On Coverings of the Complete Graph with 4 Vertices

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Abstract. We present a permutation group whose orbits classify isomorphism of covering projections of the complete graph with 4 vertices. Two structure theorems concerning with this group are proved.

1. Introduction.

Let G be a simple graph. The graph H is called an *r-fold covering* of G if there is an r-to-one epimorphism p: $H \rightarrow G$, called the *covering projection*, which is "local homeomorphic", that is, which sends the neighbors of each vertex v of H bijectively to the neighbors of $p(v)$ of G.

Now let $Aut\,(G)$ be the automorphism group of G. An *isomorphism* of covering projections of G is a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\;\psi\;} & \tilde{H} \\ p \big\downarrow & & \big\downarrow \tilde{p} \\ G & \xrightarrow[\;\varphi\;]{} & G \end{array} \qquad (1)$$

with an isomorphism ψ and $\varphi \in Aut\,(G)$; we write $p \sim_G \tilde{p}$.

During the last years one may observe a lot of interest in enumerative aspects of topological graph theory. For example, Mohar counted the akempic triangulations of the 2-sphere with 4 vertices of degree 3, which correspond to certain coverings of the complete graph K_4 with 4 vertices via duality [7, 8]. Negami counted embeddings of a 3-connected nonplanar graph G into a projective plane by establishing a bijection between such embeddings and nonisomorphic planar 2-fold coverings of G [9]. Hong and Kwak counted regular 4-fold coverings of an identity graph G [6]. Nonisomorphic concrete graph coverings are counted in [5].

Nevertheless, the general counting problem for covering projections of G up to isomorphism is unsolved except in the cases $r = 2$ [3] or trivial automorphism group of G [4]. Our purpose is to present a permutation group, whose orbits correspond to the isomorphism classes of r-fold covering projections of the complete graph with 4 vertices K_4. We will prove two structure theorems concerning this permutation group.

But first let us state some general results.

[1] Supported by the DEUTSCHE FORSCHUNGSGEMEINSCHAFT, SFB 303

2. Permutation voltage assignments.

Let S_r denote the symmetric group on the set $\{1, \ldots, r\}$. Let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. The arc set of the corresponding symmetric directed graph will be denoted by $A(G)$.

A *permutation voltage assignment* in S_r for G is a mapping $f\colon A(G) \to S_r$ such that $f(x, y) = f(y, x)^{-1}$ for every edge $[x, y]$ of G. The pair (G, f) is called a *permutation voltage graph*. The set of the permutation voltage assignments in S_r for G will be denoted by $\mathcal{F}_r(G)$.

Given such a permutation voltage graph (G, f), we construct the *derived graph* G_f as follows. Its vertex set is $V(G) \times \{1, \ldots, r\}$; two vertices $(x, i), (y, j)$ are adjacent in G_f iff x and y are adjacent in G and $f(x, y)(i) = j$.

In order to deal with covering projections it is admissible to restrict attention to derived graphs of permutation voltage graphs. This was shown by Gross and Tucker [1]; we present their main theorem.

Theorem 1. *(1) Let f be a permutation voltage assignment for G in S_r. Then the natural projection $p_f\colon G_f \to G$ (sending vertex (u, i) of G_f to vertex u of G) is an r-fold covering projection of G.*

(2) Let $p\colon H \to G$ be an r-fold covering projection. Then there is an assignment f of voltages in S_r for G such that the covering projections p and p_f are isomorphic with the trivial automorphism of G.

Permutation voltage assignments are a powerful tool to handle coverings (for a general development of the theory see [2]). In particular, isomorphism of covering projections can be classified by them. Two permutation voltage assignments $f, \tilde{f}$ in S_r for G are said to be *equivalent* if there is a family $(\pi_u)_{u \in V(G)}$ in $S_r^{V(G)}$ and $\varphi \in Aut(G)$ such that

$$\tilde{f}(x, y) = \pi_y f(\varphi(x), \varphi(y))\, \pi_x^{-1} \qquad (2)$$

for every edge $[x, y]$ of G; we write $f \sim_G \tilde{f}$. (Note that composition of permutations is read from right to left.) The permutations π_u are to allow relabeling of the fibers $p^{-1}(u)$. The equivalence class of f will be denoted by $[f]$, and the set of equivalence classes by $\mathcal{F}_r(G)/\sim_G$. The following theorem can be found in [4].

