The Factor-Indices of Complete n-Partite Graphs

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ABSTRACT

Let G be a graph with minimum degree δ. For each $i = 1, 2, \ldots, \delta$, let $\alpha_i(G)$ (resp. $\alpha_i^*(G)$) denote the smallest integer k such that G has an $[i, k]$-factor (resp. a connected $[i, k]$-factor). Denote by G_n a complete n-partite graph. In this paper, we determine the value of $\alpha_i(G_n)$, and show that $0 \leq \alpha_1^*(G_n) - \alpha_1(G_n) \leq 1$ and $\alpha_i^*(G_n) = \alpha_i(G_n)$ for each $i = 2, 3, \ldots, \delta$.

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§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(x)$ the degree of a vertex x of G. Let a and b be integers such that $0 \leq a \leq b$. A graph G is called an $[a, b]$-*graph* if $a \leq d_G(x) \leq b$ for all $x \in V(G)$. A subgraph F of a graph G is called an $[a, b]$-*factor* of G if $V(F) = V(G)$ and $a \leq d_F(x) \leq b$ for all $x \in V(G)$. An $[a, a]$-factor of G is also called an *a-factor* of G.

Let $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ) denote the maximum and minimum degree of G, respectively. We now introduce the following new concepts. For $i = 1, 2, \ldots, \delta$, let $\alpha_i(G)$

(resp., $\alpha_i^*(G)$) denote the smallest integer k such that G has an $[i,k]$-factor (resp., a connected $[i,k]$-factor). We call $\alpha_i(G)$ (resp. $\alpha_i^*(G)$) the i^{th} *factor-index* (resp. the i^{th} *connected factor-index*) of G.
Evidently:

(a) $i \le \alpha_i(G) \le \alpha_i^*(G) \le \Delta$, for each $i = 1, 2, \ldots, \delta$;
(b) $\alpha_i(G) \le \alpha_{i+1}(G)$ and $\alpha_i^*(G) \le \alpha_{i+1}^*(G)$, for each $i = 1, 2, \ldots, \delta - 1$;
(c) a graph G has an i-factor if and only if $\alpha_i(G) = i$;
(d) a graph G is hamiltonian if and only if $\alpha_2^*(G) = 2$; and
(e) a graph G contains a hamiltonian path if and only if $\alpha_1^*(G) = 2$.

Throughout this paper, we write $G_n = K_n(m_1, m_2, \ldots, m_n)$ to denote a complete n-partite graph with n partite sets $V_1, V_2, \ldots, V_n$ such that $|V_i| = m_i \ge 1$, for each $i = 1, 2, \ldots, n$. For the sake of convenience, we assume that

$$m_1 \le m_2 \le \ldots \le m_n.$$

Let p denote the order of G_n. Then we have

$$p = \sum_{i=1}^{n} m_i \quad \text{and} \quad \delta = \delta(G_n) = p - m_n.$$

The problem of determining the exact values of $\alpha_i(G)$ and $\alpha_i^*(G)$ for an arbitrary graph G seems formidable. The purpose of this paper is to determine $\alpha_i(G_n)$ and $\alpha_i^*(G_n)$ of the graph G_n. For a real x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x and $\lceil x \rceil$ denote the smallest integer not less than x. Denote $h = \lfloor (\delta - m_n)/2 \rfloor + 1$ if $\delta > m_n$. Then the main results we obtain may be summarized in Table 1.

We further introduce a special class of G_n for which the exact value of $\alpha_i(G_n)$ can be determined. Finally, we show that $\alpha_i(G_n)$ and $\alpha_i^*(G_n)$ are almost identical in the sense that $0 \le \alpha_1^*(G_n) - \alpha_1(G_n) \le 1$ and $\alpha_i^*(G_n) = \alpha_i(G_n)$, for each $i = 2, 3, \ldots, \delta$.

For terminologies and notation not explained here, we refer to [1].

