Steiner Triple Systems with a Given Autormorphism

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Abstract. We give necessary and sufficient conditions on the order of a Steiner triple system admitting an automorphism π, consisting of 1 large cycle, several cycles of length 4 and a fixed point.

1. Introduction.

A Steiner triple system of order v, denoted $S(v)$, is an ordered pair (X, B), where X is a set of cardinality v, and B is a set of 3-subsets of X, called blocks, such that any 2-subset of X is contained in a unique block. A Steiner triple system of order v with a hole of size w, denoted $S(v)_w$, can be defined as follows:

Let X be a set of size v, $X' \subset X$, $|X'| = w$ and B a set of 3-subsets of X, called blocks. Then (X, X', B) is an $S(v)_w$ if

i. no 2-subset of X' is contained in any block; and
ii. all other 2-subsets of X are contained in a unique block.

An automorphism of an $S(v)$, or an $S(v)_w$, is a permutation π of the set X that preserves the blocks in B. π is said to be of type $[\pi] = [\pi_1, \pi_2, \ldots, \pi_v]$ if the disjoint cyclic decomposition of π has π_i cycles of length i. So $\sum i\pi_i = v$. A question of concern has been that of given a particular automorphism type, does there exist an $S_\pi(v)$? A more general question is: for a particular automorphism type, does there exist an $S_\pi(v)_w$? Then when possible, extend the $S_\pi(v)_w$ to an $S_\pi(v)$. For $[\pi] = [1, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ and $[0, 0, 1, 0, \ldots, 0, 1, 0, 0, 0]$, necessary and sufficient conditions have been shown for the existence of $S_\pi(v)_w$ and $S_\pi(v)$ [1]. In this paper, we give necessary and sufficient conditions for the existence of $S_\pi(v)_w$ and $S_\pi(v)$ when $[\pi] = [1, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$.

2. $S_\pi(v)_w$ with $[\pi] = [1, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$.

The automorphism has 1 fixed point, t 4-cycles and 1 cycle of length d. So $v = d + 4t + 1$. For the $S_\pi(v)_w$, let:

$$X = \{\infty, x_1, y_1, z_1, w_1, \ldots, x_t, y_t, z_t, w_y, 0, 1, 2, \ldots, d-1\}$$
$$X' = \{\infty, x_1, y_1, z_1, w_1, \ldots, x_t, y_t, z_t, w_t\}$$
$$(\pi) = (\infty)(x_1, y_1, z_1, w_1)\ldots(x_t, y_t, z_t, w_t)(0, 1, 2, \ldots, d-1)$$

Lemma 2.1. *Let* (π): $(0_0, 1_0, \ldots, s_0-1)$ $(0_1, 1_1, \ldots, s_1-1) \ldots, (0_k, 1_k, \ldots, s_k-1)$ *be a permutation of the* v*-element set* $X = \{0_0, 1_0, \ldots, s_0-1, 0_1, 1_1, \ldots,$

$s_1 - 1, \ldots, 0_k, 1_k, \ldots, s_k - 1\}$. *Let* $X' = \{0_1, 1_1, \ldots, s_1 - 1, \ldots, 0_k, 1_k, \ldots, s_k - 1\}$, $|X'| = w$. *Then if* $S_\pi(v)_w$ *on* X *with hole on* X' *exists,* $s_i \mid s_0$ *for each* $1 \leq i \leq k$.

Proof: Let $a_i \in \{0_i, 1_i, \ldots, s_i - 1\}, 1 \leq i \leq k$. Then $\{0_0, a_i\}$ must be in a block of the form $\{a_i, 0_0, x_0\}$. Apply π s_0 times to get $\{a_i + s_0, 0_0, x_0\}$ as a block in the $S_\pi(v)_w$. Then $a_i = (a_i + s_0) \pmod{s_i}$. So $s_i \mid s_0$. ∎

By the above lemma, $4 \mid d$. So let $d = 4e$. Since every 2-subset of X/X' must appear in a block, each of the differences in the set $S = \{1, 2, \ldots, 2e\}$ must be in a difference triple. For each of the 4-cycles, either 0 or 2 odd differences are required, hence, an even number of odd differences in S will be in blocks of the form $\{x_i, a, b\}, x_i \in X', a, b, \in X/X'$. The remaining odd differences in S must occur in pairs in difference triples since any difference triple has 2 odd differences or 0 odd differences. S contains e odd differences. If $2j$ odd differences are removed, $e - 2j$ must be even and so e must be even.