Theorem 2. *The equivalence classes of permutation voltage assignments for S_r in G are in one-to-one correspondence with the isomorphism classes of r-fold covering projections of G.*

Theorem 2 could be a basis for the enumeration of nonisomorphic covering projections, but as far as we know, no good counting formula for this problem is known.

It is natural to look for "simple" representatives of the equivalence classes of permutation voltage assignments. The following theorem states their existence [4].

Theorem 3. *Let T be a spanning forest of G, and let f be a permutation voltage assignment in S_r for G. Then there is an $\bar{f} \in [f]$ such that $\bar{f}(x,y) = (1)$ for every edge $[x,y]$ of T.*

Motivated by Theorem 3, we call such assignments which are trivial on the arcs of a spanning forest T of G *rooted* at T. The set of permutation voltage assignments in S_r for G that are rooted at T will be denoted by $\mathcal{W}_r(G;T)$.

3. The case of K_4.

From now on, we consider the complete graph K_4 with vertex set $\{0,1,2,3\}$. Let T^0 be the spanning star of K_4 centered at 0. Define the mapping $\mathcal{P}_r^0 \colon \mathcal{F}_r(K_4) \to \mathcal{W}_r(K_4;T^0)$ by setting

$$\mathcal{P}_r^0(f)(x,y) = \begin{cases} f(y,0)f(x,y)f(0,x), & \text{if } (x,y) \notin A(T^0); \\ (1) & \text{if } (x,y) \in A(T^0). \end{cases} \tag{3}$$

(Remember that composition of permutations is read from right to left.) It is easy to see that $f \sim_{K_4} \mathcal{P}_r^0(f)$; therefore, $\mathcal{P}_r^0(f)$ is a "simple" representative in $[f]$. Now define a bijection $\Delta_r \colon \mathcal{W}_r(K_4;T^0) \to S_r^3$ by setting

$$\Delta_r(g) = (g(1,2), g(2,3), g(3,1)) .$$

Then $\Delta_r \circ \mathcal{P}_r^0$ is a mapping from $\mathcal{F}_r(K_4)$ onto S_r^3 that induces an equivalence relation on S_r^3 from $\sim_{K_4}$ in an obvious way.

The automorphism group of K_4 is S_4, considered as a permutation group of the set $\{0,1,2,3\}$. Observe that S_4 acts on $\mathcal{F}_r(K_4)$ via

$$\varphi(f)(x,y) = f\left(\varphi^{-1}(x), \varphi^{-1}(y)\right),$$

and that S_r acts on $\mathcal{W}_r(K_4;T^0)$ via conjugation:

$$\pi(f)(x,y) = \pi f(x,y)\pi^{-1}.$$

It is well known that S_4 can be generated by the three transpositions

$$\varphi_1 = (0\ 1), \quad \varphi_2 = (1\ 2), \quad \varphi_3 = (2\ 3).$$

For $i \in \{1,2,3\}$ and $\pi \in S_r$ define permutations of the elements of S_r^3 as follows (the permutations are not, in general, automorphisms of S_r^3, except for τ_π):

$$\begin{aligned} \tau_1(\alpha,\beta,\gamma) &= (\alpha^{-1}, \alpha\beta\gamma, \gamma^{-1}), \\ \tau_2(\alpha,\beta,\gamma) &= (\alpha^{-1}, \gamma^{-1}, \beta^{-1}), \\ \tau_3(\alpha,\beta,\gamma) &= (\gamma^{-1}, \beta^{-1}, \alpha^{-1}), \\ \tau_\pi(\alpha,\beta,\gamma) &= \pi(\alpha,\beta,\gamma)\pi^{-1}, \\ &:= (\pi\alpha\pi^{-1}, \pi\beta\pi^{-1}, \pi\gamma\pi^{-1}), \end{aligned}$$

and set

$$S = \{\tau_1, \tau_2, \tau_3\} \cup \{\tau_\pi \mid \pi \in S_r\}.$$

Consider the subgroups of the full symmetric group on the elements of S_r^3 given by the following generating sets:

$$\mathcal{G}_r = \langle S \rangle, \ \ \mathcal{H}_r = \langle \{\tau_1, \tau_2, \tau_3\} \rangle, \ \ \mathcal{I}_r = \langle \{\tau_\pi \mid \pi \in S_r\} \rangle.$$