If	then for $i =$	$\alpha_i(G_n) =$
$\delta \le m_n$	$1, 2, \ldots, \delta$	$\lceil \frac{im_n}{\delta} \rceil$
	$1, 2, \ldots, \lfloor \frac{p+5}{4} \rfloor$	$\begin{cases} i & \text{if } ip \text{ is even} \\ i+1 & \text{if } ip \text{ is odd} \end{cases}$
	$1, 3$	$\begin{cases} i & \text{if } p \text{ is even} \\ i+1 & \text{if } p \text{ is odd} \end{cases}$
	2	2
	4	$\begin{cases} 4 & \text{if } G_n \neq K_3(1,3,3) \\ 6 & \text{if } G_n = K_3(1,3,3) \end{cases}$
$\delta > m_n$	$5, \ldots, h$ (if $h \ge 5$)	$\begin{cases} i & \text{if } ip \text{ is even} \\ i+1 & \text{if } ip \text{ is odd} \end{cases}$
	$h+1$	$\begin{cases} \le h+2 & \text{if } h \text{ is even} \\ = h+1 & \text{if } h \text{ is odd} \end{cases}$
	$h+2, \ldots, \delta - 1$	$\le 2(i - \lceil \frac{h}{2} \rceil)$
	δ	$\le \begin{cases} \delta + 1 & \text{if } m_{n-1} = m_1 \\ \delta + s & \text{if } m_{n-1} > m_1 \end{cases}$
		$s = \left\lceil \frac{(\delta - m_n)(m_{n-1} - m_1)}{\sum_{i=1}^{n-2} m_i} \right\rceil$

Table 1. Factor-Indices of Complete n-partite Graph

§2. Basic Results

We first state the following three results which will play key roles in the subsequent sections.

Theorem A (Dirac [2]). Let s be a positive integer. If G is a graph of order $p \geq 3$ such that $\deg_G x \geq p/2 + s$ for each $x \in V(G)$, then G has $\lfloor s/2 \rfloor + 1$ edge-disjoint hamiltonian cycles.

Theorem B (Katerinis [4]). Let G be a graph of order p and let k be a positive integer such that kp is even. If $\delta(G) \geq p/2$ and $p \geq 4k - 5$, then G has a k-factor.

The following corollary follows immediately from Theorem B.

Corollary. Let G be a graph of order p such that $\delta(G) \geq p/2$. Then for $i = 1, 2, \ldots, \lfloor (p+5)/4 \rfloor$,

$$\alpha_i(G) = \begin{cases} i & \text{if } ip \text{ is even,} \\ i+1 & \text{if } ip \text{ is odd.} \end{cases}$$

In the remainder of this section, we shall obtain, for an arbitrary graph G, an upper bound of $\alpha_i(G)$, which will be found useful in the sequel. To get an upper bound, we need the following known result:

Theorem C (Kano and Saito [3]). Let k, r, s and t be integers such that $0 \leq k \leq r$, $s \geq 0$ and $t \geq 1$. If $ks \leq tr$, then any $[r, r+s]$-graph has a $[k, k+t]$-factor.

As an immediate consequence of Theorem C, we have:

Corollary. Let k, r and s be integers such that $0 \leq k \leq r$, $r > 0$, $s \geq 0$, and let

$$t = \begin{cases} \left\lceil \frac{ks}{r} \right\rceil & \text{if } ks \neq 0, \\ 1 & \text{if } ks = 0. \end{cases}$$

Then any $[r, r+s]$-graph has a $[k, k+t]$-factor.

Theorem 1. Let G be a graph having a c-factor, where $0 \leq c < \delta$. Then

$$\alpha_i(G) \leq \begin{cases} i + \left\lceil \dfrac{(i-c)(\Delta - \delta)}{\delta - c} \right\rceil & \text{if } \Delta \neq \delta, \\ i+1 & \text{if } \Delta = \delta, \end{cases}$$

for each $i = c+1, c+2, \ldots, \delta$.