We have the following necessary conditions for the existence of an $S_\pi(v)_w$ on X with hole on X':

(1) $d = 4e$ and e must be even;
(2) $t \leq \lfloor \frac{3}{4}e \rfloor$, with strict inequality if $e = 6, 12, 14, 18$ or $20 \pmod{24}$;
(3) If 3 divides e, then $3 \mid (2e - 2t - 2)$.
If 3 does not divide e, then $3 \mid (2e - 2t - 1)$.

For a complete proof of 2. and 3. see [1]. These necessary conditions are equivalent to partitioning the set $S = \{1, 2, \ldots, 2e - 1\}$, $e = \frac{d}{4}$, $d = v - 4t - 1$, into sets A, B, and C such that

(1) A can be partitioned into difference triples $\pmod{d}$;
(2) B is $2t$-set of even number of odd numbers and an even number of even numbers congruent to $2 \pmod{4}$;
(3) $C = \emptyset$ or $C = \{\frac{1}{3}e\}$.

Given the partition of S into sets A, B, and C, form started blocks of the following types:

Type I: For each $a, b, c \in A$ such that (a, b, c) is a difference triple, with $a + b = c$ or $a + b + c \equiv 0 \pmod{4e}$, form $\{0, a, a + b\}$.

Type II: For each i, $1 \leq i \leq t$, take 2 numbers a and b from B, with a and b both even or both odd.
If $a, b \equiv 2 \pmod{4}$ or if $a \equiv 3 \pmod{4}$ and $b \equiv 1 \pmod{4}$, take $\{x_i, 0, a\}$ and $\{x_i, 1, 1 + b\}$.
If $a, b \equiv 1 \pmod{4}$ or $a, b \equiv 3 \pmod{4}$, take $\{x_i, 0, a\}$ and $\{x_i, 2, 2 + b\}$.
If $a \equiv 1 \pmod{4}$ and $b \equiv 3 \pmod{4}$, take $\{x_i, 0, a\}$ and $\{x_i, 3, 3 + b\}$.

Type III: If $C = \{\frac{1}{3}e\}$, form $\{0, \frac{e}{3}, \frac{2e}{3}\}$.

Type IV: $\{\infty, 0, 2e\}$.

For a proof that these blocks form an $S_\pi(v)_w$ see [1].

According to the necessary conditions, the following table gives values of e and corresponding values of t for which $S = \{1, 2, \ldots, 2e-1\}$ must be partitioned. The third column gives the range of t since $0 \leq t \leq \lfloor \frac{3e}{4} \rfloor$, with strict inequality for $e \equiv 6, 12, 14, 18$ or $20 \pmod{24}$.

$t=3k+n, n \in Z_3$	$e=6s+m, m \in Z_6$	range of t
n	m	
0	2	$s \equiv 0 \pmod 4): 0 \leq k \leq \frac{3s}{2}$
		$s \equiv 1 \pmod 4): 0 \leq k \leq \frac{3s+1}{2}$
		$s \equiv 2 \pmod 4): 0 \leq k \leq \lfloor \frac{3s}{2} \rfloor - 1$
		$s \equiv 3 \pmod 4): 1 \leq k \leq \lfloor \frac{3s}{2} \rfloor$
1	0	$0 \leq k \leq \lfloor \frac{3s}{2} \rfloor / 2$
2	0	$s \equiv 0 \pmod 2): 0 \leq k \leq \frac{3s-2}{2}$
		$s \equiv 1 \pmod 2): 0 \leq k \leq \frac{3(s-1)}{2}$
	4	$0 \leq k \leq \lfloor \frac{3s}{2} \rfloor$