Clearly $\mathcal{I}_r$ is isomorphic to S_r for $r \geq 3$, while it is of order one for $r = 1, 2$. It will turn out that the orbits of $\mathcal{G}_r$ are the equivalence classes of S_r^3 induced by equivalence of permutation voltage assignments via the mapping $\Delta_r \circ \mathcal{P}_r^0$.

The proof of the following lemma is straightforward.

Lemma 4. *Let $f \in \mathcal{F}_r(K_4)$ and $(\Delta_r \circ \mathcal{P}_r^0)\ (f) = (\alpha, \beta, \gamma)$.*

(1) *If $\bar{f}(x,y) = \pi_y f(x,y) \pi_x^{-1}$ for every $[x,y] \in E(K_4)$, then $(\Delta_r \circ \mathcal{P}_r^0)(\bar{f}) = \tau_{\pi_0}(\alpha, \beta, \gamma)$.*
(2) *If $\bar{f} = \varphi_1^{-1}(f)$, then $(\Delta_r \circ \mathcal{P}_r^0)(\bar{f}) = (\tau_{f(0,1)} \tau_1)(\alpha, \beta, \gamma)$.*
(3) *If $\bar{f} = \varphi_2^{-1}(f)$, then $(\Delta_r \circ \mathcal{P}_r^0)(\bar{f}) = \tau_2(\alpha, \beta, \gamma)$.*
(4) *If $\bar{f} = \varphi_3^{-1}(f)$, then $(\Delta_r \circ \mathcal{P}_r^0)(\bar{f}) = \tau_3(\alpha, \beta, \gamma)$.*

For $(\alpha, \beta, \gamma) \in S_r^3$, let $[(\alpha, \beta, \gamma)]$ denote the orbit of (α, β, γ) under $\mathcal{G}_r$.

Theorem 5. *The mapping $\Delta_r \circ \mathcal{P}_r^0$ establishes a bijection between $\mathcal{F}_r(K_4)/ \sim_{K_4}$ and $S_r^3/\mathcal{G}_r$.*

Proof: Let $f, \bar{f} \in \mathcal{F}_r(K_4)$ such that $f \sim_{K_4} \bar{f}$. By (2) there is a family $(\pi_u)_{u \in V(K_4)}$ in $S_r^{V(K_4)}$ and $\varphi \in S_4$ such that

$$\bar{f}(x,y) = \pi_y f(\varphi(x), \varphi(y))\, \pi_x^{-1}$$

for every edge $[x,y]$ of K_4. We first have to show that $(\Delta_r \circ \mathcal{P}_r^0)(f)$ and $(\Delta_r \circ \mathcal{P}_r^0)(\bar{f})$ are in the same orbit of $\mathcal{G}_r$ We proceed by induction on $\ell(\varphi)$, where $\ell(\varphi)$ is the smallest number of factors in a presentation of φ by the generators φ_1, φ_2, φ_3. If $\ell(\varphi) = 1$, then $\bar{f}(x,y) = \pi_y f(\varphi_i(x)\ \varphi_i(y))\, \pi_x^{-1}$ for some $i \in \{1, 2, 3\}$. It follows by Lemma 4 that

$$(\Delta_r \circ \mathcal{P}_r^0)(\bar{f}) = \begin{cases} (\tau_{\pi_0} \tau_{f(0,1)} \tau_1)(\alpha, \beta, \gamma) & \text{if } i = 1, \\ (\tau_{\pi_0} \tau_i)(\alpha, \beta, \gamma) & \text{if } i = 2, 3. \end{cases} \tag{4}$$

