Partitioning Graphs into Induced Stars

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Abstract. A partition $\mathcal{D} = \{V_1, \ldots, V_m\}$ of the vertex set $V(G)$ of a graph G is said to be a *star decomposition* if each $V_i (1 \leq i \leq m)$ induces a star of order at least two. In this note, we prove that a connected graph G has a star decomposition if and only if G has a block which is not a complete graph of odd order.

A star is a bipartite graph $K_{1,n}$ with $n \geq 1$. Several works have been done on decomposing graphs into stars. For example, a 1-factor can be viewed as a decomposition of a graph into $K_{1,1}$. In [1] Amahashi and Kano have considered a generalization of 1-factors. A spanning subgraph H is said to be a $\{K_{1,1}, K_{1,2}, \ldots K_{1,n}\}$-factor if each component of H is isomorphic to one of $\{K_{1,1}, \ldots, K_{1,n}\}$. This is considered to be a decomposition of a graph into stars of order at most $n+1$. On this decomposition, they have proved the following theorem.

Theorem A. ([1]). *Let G be a graph and let $n \geq 2$ be an integer. Then G has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor if and only if $i(G-S) \leq n|S|$ for every $S \subset V(G)$, where $i(G-S)$ is the number of isolated vertices in $G-S$.*

It is easy to see that every graph has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor for some $n \geq 1$.

In this note, we also consider a decomposition of a graph into stars, but we put a further requirement that each vertex set in the decomposition *induces* a star. If we add this requirement, not every graph has a required decomposition. For example, a complete graph of odd order cannot be decomposed into induced stars.

Let G be a graph. For $S \subset V(G)$, we write $G[S]$ for the subgraph of G induced by S. Then a partition $\mathcal{D} = \{V_1, \ldots, V_m\}$ of $V(G)$ is said to be a *star*

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decomposition if for each $V_i(1 \leq i \leq m)$ $G[V_i]$ is a star. We consider that the null graph (the graph whose vertex set is empty) has a *star decomposition.* In this note we give a necessary and sufficient condition for a graph to have a star decomposition.

A connected graph G is said to be an *odd complete-cactus*, or an OCC, if each block of G is a complete graph of odd order. Note that every OCC has an odd order. By the definition, if x is a non-cutvertex of an OCC, then the neighborhood of x induces a complete graph of even order.

The result of this note is the following theorem.

Theorem 1. *Let G be a connected graph. Then G has a star decomposition if and only if G is not an OCC.*

In this paper we write $\Gamma_G(x)$ for the neighborhood of $x \in V(G)$ in G. For other notation we refer the reader to [2].

In order to prove Theorem 1, we use the following two easy lemmas.

Lemma 2. *Let G be an OCC of order at least three, and let $x \in V(G)$. Then no component of $G - x$ is an OCC.* ∎

Proof. Each component of $G - x$ has an even order.

Lemma 3. *Let G be a 2-connected graph. Then for some $xy \in E(G)$, $G - \{x, y\}$ is connected.*

Proof. Assume $G - \{x, y\}$ is disconnected for an edge $xy \in E(G)$. Since G is 2-connected, $G - x$ is connected, and y is a cutvertex of $G - x$. Let B be an end-block of $G - x$ and let c be the unique cutvertex of $G - x$ contained in B. Because of the 2-connectedness of G, $\Gamma_G(x) \cap (B - c) \neq \emptyset$. Let $y' \in \Gamma_G(x) \cap (B - c)$. Then $G - \{x, y'\}$ is connected. ∎

Proof of Theorem 1. First, we prove the necessity by induction on $|V(G)|$. Let G be an odd cactus. Suppose G has a decomposition $\mathcal{D}$. Since a complete graph of odd order has no star decomposition, we may assume that G has an end-block B with a cutvertex v. It is easy to see that $\mathcal{D}$ induces a 1-factor in $B - \{v\}$, so the family obtained from $\mathcal{D}$ by removing this 1-factor is a star decomposition of $G - (B - \{v\})$. However, $G - (B - \{v\})$ is an OCC and thus it has no star decomposition by the induction hypothesis. This is a contradiction.

Next, we prove the sufficiency. Again we proceed by induction on $|V(G)|$. Suppose G is not an OCC. If $|V(G)| \leq 4$, it is easy to see that G has a star decomposition. Therefore, we may assume $|V(G)| \geq 5$. We consider two cases.

Case 1. G is not 2-connected.

In this case G has a cutvertex c and let the components of $G - c$ be $C_1, \ldots, C_r$ $(r \geq 2)$. Let $H_i = G[C_i \cup \{c\}]$ $(1 \leq i \leq r)$. If every $H_i(1 \leq i \leq r)$ is an OCC, then G itself is an OCC, a contradiction. Therefore, we may asume that H_1 is not an OCC. We may also assume that $C_2, \ldots, C_s$ are OCC's and that $C_{s+1}, \ldots, C_r$

are not OCC's ($1 \leq s \leq r$). Since G is connected, $\Gamma_G(c) \cap C_i \neq \emptyset$ ($1 \leq i \leq r$). Let $x_i \in C_i \cap \Gamma_G(C)$ and let $J_i = C_i - x_i$ ($2 \leq i \leq s$). By Lemma 2, if $|C_i| \geq 3$, then no component of J_i is an OCC. Let $K_0 = J_2 \cup \cdots \cup C_{s+1} \cup \cdots \cup C_r$. (Possibly it is a null graph.) By the induction hypothesis K_0 has a star decomposition.

Let $H_0 = G[V(H_1) \cup \{x_2, \ldots, x_s\}]$. Then H_0 is connected and $G - V(H_0) = K_0$. Since H_1 is not an OCC, H_0 is not an OCC. Therefore, if K_0 is not a null graph, by the induction hypothesis H_0 has a star decomposition. Since both K_0 and H_0 have star decompositions, G also has a star decomposition. Therefore, we may assume that K_0 is a null graph and $G = H_0$. This implies $r = s$.