Proof. Let F be a c-factor of G. Then for each $i = c+1, c+2, \ldots, \delta$,

$$\alpha_i(G) \le c + \alpha_{i-c}(G - E(F)).$$

Observe that $G - E(F)$ is a $[\delta - c, \Delta - c]$-graph. If we put $r = \delta - c$, $s = \Delta - \delta$ and $k = i - c$, then by the corollary to Theorem C, we have

$$\alpha_{i-c}(G - E(F)) \le \begin{cases} i - c + \left\lceil \dfrac{(i-c)(\Delta-\delta)}{\delta - c} \right\rceil & \text{if } \Delta \ne \delta, \\ i - c + 1 & \text{if } \Delta = \delta. \end{cases}$$

The result thus follows. □

Remark 1. If G is a graph, then by Theorem 1,

$$\alpha_i(G) \le \begin{cases} \left\lceil \frac{i\Delta}{\delta} \right\rceil & \text{when } \Delta \ne \delta, \\ i+1 & \text{when } \Delta = \delta, \end{cases}$$

for $i = 1, 2, \ldots, \delta$.

Remark 2. The bound given in Remark 1 can be achieved. For instance,

$$\alpha_i(K_2(m_1, m_2)) = \left\lceil \frac{im_2}{m_1} \right\rceil,$$

for each $i = 1, 2, \ldots, m_1$.

§3. Complete n-Partite Graphs

The aim of this section is to determine the value of $\alpha_i(G_n)$ as shown in Table 1. For a graph G and for any two subsets A and B of $V(G)$, let $e_G(A, B)$ denote the number of edges of G joining a vertex of A to a vertex of B.

To begin with, we deal with the case when $\delta = \delta(G_n) \le m_n$.

Theorem 2. If $\delta \le m_n$, then $\alpha_i(G_n) = \left\lceil \frac{im_n}{\delta} \right\rceil$ for any $i = 1, 2, \ldots, \delta$.

Proof. Let $i = 1, 2, \ldots, \delta$. Construct an $[i, k]$-factor F of G_n with $k = \Delta(F) = \left\lceil \frac{im_n}{\delta} \right\rceil$ by joining all vertices of V_n to the vertices of $\bigcup_{i=1}^{n-1} V_i$ such that

$$\deg_F x = i, \text{ for all } x \in V_n,$$

$$\text{and} \quad |\deg_F x - \deg_F y| \le 1, \text{ for all } x, y \in \bigcup_{i=1}^{n-1} V_i.$$

We note that the factor F satisfies

$$(\Delta(F) - 1)\delta < \sum (\deg_F x \mid x \in \bigcup_{i=1}^{n-1} V_i) \le \Delta(F)\delta,$$

as there must be a vertex x in $\bigcup_{i=1}^{n-1} V_i$ such that $\deg_F x = \Delta(F)$. Since

$$\sum (\deg_F x \mid x \in \bigcup_{i=1}^{n-1} V_i) = \sum_{x \in V_n} \deg_F x = im_n,$$

we have

$$(\Delta(F) - 1)\delta < im_n \le \Delta(F)\delta,$$

or

$$\Delta(F) = \left\lceil \frac{im_n}{\delta} \right\rceil.$$

Now, we show that $\alpha_i(G_n) \ge \lceil im_n/\delta \rceil$, for any $i = 1, 2, \ldots, \delta$. Suppose $\alpha_i = \alpha_i(G_n) \le \lceil im_n/\delta \rceil - 1$. Let H be any $[i, \alpha_i]$-factor of G_n. Then we have

$$im_n \le e_H(V_n, \bigcup_{i=1}^{n-1} V_i) \le \delta\alpha_i \le \delta(\left\lceil \frac{im_n}{\delta} \right\rceil - 1) < im_n,$$

which is impossible. □

From now on, we consider in the remainder of this section the case when $\delta > m_n$.

The following theorem is an easy consequence of Theorem B.

Theorem 3. If $\delta > m_n$, then for $i = 1, 2, \ldots, \lfloor (p+5)/4 \rfloor$,

$$\alpha_i(G_n) = \begin{cases} i & \text{if } ip \text{ is even,} \\ i+1 & \text{if } ip \text{ is odd.} \end{cases}$$

If δ is much larger than m_n, then the following theorem gives the exact value of $\alpha_i(G_n)$ for more i.

