On Decomposition of Graphs into Nonisomorphic Independent Sets

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Abstract. A decomposition into non-isomorphic matchings, or DINIM for short, is a partition of the edges of a graph G into matchings of different sizes. As a special case of our results, we prove that if a graph G has at least $(2\chi' - 2)\chi' + 1$ edges, where $\chi' = \chi'(G)$, is the chromatic index of G, then G has a DINIM. In particular, the n-dimensional cube, Q_n, $n \geq 4$, has a DINIM. These results confirm two conjectures which appeared in Chinn and Richter [3].

1. Introduction

Graphs in this paper are finite and have no multiple edges nor loops. We set $\Delta = \Delta(G)$ to be the maximum degree of vertices. By $V(G)$ and $E(G)$ we denote the vertex and the edge set of a graph G, respectively.

Given a graph G, a *decomposition into non-isomorphic matchings*, or DINIM for short, is a partition of $E(G)$ into sets $E_1, E_2, \ldots, E_t$ such that, for each i, E_i is a matching in G and if $i \neq j$, then $|E_i| \neq |E_j|$. The problem of having a DINIM in a graph was raised by Pavol Hell and discussed by Chinn and Richter in [3], where they proved that every sufficiently large 2-connected 3-regular graph has a DINIM.

The subject was considered also by Buhler, Chinn, Richter and Truszczynski [2],[4].

Pavol Hell, Chinn and Richter [3] raised several problems concerning DINIM. Two of them we solve here, namely,

Conjecture 1: Is it true that the n-dimensional cube, Q_n, $n \geq 4$ has a DINIM?

Conjecture 2: Is it true that for a given maximum degree of a graph G, there are only finitely many graphs (having no isolated vertices), having no DINIM?

These two conjectures will be an easy consequence of our results on decomposition into non-isomorphic independent sets.

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An *edge-coloring* of a graph G is a map $\phi : E(G) \to C$, where C is a set of colors, such that no two edges with the same color have a common vertex. The *chromatic index* $\chi'(G)$ is the least number j of colors such that G can be edge-colored with j colors. Similarly we define the *chromatic number* $\chi(G)$ to be the least number j of colors such that G has a *vertex-coloring*, namely, no two adjacent vertices has the same color, with j colors. A *balanced t-coloring* is a vertex coloring, using t colors, in which the color classes are almost of equal sizes.

2. Results and Proofs

We start with two simple but essential lemmas.

Lemma 2.1. *Let $a_1 \le a_2 \le \cdots \le a_n$ be positive integers such that $a_1 \ge 2n-1$. Then we can decompose a_i in the following form: $a_i = b_i + c_i$ such that b_i and c_i are non-negative integers satisfying $b_i \ne b_j$, $c_i \ne c_j$, $b_i \ne c_j$. (i.e. all the members of the decomposition are distinct).*

Proof: Choose $b_i = n - i$ and $c_i = a_i - n + i$. ∎

Lemma 2.2. *Let $a_1 \le a_2 \le \cdots \le a_n$ be positive integers such that*

(i) $0 \le a_n - a_1 \le 1$
(ii) *either* $a_1 \ge 2n-1$ *or* $a_1 = 2n-2$ *and* $a_n = 2n-1$.

Then the sequence admits a decomposition as in Lemma 2.1.

Proof: If (i) is satisfied with $a_1 \ge 2n-1$ then we are in the situation of Lemma 2.1 and we are through. So that we assume (i) is satisfied with $a_1 = 2n-2$ and $a_n = 2n-1$. For $i = 2,3,\ldots,n-1$ denote $b_{i-1} = a_i$. Let m be the first place such that $j \ge m$, $b_j = 2n-1$ (it might be that all the b's are exactly $2n-2$). Then we decompose the sets in the following way:

$$b_i \to (i, 2n-2-i), \qquad i = 1,2,\ldots,m-1,$$

and

$$b_i \to (i+1, 2n-2-i), \qquad i = m, m+1, \ldots, n-2,$$

where the arrow indicates the exact decomposition. One can see that this decomposition with the integers a_1 and a_n admits the required decomposition. ∎

Lemma 2.3. *Let $a_1 \le a_2 \le \cdots \le a_n$ be positive integers such that*

(i) $0 \le a_n - a_1 \le 1$ *and*
(ii) $\sum_{i=1}^{n} a_i \ge n(2n-2) + 1$.

Then the sequence admits a decomposition as in Lemma 2.1.