Assume $C_1 = H_1 - c$ is not an OCC. Then by the induction hypothesis C_1 has a star decomposition $\mathcal{D}_0$. Let $V_0 = \{c, x_2, \ldots, x_r\}$. Then $G[V_0] \simeq K_{1,r-1}$ and this is a star since $r \geq 2$. Therefore, $\mathcal{D}_0 \cup \{V_0\}$ is a star decomposition of G. Thus we may assume that C_1 is an OCC. Let $x_1 \in \Gamma_G(c) \cap C_1$. By Lemma 2, no component of $C_1 - x_1$ is an OCC. Then by the induction hypothesis $C_1 - x_1$ has a star decomposition $\mathcal{D}_1$. (Possibly $C_1 - x_1$ is a null graph.) Let $V_1 = \{c, x_1, x_2, \ldots, x_r\}$. Then $G[V_1] \simeq K_{1,r}$ which is a star. Thus $\mathcal{D}_1 \cup \{V_1\}$ is a star decomposition of G.

Case 2. G is 2-connected.

By Lemma 3, $G - \{x, y\}$ is connected for some $xy \in E(G)$. If $G - \{x, y\}$ is not an OCC, then $G - \{x, y\}$ has a star decomposition $\mathcal{D}$ by the induction hypothesis. Then $\{\{x, y\}\} \cup \mathcal{D}$ is a star decomposition of G. Therefore, we may assume that $G - \{x, y\}$ is an OCC. In particular, $|V(G)| \equiv |V(G) - \{x, y\}| \equiv 1 \pmod 2$.

We consider two subcases.

Subcase 2.1. $G - \{x, y\}$ is 2-connected.

Since $G - \{x, y\}$ is an OCC and $|V(G)| \geq 5$, $G \simeq K_{2n+1}$ for some $n > 2$. If $|\Gamma_G(v) \cap \{x, y\}| = 1$, say $vx \in E(G)$ and $vy \notin E(G)$, for some $v \in V(G) - \{x, y\}$, then let $V_0 = \{x, y, v\}$. Since $|V(G) - V_0| \equiv 0 \pmod 2$, $G - V_0$ is not an OCC. Since $G - \{x, y\}$ is 2-connected, $G - V_0$ is connected. Then by the induction hypothesis $G - V_0$ has a star decomposition $\mathcal{D}$. Hence $\{V_0\} \cup \mathcal{D}$ is a star decomposition of G since $G[V_0] \simeq K_{1,2}$. Therefore, we may assume $|\Gamma_G(v) \cap \{x, y\}| = 0$ or 2 for all $v \in V(G) - \{x, y\}$. Let

$$U_i = \{v \in V(G) - \{x, y\} : |\Gamma_G(v) \cap \{x, y\}| = i\} \quad (i = 0, 2).$$

Since G is 2-connected, $|U_2| \geq 2$. On the other hand, if $U_0 = \emptyset$, then $V(G) - \{x, y\} = U_2$. This means $G \simeq K_{2n+3}$, and hence G is an OCC. This contradicts the assumption. Therefore, we have $U_0 \neq \emptyset$.

Let $v \in U_0$ and $u \in U_2$. Since $G - \{x, y\} \simeq K_{2n+1}$, $uv \in E(G)$. Let $V_0 = \{x, u, v\}$. Since $n \geq 2$ and $|U_2| \geq 2$, $G - V_0$ is connected. Since $|V(G) - V_0| \equiv 0 \pmod 2$, $G - V_0$ is not an OCC. By the induction hypothesis $G - V_0$ has a star decomposition. Since $G[V_0] \simeq K_{1,2}$, G also has a star decomposition. Therefore, the theorem follows in this subcase.

Subcase 2.2. $G - \{x, y\}$ is not 2-connected.

In this case $G - \{x, y\}$ has at least two end-blocks B_1 and B_2. Since B_i is a complete graph of odd order, $|B_i| \geq 3$ $(i = 1, 2)$. Let c_i be the unique cutvertex of G contained in B_i $(i = 1, 2)$. (Possibly $c_1 = c_2$.)

Since G is 2-connected, $(\Gamma_G(x) \cup \Gamma_G(y)) \cap (B_i - c_i) \neq \emptyset$ $(i = 1, 2)$. Then by arguments similar to those in Subcase 2.1, we have

$$(B_1 - c_1) \cup (B_2 - c_2) \subset \Gamma_G(x) \cap \Gamma_G(y).$$

Let $v \in B_1 - c_1$ and $H = G - \{x, v\}$. Then $xv \in E(G)$ and H is connected.

We claim that H is not an OCC. Assume that H is an OCC. Since $H - y = G - \{x, y\} - v$ is connected, y is not a cutvertex of H. Then $H[\Gamma_H(y)]$ is a complete graph (of even order). However, since $\Gamma_G(y) \supset B_1 - \{c_1, v\} \neq \emptyset$ and $\Gamma_G(y) \supset B_2 - c_2 \neq \emptyset$, this is impossible. Therefore, the claim follows.

By the induction hypothesis H has a star decomposition $\mathcal{D}$. Then $\mathcal{D} \cup \{\{x, v\}\}$ is a star decomposition of G.

Therefore, Case 2 is proved, and hence the proof of the theorem is complete. ∎

References

[1] A. Amahashi and M. Kano, *On factors with given components*, Discrete Math. 42 (1982), 1–6.

[2] G. Chartrand and L. Lesniak, "Graphs and Digraphs, 2nd Ed.", Wadsworth, Belmont, CA, 1986.