Theorem 4. Let $\delta > m_n$ and $h = \lfloor (\delta - m_n)/2 \rfloor + 1$. Then

(1) for $i = 1, 2, \ldots, h$,

$$\alpha_i(G_n) = \begin{cases} i & \text{if } ip \text{ is even,} \\ i+1 & \text{if } ip \text{ is odd;} \end{cases}$$

(2)

$$\alpha_{h+1}(G_n) \begin{cases} \le h+2 & \text{if } h \text{ is even,} \\ = h+1 & \text{if } h \text{ is odd;} \end{cases}$$

(3) for $i = h+2, \ldots, \delta$,

$$\alpha_i(G_n) \le 2(i - \lceil h/2 \rceil).$$

Proof. Since $\lfloor (\delta - m_n)/2 \rfloor = h - 1$ and $p = m_n + \delta$, we have $\delta \ge p/2 + (h-1)$. By Theorem A, G_n has $\lceil h/2 \rceil$ edge-disjoint hamiltonian cycles.

We now prove (1). Let i be an integer with $1 \le i \le h$. If ip is odd, we may take the edge sum of $(i+1)/2$ of these hamiltonian cycles. It follows that $\alpha_i(G_n) = i+1$. On the other hand, if i is even, we take the edge sum of $i/2$ of these hamiltonian cycles; and if i is odd and p is even, we take the edge sum of $(i-1)/2$ of these hamiltonian cycles plus a 1-factor. Thus $\alpha_i(G_n) = i$ if ip is even.

We next prove (3). Let $i = h+2, \ldots, \delta$, and let the edge sum of the $\lceil h/2 \rceil$ hamiltonian cycles be denoted by F. Then we have

$$\alpha_i(G_n) \le 2\left\lceil \frac{h}{2} \right\rceil + \alpha_{i - 2\lceil \frac{h}{2} \rceil}(G_n - E(F)).$$

Note that $G_n - E(F)$ is a graph with $\delta' = \delta(G_n - E(F)) = \delta - 2\lceil h/2\rceil$ and $\Delta' = \Delta(G_n - E(F)) = \Delta - 2\lceil h/2\rceil$. Also $\Delta' - \delta' = \Delta - \delta = m_n - m_1$.

If $m_n > m_1$, then since $i - 2\lceil h/2\rceil \geq 1$, we have by Theorem 1,

$$\alpha_{i-2\lceil\frac{h}{2}\rceil}(G_n - E(F)) \leq i - 2\left\lceil\frac{h}{2}\right\rceil + \left\lceil\frac{(i-2\lceil\frac{h}{2}\rceil)(m_n - m_1)}{\delta - 2\lceil\frac{h}{2}\rceil}\right\rceil .$$

It can be checked that $m_n - m_1 \leq m_n - 1 \leq \delta - 2\lceil h/2\rceil$. Thus,

$$\alpha_i(G_n) \leq i + \left\lceil i - 2\left\lceil\frac{h}{2}\right\rceil\right\rceil = 2\left(i - \left\lceil\frac{h}{2}\right\rceil\right).$$

If $m_n = m_1$, by Theorem 1 again, we have

$$\begin{aligned}\alpha_i(G_n) &\leq 2\left\lceil\frac{h}{2}\right\rceil + \left(i - 2\left\lceil\frac{h}{2}\right\rceil\right) + 1 = i + 1 \\ &\leq 2\left(i - \left\lceil\frac{h}{2}\right\rceil\right),\end{aligned}$$

as $i \geq h + 2$.

Finally, we prove (2). If h is odd, we may take the edge sum of the $\lceil h/2\rceil$ hamiltonian cycles and we have $\alpha_{h+1}(G_n) = h + 1$. Assume that h is even. If $m_n > m_1$, then as shown above, we have

$$\alpha_{h+1}(G_n) \leq 2\left(h + 1 - \left\lceil\frac{h}{2}\right\rceil\right) = h + 2.$$

If $m_n = m_1$, then

$$\alpha_{h+1}(G_n) \leq (h+1) + 1 = h + 2.$$

The proof is now complete. □

Remark. The upper bound in Theorem 4 is sharp. For instance,

$$\alpha_i\left(K_3\left(\left\lceil\frac{h}{2}\right\rceil, i - \left\lceil\frac{h}{2}\right\rceil, i - \left\lceil\frac{h}{2}\right\rceil\right)\right) = 2\left(i - \left\lceil\frac{h}{2}\right\rceil\right),$$

for any i with $i \geq h$ if h is even, and for $i \geq h+1$ if h is odd.