Proof: If $a_1 \ge 2n-1$ then we have a decomposition as in Lemma 2.1. So we may assume $a_1 = 2n-2$. If also $a_n = 2n-2$ then we get $\sum_{i=1}^{n} a_i = n(2n-2)$

a contradiction to (ii). Hence, $a_n = 2n - 1$ so that by Lemma 2.2 we have the required decomposition.

If $a_1 \leq 2n - 3$ then it follows that $a_n \leq 2n - 2$ so that we get $\sum_{i=1}^{n} a_i < n(2n - 2)$ a contradiction to (ii). Hence the Lemma is proved. ∎

The next result is of interest and will be useful in the sequel. It is also related to some earlier work of Favaron [6].

Theorem 2.4. *Let G be a $K_{1,m}$-free graph. Then,*

(i) *If A and B are two independent sets in G then $\Delta(A \cup B) \leq m - 1$.*
(ii) $\chi - 1 \leq \Delta(G) \leq (m - 1)(\chi - 1)$.
(iii) *Let β_0 and β_0' be the maximum and the minimum sizes of a maximal vertex independent set, respectively. Then, $\beta_0' \leq \beta_0 \leq (m - 1)\beta_0'$.*

Proof:

(i) Consider a vertex of maximum degree in $A \cup B$. Since G is a $K_{1,m}$-free graph and both A and B are independent, the result follows immediately.
(ii) The left hand side follows immediately. For the right hand side consider the χ coloring classes of G. Let v be a vertex that realizes the maximum degree, $\Delta(G)$. By (i) v has at most $m - 1$ adjacent vertices in each color class. Hence, $\Delta(G) \leq (m - 1)(\chi - 1)$.
(iii) The proof uses similar arguments as in (ii). ∎

Theorem 2.5. *Let be a $K_{1,3}$-free graph. Then there is a balanced $\chi(G)$-coloring of $V(G)$.*

Proof: Let $V_1, V_2, \ldots, V_\chi$ be the color classes with cardinalities $a_1 \leq a_2 \leq \cdots \leq a_\chi$. Suppose $a_\chi - a_1 \geq 2$. Consider the bipartite graph on the vertex set $V_1 \cup V_\chi$. By Theorem 2.4(i) $\Delta(V_1 \cup V_\chi) \leq 2$. Hence, the components of this graph consist of paths and even cycles. Since $a_\chi - a_1 \geq 2$, there must exist a path of odd length in which the number of vertices from V_χ is greater by one then that from V_1. Interchange the colors on this path to reduce the difference between a_χ and a_1. Repeat this argument as long as neccesary. ∎

Theorem 2.6. *Let G be a $K_{1,3}$-free graph such that $|V(G)| \geq (2\chi - 2)\chi + 1$. Then $V(G)$ has a decomposition into independent sets of distinct orders.*

Proof: By Theorem 2.5 G has a balanced χ coloring of $V(G)$. Since $|V(G)| \geq (2\chi - 2)\chi + 1$, we may use Lemma 2.3 and the result follows. ∎

Corollary. *Let G be a graph such that $|E(G)| \geq (2\chi' - 2)\chi' + 1$. Then G has a DINIM.*

Proof: Take the line graph of G, $L(G)$. Then it is well known that $L(G)$ is a $K_{1,3}$-free, and by Theorem 2.6 we are done. ∎

Remark 1: Using Vizing's Theorem [8], (in the "worst" case, namely, $\chi' = \Delta + 1$) one may get a more tractable condition namely, if, $|E(G)| \geq 2\Delta(\Delta+1)+1$ then G has a DINIM. In particular if $\Delta(G) \leq 3$ and $|E(G)| \geq 25$ then G has a DINIM, a result which improves significantly that of [3]. ∎

Remark 2: Observe that in the n-dimensional cube, Q_n, $|E(Q_n)| = 2^{n-1}n$ while $\Delta(Q_n) = n$. But, $2^{n-1}n > 2n(n+1) + 1$ for $n \geq 5$. Hence Q_n, $n \geq 5$ has a DINIM. The case $n = 4$ was solved in [3].

Remark 3: The bounds of Theorem 2.6 and its corollary are tight as the following extremal graphs show.