3. Extending the $S_\pi(v)_w$ to a $S_\pi(v)$.

Since $v \equiv 1$ or $3 \pmod 6$ for an $S_\pi(v)$, the only possible values of t are $t \equiv 0$ or $2 \pmod 3$, if the $S_\pi(v)_w$ is to be extended to an $S_\pi(v)$. Also, in order to extend to an $S_\pi(v)$, an $S_{\pi_1}(w)$ is needed with $[\pi_1] = [1, 0, 0, \frac{w-1}{4}, 0, \ldots, 0]$. If an $S_{\pi_1}(w)$ exists, then an $S_{(\pi_1)^2}(w)$ exists with $[(\pi_1)^2] = [1, \frac{w-1}{2}, 0, \ldots, 0]$, which is a reverse Steiner triple system and has been shown to exist iff $w \equiv 1, 3, 9$ or $19 \pmod{24}$ [7]. Hence, $t = \frac{w-1}{4}$ results in $t \equiv 0$ or $2 \pmod 6$ for the $S_{\pi_i}(w)$. If $t \equiv 0 \pmod 6$, $w \equiv 1 \pmod{24}$. By [4], a 2-rotational Steiner triple system exists for $w \equiv 1 \pmod{24}$. If π is 2-rotational, $\pi^{\frac{w-1}{8}}$ is of type $[1, 0, 0, \frac{w-1}{4}, 0, \ldots, 0]$ obtaining the desired automorphism. If $t \equiv 2 \pmod 6$, $w \equiv 9 \pmod{24}$. Again a 2-rotational $S_\pi(w)$ exists for $w \equiv 9 \pmod{24}$ and $\pi^{\frac{w-1}{8}}$ is of the form $[1, 0, 0, \frac{w-1}{4}, 0, \ldots]$.

In conclusion, we have that a $S_\pi(w)$, $[\pi] = [1, 0, 0, t, 0, \ldots, 0, 1, 0, \ldots, 0]$ exists iff $v \equiv 1 \pmod{24}$ and $t \equiv 2 \pmod 6$ or $v \equiv 9 \pmod{24}$ and $t \equiv 0$ or $2 \pmod 6$, $t \leq \lfloor \frac{3}{4e} \rfloor$ with strict inequality if $e \equiv 6, 12, 14, 18$ or $20 \pmod{24}$.

The size of t varies with each value of e. In most cases as t increases by 3, each new partition of S is based on the previous partition. Hence, it is possible to set up a recursion form based on varying sizes of t corresponding to particular values of s. So for $S = \{1, 2, \ldots, 2e-1\}$, $0 \leq k \leq j$, j as determined by the range of

k for values of e, let:

$A_0 = \{ (x,y,z) | x,y,z \in A$ and (x,y,z) is a difference triple (mod $4e$)$\}$
$B_0 = \{a_1, a_2, \ldots, a_{2t}$ where $a_1, \ldots, a_{2t}$ are elements of S satisfying the conditions of set $B\}$
$C_0 = \{4e/3\}$ or $\emptyset$
$A_k = A_{k-1} / \{(a_1, b_1, c_1), (a_2, b_2, c_2)$ with the set $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ containing an even number of odds or an even number of numbers congruent to 2 (mod 4) $\}$
$B_k = B_{k-1} \cup \{a_1, b_2, c_1, a_2, b_2, c_2\}$
$C_k = C_0$ for $0 \le k \le j$

We show here a partition of $S = \{1,2,\ldots,2e-1\}$ for $t \equiv 1 \pmod 3$ and $e \equiv 0 \pmod 6$. The remaining partitions for each value of e and corresponding values of t are given in [1], hence, showing that the necessary conditions for the $S_\pi(v)_w$, are also sufficient.

For the partition of S, let $e = 6s$, $t = 3k+1$ and $S = \{1,2,\ldots,12s-1\}$. There are 4 cases:

$$s \equiv 0 \pmod 4, \quad t = 3k+1 \text{ for } 0 \le k \le \frac{3s}{2} - 1$$
$$s \equiv 1 \pmod 4, \quad t = 3k+1 \text{ for } 0 \le k \le \left\lfloor \frac{3s}{2} \right\rfloor - 1$$
$$s \equiv 2 \pmod 4, \quad t = 3k+1 \text{ for } 0 \le k \le \left\lfloor \frac{3s}{2} \right\rfloor - 1$$
$$s \equiv 3 \pmod 4, \quad t = 3k+1 \text{ for } 0 \le k \le \left\lfloor \frac{3s}{2} \right\rfloor - 1$$
$$C_k = \emptyset \text{ for all } k$$

$s \equiv 0 \pmod 4$.

i.