By Theorem 3, one can obtain the exact value of $\alpha_i(G_n)$ for any positive integer i, $i \leq \lfloor (p+5)/4 \rfloor$. If h is larger, then Theorem 4(1) gives the exact value of $\alpha_i(G_n)$ for more i. Indeed, whatever the value of h is, the exact values of $\alpha_i(G_n)$, for $i = 1, 2, 3, 4$, can be determined as shown below.

Theorem 5. If $\delta > m_n$, then

(1) $\alpha_1(G_n) = \begin{cases} 1 & \text{if } p \text{ is even,} \\ 2 & \text{if } p \text{ is odd;} \end{cases}$

(2) $\alpha_2(G_n) = 2$;

(3) $\alpha_3(G_n) = \begin{cases} 3 & \text{if } p \text{ is even,} \\ 4 & \text{if } p \text{ is odd;} \end{cases}$

(4) $\alpha_4(G_n) = \begin{cases} 4 & \text{if } G \neq K_3(1,3,3), \\ 6 & \text{if } G = K_3(1,3,3). \end{cases}$

Proof. Since $\delta > m_n$, (1) and (2) follow immediately from Theorem A. Note that (3) follows immediately from Theorem 3, if $p \geq 7$. If $p < 7$, then it can easily be checked that $\alpha_3(G_n) = 3$ when $p = 4$ or 6, and $\alpha_3(G_n) = 4$ when $p = 5$. We now give a proof of (4).

If $\delta - m_n \geq 4$, then $\delta \geq p/2+2$. So by Theorem A, G_n has two edge-disjoint hamiltonian cycles. Therefore $\alpha_4(G_n) = 4$. If $\delta - m_n < 4$, then we consider three cases. Before we proceed, let us agree on the following notation:

We denote the vertices in $\cup_{i=1}^{n-1} V_i$ by v_i such that if $1 \leq i < j \leq \delta$, $v_i \in V_k$ and $v_j \in V_l$, then $k \geq l$. Thus $v_1 \in V_{n-1}$ and $v_\delta \in V_1$.

Case 1. $\delta - m_n = 1$.

Since $\delta \geq 4$ (otherwise $\alpha_4(G_n)$ is not defined), $m_n \geq 3$. If $m_n = 3$, then G_n is one of $K_3(1,3,3)$, $K_3(2,2,3)$, $K_4(1,1,2,3)$ or $K_5(1,1,1,1,3)$, and all of them satisfy $\alpha_4(G_n) = 4$ except $\alpha_4(K_3(1,3,3)) = 6$. If $m_n \geq 4$, then we let

$$H = (G_n - \{v_\delta\}) - \{xy \in E(G_n) \mid x, y \in \bigcup_{i=1}^{n-1} V_i\}.$$

Note that $H = K_2(m_n, m_n)$. Since $m_n \geq 4$, H has a 4-factor F'.

Let the vertices v_1 and v_2 be adjacent to two distinct vertices a and b of V_n in F', respectively. We now delete the edges v_1a and v_2b from F' and join v_δ to v_1, v_2, a and b. This is possible because v_1 and v_δ (resp., v_2 and v_δ) are in different partite sets of G_n. This shows that $\alpha_4(G_n) = 4$.

Case 2. $\delta - m_n = 2$.