(i) For each $\chi \geq 3$ consider the graph G to be a disjoint union of $2\chi - 2$ copies of K_χ. Then $|V(G)| = (2\chi-2)\chi$, and we do not have the required vertex-decomposition.
(ii) For each $\chi' \geq 3$ consider the graph G to be a disjoint union of $2\chi' - 2$ copies of the star $K_{1,\chi'}$. Then $|E(G)| = (2\chi'-2)\chi'$, and we do not have the required DINIM. ∎

Theorem 2.7. *Let G be a $K_{1,m}$-free graph such that $|V(G)| \geq 1 + (2\chi - 2)(1+(\chi-1)(m-1))$. Then $V(G)$ has a decomposition into independent sets of distinct orders.*

Proof: Let $V_1, V_2, \ldots, V_\chi$ be the color classes with cardinalities $a_1 \leq a_2 \leq \cdots \leq a_\chi$. Suppose $a_1 \leq 2\chi - 2$. Recall Theorem 2.4(iii). Hence we may assume that $a_\chi \leq (2\chi-2)(m-1)$. Thus, $\sum_{i=1}^{i=\chi} a_i \leq (2\chi-2)(1+(\chi-1)(m-1))$, a contradiction to the order of $V(G)$. Hence, $a_1 \geq 2\chi - 1$ so that by Lemma 2.1 we are done. ∎

The next result is a general theorem concerning a vertex decomposition into independent sets of distinct order.

Theorem 2.8. *Let G be a graph such that $|V(G)| \geq (\Delta+1)2\Delta + 1$. Then $V(G)$ has a decomposition into independent sets of distinct orders.*

Proof: Let $V_1, V_2, \ldots, V_{\Delta+1}$ be color classes with cardinalities $a_1 \leq a_2 \leq \cdots \leq a_{\Delta+1}$. Since we use $\Delta + 1$ colors we may apply the deep Theorem of Hajnal and Szemeredi [7] which claims that $V(G)$ has a balanced-$\Delta+1$-coloring. Since $V(G)$ satisfy the condition of the theorem we have the required decomposition by Lemma 2.3. ∎

Remark 4: The bound of Theorem 2.8 is tight. Consider the graph G to be a union of 2Δ copies of the complete graph $K_{\Delta+1}$. Then $|V(G)| = 2\Delta(\Delta+1)$ and we do not have the required decomposition.

Remark 5: The notion of balanced-coloring is perfectly suitable to deal with matroids (see e.g. [9]). By a "well known" theorem of Edmonds [5] the minimum number of independent sets that cover the matroid M is given by the formula

$\chi(M) = \max_{A \subset M} \lceil \frac{|A|}{\rho(A)} \rceil$, where $\rho(A)$ is the rank function of the matroid. Moreover this equality is satisfied by a balanced-$\chi(M)$-coloring. Hence by Lemma 2.3 if M is a matroid such that $|M| \geq (2\chi(M) - 2)\chi(M) + 1$ then there is a decomposition of M into independent subsets of distinct orders. ∎

The following theorem is a particular example of a result that belongs to both Matroid Theory and Graph Theory.

Theorem 2.9. *Let G be a graph with nonempty edge set. Then $E(G)$ has a decomposition into forests of distinct sizes.*

Proof: The proof is by induction on the number of edges in G and in particular we shall show that such a decomposition exists using at least $\delta(G)$ forests.

For $|E(G)| = 1$ the claim is obvious. Let v be a vertex of minimum degree. Consider three cases.

Case 1. *$G\backslash v$ is empty.*

In this case it follows that $G = K_2$ and the result follows.

Case 2. $\delta(G\backslash v) \geq \delta(G)$.

In this case by the induction hypothesis $E(G\backslash v)$ has a decomposition into forests of distinct sizes using at least $\delta(G)$ forests. Now we add each of the edges incident with v to the $\delta(G)$ largest forests and we are done.

Case 3. $\delta(G\backslash v) < \delta(G)$. ∎

In this case it follows that $\delta(G\backslash v) = \delta(G) - 1$. Then by the induction hypothesis we have a decomposition using at least $\delta(G) - 1$ forests of distinct sizes. If this decomposition uses at least $\delta(G)$ forests then we apply the same argument as in Case 2. Otherwise there are exactly $\delta(G) - 1$ forests. Add to each of those forests one edge incident to v. Then the sizes of the forests increase by one and we have an additional forest of size exactly one coming from the remaining edge incident to v. Notice that now the number of forests is $\delta(G)$. ∎

Remark 6: One can see that in Theorem 2.9 the number of forests we need is bounded by $\max_{H \subset G} \delta(H)$, a parameter of the graph that often appears in coloring problems. In particular, for planar graphs 5 forests will suffice. ∎

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