$A_0 = \{(1+2m,\ 12s-2-4m,\ 12s-1-2m);\ m=2,3,\ldots,3s-1,$
$(4, 12s-6,\ 12s-2),\ (1, 12s-4,\ 12s-3),\ (4r, 4(a_r+s),\ 4(b_r+s))\,;$
$r = 2,3,\ldots,s$ with (a_r, b_r) as defined below $\}$

$B_0 = \{3, 12s-1\}$

(a_r, b_r), $r = 2,3,\dots,s$[15]

(1) $\left(\frac{s}{4}, \frac{5s}{4}-1\right), \left(\frac{s}{2}, \frac{3s}{2}\right), \left(s-1, \frac{3s}{2}-1\right)$

(2) $\left(\frac{s}{4}+m,\ \frac{3s}{4}-m\right);\ m = 1,2,\dots,\frac{s}{4}-1$

(3) $(s-1+m,\ 2s-1-m);\ m = 1,2,\dots,\frac{s}{4}-1$

(4) $(m, s-1-m);\ m = 1,2,\dots,\frac{s}{4}-1$

(5) $\left(\frac{5s}{4}-1+m,\ \frac{7s}{4}-m\right);\ m = 1,2,\dots,\frac{s}{4}-1$

This is based on Davies [3] using Theorem 2 with the case $n = 4m-1$.

$$
\begin{aligned}
A_k &= A_{k-1} / \{(1+4k, 12s-2-8k, 12s-1-4k),\\
&\quad (3+4k, 12s-8k-6, 12s-4k-3)\}\\
B_k &= B_{k-1} \cup \{1+4k, 12s-2-8k, 12s-1-4k,\\
&\quad 3+4k, 12s-8k-6, 12s-4k-3\}
\end{aligned}
$$

$s \equiv 1 \pmod 4$, $s \geq 13$.

$$
\begin{aligned}
A_0 = \{ &(1+2m, 12s-2-4m, 12s-1-2m);\ m = 2,3,\dots,3s-1,\\
&(4, 12s-6, 12s-2), (8, 12s-16, 12s-8),\\
&(12, 12s-24, 12s-12), (1, 12s-4, 12s-3), (4r, 4(a_r+s), 4(b_r+s));\\
&r = 4,5,\dots,s \text{ with } (a_r, b_r) \text{ as defined below}\}\\
B_0 = \{&3, 12s-1\}
\end{aligned}
$$

For $1 \leq k \leq \lfloor \frac{3s}{2} \rfloor - 1$:

$$
\begin{aligned}
A_k &= A_{k-1} / \{(1+4k, 12s-2-8k, 12s-1-4k),\\
&\quad (3+4k, 12s-8k-6, 12s-4k-3)\}
\end{aligned}
$$

$B_k = B_{k-1} \cup \{1+4k, 12s-1-4k, 12s-2-8k,$
$3+4k, 12s-4k-3, 12s-8k-6\}$
$(a_r, b_r),\ r = 4,5,\dots,s$

(1) $\left(\frac{s-9}{2}-m, \frac{s-1}{2}+m\right); m=0,1,\dots,\frac{s-13}{4}$

(2) $\left(\frac{s-9}{4}-m, \frac{3s-3}{2}+m\right); m=0,\dots,\frac{s-13}{4}$

(3) $\left(\frac{3s-13}{2}-m, \frac{3s-3}{2}+m\right); m=0,\dots,\frac{s-13}{4}-1$

(4) $\left(\frac{5s-17}{4}-m, \frac{7s-15}{4}+m\right); m=0,\dots,\frac{s-13}{4}$

(5) $(s-, 2s-5), \left(s-3, \frac{3s-7}{2}\right), \left(\frac{3s-11}{4}, \frac{7s-19}{4}\right),$
$\left(\frac{3s-7}{9}, \frac{5s-13}{4}\right), \left(\frac{s-3}{2}, \frac{3s-11}{2}\right)$

(6) $\left(\frac{s-7}{2}+m, \frac{3s-9}{2}+2m\right); m=0,1.$

This is based on Simpson [6] using a hooked sequence $\{4,5,\dots,4+m-1\}$, $m=s-3$.

$s = 1$

$A_0 = \{(4,6,10),(1,8,9),(2,5,7)\}$
$B_0 = \{3,11\}$

$s = 5$

$A_0 = \{ (1,56,57),(4,54,58),(8,40,48),(12,24,36),$
$(16,28,44),(20,32,52),(1+2m, 12s-2-4m, 12s-1-2m);$
$m = 2,\dots,3s-1 \}$
$B_0 = \{3,59\}$

For $1 \le k \le 6$:

$$A_k = A_{k-1} / \{(1+4k, 58-8k, 59-4k), (3+4k, 54-8k, 57-4k)\}$$
$$B_k = B_k \cup \{1+4k, 59-4k, 58-8k, 3+4k, 57-4k, 54-8k\}$$