For $\varphi^{-1} = \varphi^{(1)} \varphi^{(2)} \ldots \varphi^{(k+1)}$, $\varphi^{(i)} \in \{\varphi_1, \varphi_2, \varphi_3\}$ for $1 \leq i \leq k+1$, we have

$$\begin{aligned} [(\Delta_r \circ \mathcal{P}_r^0)(\bar{f})] &= [\tau_{\pi_0}\ ((\Delta_r \circ \mathcal{P}_r^0)(\varphi^{-1}(f)))] \\ &= [(\Delta_r \circ \mathcal{P}_r^0)\ ((\varphi^{(1)} \ldots \varphi^{(k+1)})(f))] \\ &= [(\Delta_r \circ \mathcal{P}_r^0)\ ((\varphi^{(2)} \ldots \varphi^{(k+1)})(f))] \\ &= [(\Delta_r \circ \mathcal{P}_r^0)(f)] \end{aligned}$$

by equation (4) and induction.

Conversely, let $\varrho \in \mathcal{G}_r$. We have to show that for any $(\alpha, \beta, \gamma) \in S_r^3$

$$\Delta_r^{-1}(\alpha, \beta, \gamma) \sim_{K_4} \Delta_r^{-1}(\varrho(\alpha, \beta, \gamma)) . \quad (5)$$

If $\varrho \in S$, this follows from the equations

$$\begin{aligned} \Delta_r^{-1} \circ \tau_i &= \mathcal{P}_r^0 \circ \varphi_i \circ \Delta_r^{-1} \quad && (i \in \{1,2,3\}), \\ \Delta_r^{-1} \circ \tau_\pi &= \pi \circ \Delta_r^{-1} && (\pi \in S_r), \end{aligned}$$

which can be verified directly from the definitions.

We now conclude that equation (5) is true by induction on the minimal number of factors in a presentation of ϱ by the generators in S. ∎

By Theorem 5, it suffices to count the orbits of the permutation group $\mathcal{G}_r$ to compute the number of r-fold covering projections of the graph K_4. In principle, this can be done by Burnside's lemma; unfortunately, the enumeration fails on the unknown members of $\mathcal{G}_r$. Indeed we can not solve the counting problem, but we can present some structures of the group $\mathcal{G}_r$.

4. Two structure theorems for $\mathcal{G}_r$.

Let $(\alpha, \beta, \gamma) \in S_r^3$. Then

$$\tau_3 \tau_2(\alpha, \beta, \gamma) = (\beta, \gamma, \alpha) \quad (6)$$

and

$$(\tau_1 \tau_2)^3(\alpha, \beta, \gamma) = \alpha(\alpha, \beta, \gamma)\alpha^{-1}.$$

Define $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{H}_r$ by

$$\begin{aligned} \sigma_1 &= (\tau_1 \tau_2)^3 \\ \sigma_2 &= (\tau_3 \tau_2)^{-1}(\tau_1 \tau_2)^3(\tau_3 \tau_2) \\ \sigma_3 &= (\tau_3 \tau_2)^{-2}(\tau_1 \tau_2)^3(\tau_3 \tau_2)^2 . \end{aligned}$$

Then, for every $(\alpha_1, \alpha_2, \alpha_3) \in S_r^3$ and $i \in \{1,2,3\}$,

$$\sigma_i(\alpha_1, \alpha_2, \alpha_3) = \alpha_i(\alpha_1, \alpha_2, \alpha_3)\alpha_i^{-1}, \quad (7)$$

and for every $\pi \in S_r$,

$$\sigma_i \tau_\pi = \tau_\pi \sigma_i . \quad (8)$$

Now set $\mathcal{V}_r = \langle\{\sigma_1, \sigma_2, \sigma_3\}\rangle$. Note that $\mathcal{V}_1$, $\mathcal{V}_2$ are of order one since S_1, S_2 are commutative.

Theorem 6. *$\mathcal{V}_r$ is normal in $\mathcal{H}_r$, and for $r \geq 2$, $\mathcal{H}_r/\mathcal{V}_r \cong S_4$.*

Proof: For $r = 1$, there is nothing to show. Thus, assume that $r \geq 2$. We first show that $\mathcal{V}_r$ is normal in $\mathcal{H}_r$. It suffices to show that

$$\tau_i \sigma_j \tau_i^{-1} \in \mathcal{V}_r$$

for all $i, j \in \{1, 2, 3\}$. The following Table 1 presents all these relations; the values are easy to check. The table is to be understood that the element of ith row and jth column is $\tau_i \sigma_j \tau_i^{-1}$.