Since $\delta \geq 4$, $m_n \geq 2$. If $m_n = 2$, then G_n is $K_3(2,2,2)$, $K_4(1,1,2,2)$ or $K_5(1,1,1,1,2)$. The 4^{th} factor-index for these graphs is four. If $m_n = 3$, then G_n is $K_3(2,3,3)$, $K_4(1,1,3,3)$, $K_4(1,2,2,3)$, $K_5(1,1,1,2,3)$ or $K_6(1,1,1,1,1,3)$, and all of them satisfy $\alpha_4(G_n) = 4$. If $m_n \geq 4$, then we let

$$H = (G_n - \{v_{\delta-1}, v_\delta\}) - \{xy \in E(G_n) \mid x, y \in \bigcup_{i=1}^{n-1} V_i\}.$$

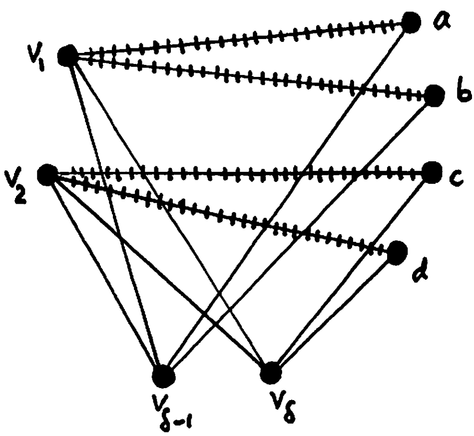
Note that H has a 4-factor F' because H is the complete bipartite graph $K_2(m_n, m_n)$ and $m_n \geq 4$. We will construct a 4-factor of G_n from F'.

Let the vertex v_1 be adjacent to two distinct vertices a and b of V_n and the vertex v_2 be adjacent to two distinct vertices c and d of V_n in F'. (The vertices a or b may be the same as c or d.) We now construct a 4-factor F of G_n from F' by joining $v_{\delta-1}$ to v_1, v_2, a and b; v_δ to v_1 ,v_2, c and d, and remove the edges v_1a, v_1b, v_2c and v_2d from F' (see Figure 1). This construction is possible because $v_{\delta-1}$ and v_1 or v_2 (resp., v_δ and v_1 or v_2) are in different partite sets of G_n. This shows that $\alpha_4(G_n) = 4$.

Case 3. $\delta - m_n = 3$.

If $m_n = 1$, then $G_n = K_5$ and clearly $\alpha_4(G_n) = 4$. If $m_n = 2$, then G_n is $K_4(1,2,2,2)$, $K_5(1,1,1,2,2)$ or $K_6(1,1,1,1,1,2)$ and these graphs satisfy $\alpha_4(G_n) = 4$. If $m_n = 3$, then G_n is one of $K_3(3,3,3)$, $K_4(1,2,3,3)$, $K_4(2,2,2,3)$, $K_5(1,1,1,3,3)$, $K_5(1,1,2,2,3)$, $K_6(1,1,1,1,2,3)$ or $K_7(1,1,1,1,1,1,3)$. It is not difficult to check that the 4^{th} factor-index of these graphs is also four. If $m_n \geq 4$, then we let

$$H = (G_n - \{v_\delta, v_{\delta-1}, v_{\delta-2}\}) - \{xy \in E(G_n) \mid x, y \in \bigcup_{i=1}^{n-1} V_i\}.$$



deleted edges of F'.

Figure 1.

We note that H has a 4-factor F', and we will construct a 4-factor F of G_n from F' as follows. Let the vertex v_1 be adjacent to two distinct vertices a and b of V_n, the vertex v_2 be adjacent to two distinct vertices c and d of V_n, and the vertex v_3 be adjacent to two distinct vertices e and f of V_n in F'. (Some of the vertices a, b, c, d, e and f may be the same.) Remove the edges $v_1 a$, $v_1 b$, $v_2 c$, $v_2 d$, $v_3 e$ and $v_3 f$ from F' and join $v_{\delta-2}$ to v_1, v_2, a and b; $v_{\delta-1}$ to v_1, v_3, c and d; and finally join v_δ to v_2, v_3, e and f (see Figure 2).

Since $v_{\delta-2}$ and v_1 or v_2 (resp., $v_{\delta-1}$ and v_1 or v_3, and v_δ and v_2 or v_3) cannot be in the same partite set of G_n, such a factor F exists. This again shows that $\alpha_4(G_n) = 4$.

