$$

Let $T_1 = \{d_i : i = 1,2,\ldots,h\}$, $T_2 = \{d'_i : i = 1,2,\ldots,h-1\}$ and let $T = T_1 \cup T_2$. Observe that if $h = 1$ then $T_2 = \emptyset$.

It is easy to verify that T is a partitioning of the set $\{1,2,\ldots,6h-2\} - \{5h-1\}$ and, therefore, a rotational OTS(v) exists for every $v \equiv 0 \pmod 6$. ∎

From Theorem 2 it follows that *for every $v \equiv 0,1 \pmod 3$ a rotational* OTS(v) *exists.*

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