Table 1

	1	2	3
1	σ_1^{-1}	$\sigma_1\sigma_2\sigma_3$	σ_3^{-1}
2	σ_1^{-1}	σ_3^{-1}	σ_2^{-1}
3	σ_3^{-1}	σ_2^{-1}	σ_1^{-1}

Next we have to prove that $\mathcal{H}_r/\mathcal{V}_r \cong S_4$. Set

$$\tau_4 = \tau_3 \tau_2 \tau_1 \tau_2 \tau_3 .$$

Then $\tau_4(\alpha, \beta, \gamma) = (\beta\alpha\gamma, \beta^{-1}, \gamma^{-1})$. Since $\tau_1 = \tau_2\tau_3\tau_4\tau_3\tau_2, \mathcal{H}_r = \langle\{\tau_2, \tau_3\tau_4\}\rangle$. The members of the group $\mathcal{H}_r/\mathcal{V}_r$ are cosets $\overline{\zeta} = \zeta\mathcal{V}_r$, $\zeta \in \mathcal{H}_r$; thus, the set $\{\overline{\tau}_2, \overline{\tau}_3, \overline{\tau}_4\}$ generates $\mathcal{H}_r/\mathcal{V}_r$.

The symmetric group S_4 is generated by

$$X_2 = (1\ 2),\ X_3 = (2\ 3),\ X_4 = (3\ 4),$$

together with the relations

$$X_i^2 = (X_j X_{j+1})^3 = (X_2 X_4)^1 = (1) \quad (i = 2, 3, 4;\ j = 2, 3).$$

We show that these relations are satisfied by $\overline{\tau}_2, \overline{\tau}_3, \overline{\tau}_4$. Clearly,

$$\overline{\tau}_2^2 = \overline{\tau}_3^2 = \overline{\tau}_4^2 = \overline{1}.$$

For $(\alpha, \beta, \gamma) \in S_r^3$,

$$(\tau_2\tau_3)^3(\alpha, \beta, \gamma) = (\tau_3\tau_2)^6(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)$$

by (6), hence, $(\overline{\tau}_2\overline{\tau}_3)^3 = \overline{1}$.

Furthermore,

$$
\begin{aligned}
(\tau_3\tau_4)^3(\alpha,\beta,\gamma) &= (\tau_3\tau_4)^2(\gamma,\beta,\gamma^{-1}\alpha^{-1}\beta^{-1}) \\
&= (\tau_3\tau_4)(\gamma^{-1}\alpha^{-1}\beta^{-1},\beta,\beta\alpha\beta^{-1}) \\
&= (\beta\alpha\beta^{-1},\beta,\beta\gamma\beta^{-1}) \\
&= \sigma_2(\alpha,\beta,\gamma)
\end{aligned}
$$

and

$$
(\tau_2\tau_4)^2(\alpha,\beta,\gamma) = (\tau_2\tau_4)(\gamma^{-1}\alpha^{-1}\beta^{-1},\gamma,\beta) = (\alpha,\beta,\gamma)
$$

what implies $(\overline{\tau}_3\overline{\tau}_4)^3 = (\overline{\tau}_2\overline{\tau}_4)^2 = \overline{1}$.

We conclude that the agreement

$$
\mu(X_i) = \overline{\tau}_i \quad (i \in \{1,2,3\})
$$

constitutes a group epimorphism $S_4 \longrightarrow \mathcal{H}_\tau/\mathcal{V}_\tau$.