$s = 9$

$$A_0 = \{\ (1+2m, 12s-2m, 12s-1-2m); m = 2, \ldots, 3s-1,$$
$$(4, 106, 102), (1, 105, 104), (8, 92, 100), (12, 52, 64), (16, 80, 96),$$
$$(20, 48, 68), (24, 60, 84), (28, 44, 72), (32, 56, 88), (36, 40, 76)\ \}$$
$$B_0 = \{3, 107\}$$

For $1 \leq k \leq 12$:

$$A_k = A_{k-1} / \{\ (1+4k, 106-8k, 107-4k), (3+4k, 102-8k, 105-4k)\ \}$$
$$B_k = B_{k-1} \cup \{1+4k, 107-4k, 106-8k, 3+4k, 105-4k, 102-8k\}$$

$s \equiv 2 \pmod 4$, $s \geq 10$

$$A_0 = \{\ (1+2m, 12s-2-4m, 12s-1-2m); m = 3, \ldots, 3s-1,$$
$$(8, 12s-12, 12s-4), (4, 12s-10, 12s-6), (5, 12s-8, 12s-3),$$
$$(1, 12s-2, 12s-1), (4r, 4(a_r+s), 4(b_r+s));$$
$$r = 3, \ldots, s \text{ with } (a_r, b_r) \text{ as defined below}\}$$
$$B_0 = \{3, 12s-5\}$$
$$A_1 = A_0 / \{(1, 12s-1, 12s-2), (7, 12s-14, 12s-7)\}$$
$$B_1 = B_0 \cup \{1, 12-1, 12s-2, 5, 12s-14, 12-7\}$$

For $2 \leq k \leq \lfloor \frac{3s}{2} \rfloor - 1$:

$$A_k = A_{k-1} / \{\ (4k+3, 12s-8k-6, 12s-4k-3),$$
$$(1+4k, 12s-2-8k, 12s-1-4k)\ \}$$
$$B_k = B_{k-1} \cup \{4k+3, 12s-8k-6, 12s-4k-3, 1+4k,$$
$$12s-2-8k, 12s-1-4k\}$$
$$(a_r, b_r), r = 3, \ldots, s$$

(1) $\left(\frac{s-2}{2}-2-m, \frac{s-2}{2}+2+m\right); m=0,\dots,\frac{s-2}{4}-3$

(2) $\left(\frac{s-2}{4}-m, \frac{3(s-2)}{4}+1+m\right); m=0,\dots,\frac{s-2}{4}-2$

(3) $\left(\frac{3(s-2)}{2}-1-m, \frac{3(s-2)}{2}+2+m\right);$

$m=0,\dots,\frac{s-2}{4}-2$

(4) $\left(\frac{5(s-2)}{4}-m, \frac{7(s-2)}{4}+2+m\right); m=0\dots,\frac{s-2}{4}-2$

(5) $\left(\frac{s-2}{2}+1, s-1\right), \left(\frac{3(s-2)}{4}, \frac{7(s-2)}{4}+1\right),$

$\left(\frac{s-2}{2}-1, \frac{3s-2)}{2}+1\right), \left(\frac{s-2}{2}, \frac{3(s-2)}{2}\right).$

Omit (1) if $s = 10$.

This is based on Simpson [6] using a perfect sequence $\{3,\dots,3+m-1\}$, $m = s-2$.

$s = 2$

$A_0 = \{(4,14,18),(8,12,20),(5,21,16),(1,22,23),$
$(7,10,17),(9,6,15),(11,13,2)\}$

$B_0 = \{3,19\}$

$A_1 = \{(4,14,18),(8,12,20),(5,21,16),(1,22,23),$
$(7,10,17)\}$

$B_1 = \{3,29,9,6,15,11,13,2\}$

$A_2 = \{(4,14,18),(8,12,20),(5,21,16)\}$

$B_2 = \{3,19,9,6,15,13,11,2,7,10,17,1,22,23\}$

$s = 6$

$A_0 = \{(1+2m, 12s-2-4m, 12-1-2m); m = 3,4,\dots,3s-1,$
$(4,62,66),(8,28,36),(12,48,60),(16,40,46),$
$(1,71,70),(20,32,52),(24,44,68),(3,64,67)\}$