The proof is now complete. □

§4. Connected Factor-Index

The theory of $[i, k]$-factor has been developed by Tutte [6,7,8] and Lovász [5]. They obtained a necessary and sufficient condition for a graph to have an $[i, k]$-factor. However, the problem of determining the exact value of $\alpha_i^*(G)$ is difficult. For example, a graph contains a hamiltonian cycle if and only if $\alpha_2^*(G) = 2$, and the problem of finding a hamiltonian cycle in a graph is NP-complete.

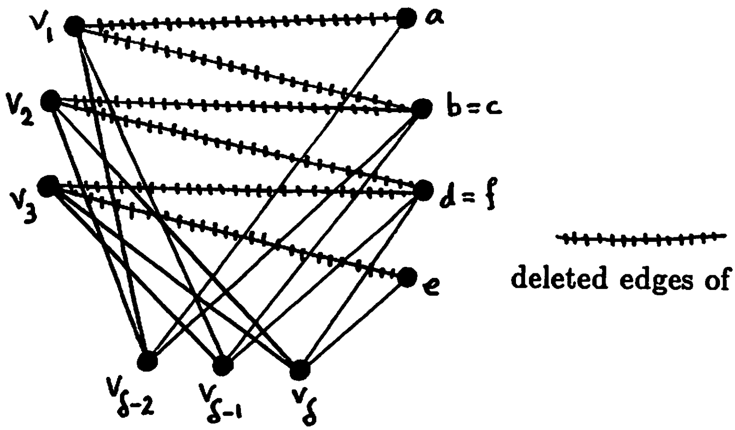


Figure 2.

It follows by definition that $\alpha_i(G) \le \alpha_i^*(G)$, for any graph G and for each $i = 1, 2, \ldots, \delta$. In this final section, we shall show that if we confine ourselves to G_n, then $\alpha_i^*(G_n)$ and $\alpha_i(G_n)$ are indeed identical for all i, except possibly when $i = 1$.

Theorem 6. Let $G_n = K_n(m_1, m_2, \ldots, m_n)$. Then

$$(1)\ \alpha_1^*(G_n) = \begin{cases} 2 & \text{if } \delta > m_n, \\ \left\lceil \dfrac{m_n + \delta - 1}{\delta} \right\rceil & \text{if } \delta \le m_n; \end{cases}$$

$$(2)\ \alpha_i^*(G_n) = \alpha_i(G_n), \text{ for each } i = 2, 3, \ldots, \delta.$$

Proof. (1) If $\delta > m_n$, then $\alpha_1^*(G_n) = 2$ by Theorem A. Assume $\delta \le m_n$. We first construct a connected $[1, k]$-factor F with $k = \Delta(F) = \lceil (m_n + \delta - 1)/\delta \rceil$ and then show that $\alpha_1^*(G_n)$ cannot be less than this value. To construct F, we take a path P of G_n which begins with a vertex in V_n with the successive vertices alternately in $\cup_{i=1}^{n-1} V_i$ and V_n until P contains all the vertices of $\cup_{i=1}^{n-1} V_i$. (This is possible because $m_n \ge \delta$.) If $\delta \le m_n \le \delta + 1$, then F is the path P with $k = 2 = \lceil (m_n + \delta - 1)/\delta \rceil$. If $m_n > \delta + 1$, then we get the connected factor F by joining all the vertices in $V_n - V(P)$ to the vertices in $\cup_{i=1}^{n-1} V_i$ such that

$$\deg_F x = 1, \text{ for each } x \in V_n - V(P),$$

$$\text{and } \quad |\deg_F x - \deg_F y| \leq 1, \text{ for all } x, y \in \bigcup_{i=1}^{n-1} V_i.$$

Thus F satisfies

$$(\Delta(F) - 1)\delta < \sum(\deg_F x \mid x \in \bigcup_{i=1}^{n-1} V_i) \leq \Delta(F)\delta,$$

as there must be a vertex x in $\bigcup_{i=1}^{n-1} V_i$ such that $\deg_F x = \Delta(F)$. But

$$\sum(\deg_F x \mid x \in \bigcup_{i=1}^{n-1} V_i) = \sum(\deg_F x \mid x \in V_n) = m_n + \delta - 1.$$