Table 2 establishes the complete isomorphism.

i	$X^{(i)}$	$\tau^{(i)}$	$\tau^{(i)}(\alpha,\beta,\gamma)$
1	(1)	1	(α,β,γ)
2	(1 2)	τ_2	$(\alpha^{-1},\gamma^{-1},\beta^{-1})$
3	(2 3)	τ_3	$(\gamma^{-1},\beta^{-1},\alpha^{-1})$
4	(3 4)	τ_4	$(\beta\alpha\gamma,\beta^{-1},\gamma^{-1})$
5	(1 3)	$\tau_3\tau_2\tau_3$	$(\beta^{-1},\alpha^{-1},\gamma^{-1})$
6	(1 4)	$\tau_2\tau_3\tau_4\tau_3\tau_2$	$(\alpha^{-1},\gamma\beta\alpha,\gamma^{-1})$
7	(2 4)	$\tau_4\tau_3\tau_4$	$(\alpha^{-1},\beta^{-1},\alpha\gamma\beta)$
8	(1 2 3)	$\tau_2\tau_3$	(γ,α,β)
9	(1 3 2)	$\tau_3\tau_2$	(β,γ,α)
10	(1 2 4)	$\tau_4\tau_2\tau_3\tau_4$	$(\alpha,\beta^{-1}\gamma^{-1}\alpha^{-1},\beta)$
11	(1 4 2)	$\tau_4\tau_3\tau_2\tau_4$	$(\alpha,\gamma,\alpha^{-1}\beta^{-1}\gamma^{-1})$
12	(1 3 4)	$\tau_3\tau_4\tau_2\tau_3\tau_4\tau_3$	$(\beta,\gamma^{-1}\alpha^{-1}\beta^{-1},\gamma)$
13	(1 4 3)	$\tau_2\tau_4\tau_3\tau_2$	$(\alpha^{-1}\beta^{-1}\gamma^{-1},\alpha,\gamma)$
14	(2 3 4)	$\tau_3\tau_4$	$(\gamma,\beta,\gamma^{-1}\alpha^{-1}\beta^{-1})$
15	(2 4 3)	$\tau_4\tau_3$	$(\beta^{-1}\gamma^{-1}\alpha^{-1},\beta,\alpha)$
16	(1 2)(3 4)	$\tau_2\tau_4$	$(\gamma^{-1}\alpha^{-1}\beta^{-1},\gamma,\beta)$
17	(1 3)(2 4)	$\tau_3\tau_2\tau_3\tau_4\tau_3\tau_4$	$(\beta,\alpha,\beta^{-1}\gamma^{-1}\alpha^{-1})$
18	(1 4)(2 3)	$\tau_2\tau_3\tau_4\tau_3\tau_2\tau_3$	$(\gamma,\alpha^{-1}\beta^{-1}\gamma^{-1},\alpha)$
19	(1 2 3 4)	$\tau_2\tau_3\tau_4$	$(\gamma^{-1},\beta\alpha\gamma,\beta^{-1})$
20	(1 2 4 3)	$\tau_4\tau_2\tau_3$	$(\alpha\gamma\beta,\alpha^{-1},\beta^{-1})$
21	(1 3 2 4)	$\tau_3\tau_2\tau_3\tau_4\tau_3$	$(\beta^{-1},\alpha\gamma\beta,\alpha^{-1})$
22	(1 3 4 2)	$\tau_3\tau_4\tau_2$	$(\beta^{-1},\gamma^{-1},\beta\alpha\gamma)$
23	(1 4 2 3)	$\tau_3\tau_2\tau_4\tau_3\tau_2$	$(\gamma^{-1},\alpha^{-1},\gamma\beta\alpha)$
24	(1 4 3 2)	$\tau_4\tau_3\tau_2$	$(\gamma\beta\alpha,\gamma^{-1},\alpha^{-1})$

Table 2

The second column contains the elements of S_4, denoted by $X^{(i)}$, the third one representatives $\tau^{(i)}$ of the cosets $\mu(X^{(i)})$, and the fourth column describes the action of $\tau^{(i)}$ on $(\alpha,\beta,\gamma) \in \mathcal{S}_r^3$.

Now let $(\alpha,\beta,\gamma) \in \mathcal{S}_r^3$. The members of $\mathcal{V}_r$ preserve the conjugacy classes of α,β,γ. Using the fourth column of Table 2, it is easy to check that no permutation $\tau^{(i)}, 2 \le i \le 24$, has this property. This implies that μ is injective and in fact an isomorphism. ∎

Theorem 7. $\mathcal{G}_r = \mathcal{H}_r \times \mathcal{I}_r$.