$B_0 = \{3,69\}$

$A_1 = A_0/\{(1,70,71),(7,58,65)\}$

$B_1 = B_0 \cup \{1,70,71,7,58,65\}$

For $2 \leq k \leq 8$:

$$A_k = A_{k-1} / \{(4k+3, 66-8k, 69-4k), (1+4k, 70-8k, 71-4k)\}$$
$$B_k = B_{k-1} \cup \{4k+3, 66-8k, 69-4k, 1+4k, 70-8k, 71-4k\}$$

$s \equiv 3 \pmod 4$, $s \geq 11$

$$A_0 = \{ (1+2m, 12-2-4m, 12-1-2m); m = 3, \ldots, 3s-1,$$
$$(1, 12s-2, 12s-1), (4, 12s-10, 12s-6), (8, 12s-12, 12s-4),$$
$$(5, 12s-8, 12s-3), (4r, 4(a_r+s), 4(b_r+s));$$
$$r = 3, \ldots, s \text{with } (a_r, b_r) \text{ as defined below}\}$$
$$B_0 = \{3, 12s-5\}$$
$$A_1 = A_0 / \{(1, 12s-1, 12s-2), (5, 12s-14, 12s-7)\}$$
$$B_1 = B_0 \cup \{1, 12s-1, 12s-2, 5, 12s-14, 12s-7\}$$

For $2 \leq k \leq \lfloor \frac{3s}{2} \rfloor - 1$:

$$A_k = A_{k-1} / \{ (4k+3, 12s-8k-6, 12s-4k-3),$$
$$(1+4k, 12s-2-8k, 12s-1-4k\}$$
$$B_k = B_{k-1} \cup \{4k+3, 12s-8k-6, 12s-4k-3, 1+4k,$$
$$12s-2-8k, 12s-1-4k\}$$

$(a_r, b_r), r = 3, \ldots, s$

(1) $(m, s-1-m); m = 1, \ldots, \frac{s-3}{4}$

(2) $\left(\frac{s-3}{4}+1, \frac{5(s-3)}{4}+3\right)$

(3) $\left(\frac{s-3}{4}+1+m, \frac{3(s-3)}{4}+2-m\right); m = 1, \ldots, \frac{s-3}{4}-1$

(4) $\left(\frac{s-3}{2}+1, \frac{3(s-3)}{2}+4\right)$

(5) $\left(\frac{s-3}{2}+2, \frac{3(s-3)}{2}+3\right)$ and $\left(s-1, \frac{3(s-3)}{2}+2\right)$

(6) $\left(\frac{5(s-3)}{4}+3+m, \frac{7(s-3)}{4}+3-m\right); m = 1, \ldots, \frac{s-3}{4}-2$

(7) $(s-1+m, 2s-3-m); m = 1, \ldots, \frac{s-3}{4}$.

This is based on Theorem 1, case iii., for $m = 4k + 1$ in [2].

$s = 3$

$$A_0 = \{(4,18,22),(8,24,32),(12,16,28),(7,20,27),$$
$$(1,34,35),(3,30,35),(5,26,31),$$
$$(1+2m, 12s-2-4m, 12s-1-2m);\ m = 5,6,7,8\}$$
$$B_0 = \{9,29\}$$
$$A_1 = A_0 / \{(1,34,35),(3,30,33)\}$$
$$B_1 = B_0 \cup \{1,35,34,3,33,30\}$$
$$A_2 = A_1 / \{(13,10,23),(15,6,21)\}$$
$$B_2 = B_1 \cup \{13,23,10,15,21,6\}$$

$s = 7$

$$A_0 = \{ (1+2m, 12s-2-4m, 12-1-2m);\ m = 3,\dots,3s-1,$$
$$(4,74,78),(8,72,80),(5,76,81),(12,40,52),(16,48,64),$$
$$(20,36,56),(24,34,58),(28,32,60),(1,82,83) \}$$
$$B_0 = \{3,79\}$$
$$A_1 = A_0 / \{(1,82,83),(7,70,77)\}$$
$$B_1 = B_0 \cup \{1,83,82,7,77,70\}$$

For $2 \leq k \leq 9$:

$$A_k = A_{k-1} / \{ (1+4k, 12-2-8k, 12s-1-4k),$$
$$(4k+3, 12-8k-6, 12-4k-3) \}$$
$$B_k = B_{k-1} \cup \{1+4k, 12s-2-8k,$$
$$12s-1-4k, 4k+3, 12s-8k-6, 12s-4k-3\}$$

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