So

$$(\Delta(F) - 1)\delta < m_n + \delta - 1 \leq \Delta(F)\delta,$$

or

$$\Delta(F) = \left\lceil \frac{m_n + \delta - 1}{\delta} \right\rceil.$$

To show that $\alpha_1^*(G_n)$ is never less than $\Delta(F)$, we suppose the contrary and let F' be a connected $[1, \alpha_1^*]$-factor of G_n. Then

$$|E(F')| \leq \delta\alpha_1^* \leq \delta(\left\lceil \frac{m_n + \delta - 1}{\delta} \right\rceil - 1) < m_n + \delta - 1.$$

Since F' is connected,

$$|E(F')| \geq m_n + \delta - 1.$$

Thus we have

$$m_n + \delta - 1 \leq |E(F')| < m_n + \delta - 1,$$

which is impossible.

(2) Let $i = 2, 3, \ldots, \delta$, and let F be an $[i, \alpha_i(G_n)]$-factor of G_n. If F is connected, then $\alpha_i^*(G_n) = \alpha_i(G_n)$. Assume that F is not connected and let C_1 and C_2 be any two of its components. We need only to show that it is possible to modify F to get an $[i, \alpha_i(G_n)]$-factor in which C_1 and C_2 are connected.

Let ab and xy be non-bridge edges of C_1 and C_2, respectively. (Such edges exist since C_1 and C_2 are not trees.) We consider the following two cases:

Case 1. b and y are in the same partite set of G_n.

In this case, a and y cannot be in the same partite set, because a and b are adjacent. Similarly, b and x cannot be in the same partite set because x and y are adjacent. So we can delete the edges ab and xy from F and add the edges ay and bx to it. Since ab and xy are not bridges, C_1 and C_2 are now connected. (See Figure 3.)

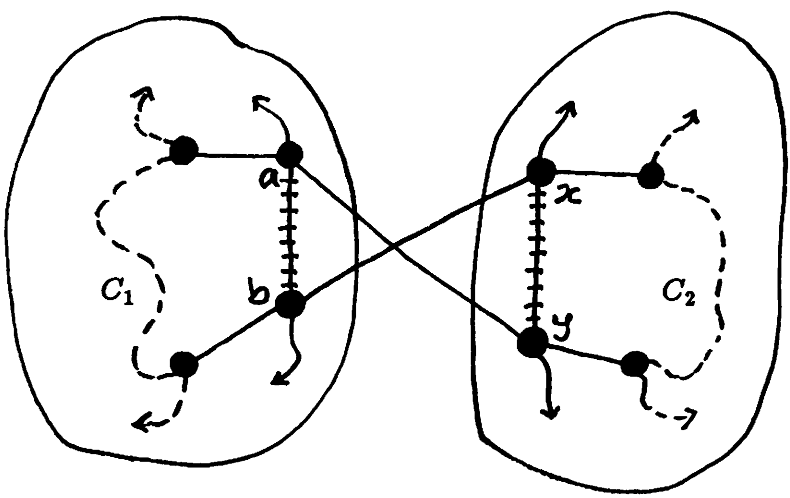


Figure 3.

Case 2. b and y are not in the same partite set of G_n.

In this case, we can assume, by Case 1, that a and x are not in the same partite set. But then we can remove the edges ab and xy from the factor and add the edges ax and by to it. Since ab and xy are not bridges, C_1 and C_2 are now connected. □

Remark 1. By comparing Theorem 9(1) with Theorems 2 and 5(1), we see that $0 \le \alpha_1^*(G_n) - \alpha_1(G_n) \le 1$.

Remark 2. The inequality in Theorem 1 also holds for $\alpha_i^*(G)$, for any graph G having a connected c-factor.

Remark 3. Theorem 2 also holds for $\alpha_i^*(G_n)$, for any $i = 2, 3, \ldots, \delta$.

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