Proof: It follows directly from Equation (8) that

$$\sigma\varrho = \varrho\sigma$$

for every $\sigma \in \mathcal{H}_r$ and $\varrho \in \mathcal{I}_r$. Since the union of the generating sets of $\mathcal{H}_r$ and $\mathcal{I}_r$ is a generating set of $\mathcal{G}_r$,

$$\mathcal{H}_r\mathcal{I}_r = \mathcal{G}_r.$$

Thus, we only have to show that

$$\mathcal{H}_r \cap \mathcal{I}_r = \{1\}. \tag{9}$$

For $r = 1,2$, the assumption is trivial, since $\mathcal{I}_r = \{1\}$. Now let $r > 2$. Every $\varrho \in \mathcal{I}_r$ preserves the conjugacy classes of all $(\alpha,\beta,\gamma) \in \mathcal{S}_r^3$. We remarked in the proof of Theorem 6 that no $\tau^{(i)}, 2 \le i \le 24$, has this property, hence, for all such $\tau^{(i)}$,

$$\tau^{(i)}\mathcal{V}_r \cap \mathcal{I}_r = \emptyset.$$

Thus, in order to prove Equation (9) it suffices to show that

$$\mathcal{V}_r \cap \mathcal{I}_r = \{1\}.$$

Let $\zeta \in \mathcal{V}_r \cap \mathcal{I}_r$. Since $\zeta \in \mathcal{V}_r$, there is a presentation

$$\zeta = \sigma_{i_1} \dots \sigma_{i_m} \quad (i_1, \dots, i_m \in \{1,2,3\}). \tag{10}$$

On the other hand,

$$\zeta = \tau_\pi \tag{11}$$

for some $\pi \in \mathcal{S}_r$ since $\zeta \in \mathcal{I}_r$. Now choose any $\alpha \in \mathcal{S}_r$. We conclude from (10) together with (7) that

$$\zeta(\alpha,\alpha,\alpha) = \alpha^m(\alpha,\alpha,\alpha)\alpha^{-m} = (\alpha,\alpha,\alpha)$$

and from (11) that

$$\zeta(\alpha,\alpha,\alpha) = \pi(\alpha,\alpha,\alpha)\pi^{-1}.$$

It follows that $\pi\alpha\pi^{-1} = \alpha$ for every $\alpha \in \mathcal{S}_r$, hence, $\pi = (1)$ since $r \ge 2$, which proves the theorem. ∎

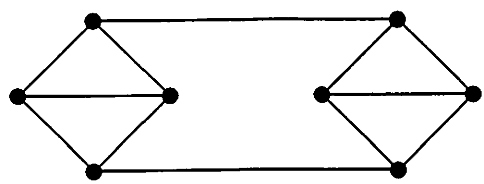
Corollary 8. *Let $r \geq 2$. Then*

$$|\mathcal{G}_r| = |S_4| \cdot |\mathcal{I}_r| \cdot |\mathcal{V}_r| = \begin{cases} 24 & \text{if } r = 2, \\ 24\, r!\, |\mathcal{V}_r| & \text{if } r \geq 3. \end{cases}$$

The group $\mathcal{G}_2$ consists precisely of the elements $\tau^{(i)}$ of Table 2. In this case we can use our results to count the 2-fold coverings of K_4 up to isomorphism. It is easy to check that the number of fixed points of $\tau^{(i)}$ is

$$\begin{array}{ll} -8 & \text{if } i = 1, \\ -4 & \text{if } 2 \leq i \leq 7 \text{ or } 16 \leq i \leq 18, \\ -2 & \text{otherwise.} \end{array}$$

Using Burnside's lemma, we obtain that the number of 2-fold covering projections of K_4 is 3. These are well known [10]: the trivial one; the 3-dimensional cube; and the following "hybrid":



This approach given here is easier than the use of the formula of [3, Theorem 3.4].

Note that there are four "obvious" equivalence class representatives in this case: $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$. What is surprising is that $(1,0,0)$ and $(1,1,0)$ are equivalent via τ_1.

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