Non-equivalent cocycles of graphs over finite fields

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Abstract

The automorphism group of a graph acts on its cocycle space over any field. The orbits of this group action will be counted in case of finite fields. In particular, we obtain an enumeration of non-equivalent edge cuts of the graph.

1 Introduction

We consider simple graphs $G = (V, E)$, that is, multiple edges and loops are not permitted. An *edge cut* of G is a subset L of E (possibly empty), such that L consists of all edges that have exactly one vertex in a given subset S of V; we write $L = (S, V \setminus S)$.

Two edge cuts L_1, L_2 are called *equivalent*, if there is an automorphism of G that maps L_1 onto L_2. We pose the problem to count the non-equivalent edge cuts of G.

More generally, let $\mathbb{F}_q$ be the field of q elements. For a given graph $G = (V, E)$ let A be the arc set of the corresponding symmetric digraph. Let $C^0(G; \mathbb{F}_q)$ be the set of functions $f : V \to \mathbb{F}_q$, and let $C^1(G; \mathbb{F}_q)$ be the set of *alternating functions* of G, i.e. functions $F : A \to \mathbb{F}_q$ such that $F(i, j) = -F(j, i)$ for every arc $(i, j) \in A$.

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The sets $C^0(G;\mathbb{F}_q)$ and $C^1(G;\mathbb{F}_q)$ form vector spaces over $\mathbb{F}_q$. Now consider the mapping

$$\delta : C^0(G;\mathbb{F}_q) \to C^1(G;\mathbb{F}_q)$$

defined by

$$\delta(f)(i,j) = f(i) - f(j)\,. \tag{1}$$

Then δ is a vector space homomorphism. The image of δ is the *cocycle space* of G over $\mathbb{F}_q$. The automorphism group Γ of G acts on $C^0(G;\mathbb{F}_q)$ and $C^1(G;\mathbb{F}_q)$ via

$$\gamma(f)(i) = f(\gamma^{-1}(i))\,,\ \gamma(F)(i,j) = F(\gamma^{-1}(i),\gamma^{-1}(j)) \tag{2}$$

for $\gamma \in \Gamma$, $f \in C^0(G;\mathbb{F}_q)$ and $F \in C^1(G;\mathbb{F}_q)$. Since every $\gamma \in \Gamma$ commutes with δ, it follows that Γ acts also on the cocycle space $im(\delta)$ by (2). Cocycles that are in the same orbit are called *equivalent.*

Cocycle spaces of graphs are classical objects in graph theory. They appear as spaces over finite fields (e.g. [4]) as well as spaces over complex numbers (e.g. [2]), and they play an important role for graphical duality (e.g. [1]).

Cocycles are well studied; however, they have not been counted up to equivalence that is induced by the automorphism group of the base graph. The purpose of this paper is to establish this enumeration.

If q is a power of 2, then every member of $\mathbb{F}_q$ is self-inverse with respect to addition; hence we may imagine $C^1(G;\mathbb{F}_q)$ as the space of all functions $F : E \to \mathbb{F}_q$. The edge cuts of G are precisely the sets $F^{-1}(1)$ for $q = 2$ and $F \in im(\delta)$. So the problem of counting non-equivalent edge cuts of G can be solved by counting the cocycle orbits of G over $\mathbb{F}_2$ under the automorphism group Γ.

2 Strategy for counting non-equivalent cocycles

First let us develop a concept for the enumeration of cocycle equivalence classes. By *Burnside*'s lemma, we have to count the cocycles that are fixed under any given automorphism γ of G. For $\gamma \in \Gamma$ we introduce the mapping

$$\nu_\gamma : C^1(G; \mathbb{F}_q) \to C^1(G; \mathbb{F}_q)$$

by setting

$$\nu_\gamma(F) = \gamma(F) - F$$

for $F \in C^1(G; \mathbb{F}_q)$. It is easy to see that ν_γ is a vector space homomorphism, and the set of fixed cocycles we are looking for is

$$N_\gamma(G; \mathbb{F}_q) := ke(\nu_\gamma) \cap im(\delta) .$$

Next define the homomorphism

$$\mu_\gamma : \ C^0(G; \mathbb{F}_q) \to C^0(G; \mathbb{F}_q)$$

by setting

$$\mu_\gamma(f) = \gamma(f) - f$$

for $f \in C^0(G; \mathbb{F}_q)$. Since γ commutes with δ, the diagram

$$\begin{array}{ccc} C^0(G; \mathbb{F}_q) & \xrightarrow{\ \delta\ } & C^1(G; \mathbb{F}_q) \\ \mu_\gamma \Big\downarrow & & \Big\downarrow \nu_\gamma \\ C^0(G; \mathbb{F}_q) & \xrightarrow[\ \delta\ ]{} & C^1(G; \mathbb{F}_q) \end{array} \quad . \qquad (3)$$

is commutative. The preimage of $N_\gamma(G; \mathbb{F}_q)$ under δ is

$$C_\gamma(G; \mathbb{F}_q) := ke(\nu_\gamma \circ \delta) = ke(\delta \circ \mu_\gamma) .$$

From $\dim C_\gamma(G;\mathbb{F}_q)$ we will obtain $\dim N_\gamma(G;\mathbb{F}_q)$, and $\dim C_\gamma(G;\mathbb{F}_q)$ can be calculated from the dimension of the space

$$M_\gamma(G; \mathbb{F}_q) := im(\mu_\gamma) \cap ke(\delta) .$$

3 Enumeration

The automorphism γ decomposes into pairwise disjoint vertex cycles σ_ν (the subscript ν suggests that γ is to be understood as a vertex permutation).
We start with a criterion for $f \in C^0(G; \mathbb{F}_q)$ to be a member of $M_\gamma(G; \mathbb{F}_q)$.

Lemma 1 *Let $f \in C^0(G; \mathbb{F}_q)$. Then $f \in M_\gamma(G; \mathbb{F}_q)$ iff it satisfies the following two conditions:*

(1) *The function f is constant on the components of G.*

(2) *For every vertex cycle σ_ν of γ, $\sum_{x \in \sigma_\nu} f(x) = 0$.*

Proof. It is easy to see that $ke(\delta)$ consists of all functions $f \in C^0(G; \mathbb{F}_q)$ that are constant on the components of G. Thus it suffices to show that the members of $im(\mu_\gamma)$ are exactly the functions $f \in C^0(G; \mathbb{F}_q)$ that satisfy Condition (2).

Let σ_ν be a vertex cycle of γ of size s, and let $i \in \sigma_\nu$. If $f = \mu_\gamma(g)$ for some $g \in C^0(G; \mathbb{F}_q)$, then we have

$$g(\gamma^{-(k+1)}(i)) - g(\gamma^{-k}(i)) = f(\gamma^{-k}(i)) ,$$

for $k = 0, \ldots, s-1$. We obtain Condition (2) by summing up these equations.

Conversely, define for $i \in V$ recursively

$$g(\gamma^{-(k+1)}(i)) := g(\gamma^{-k}(i)) + f(\gamma^{-k}(i)) .$$

Condition (2) guarantees the consistency of this procedure. So g is well defined, and $f = \mu_\gamma(g)$. □

Clearly, the automorphism group Γ of G acts not only on vertices but also on components of G. The number of component cycles of $\gamma \in \Gamma$ will be denoted by $\omega(\gamma)$. Let σ_ω be a component cycle of Γ. A vertex cycle σ_ν of γ is called *associated* to σ_ω if σ_ν permutes vertices of components in σ_ω. Now let $\kappa(\gamma)$ be the number of component cycles σ_ω of γ that have an associated vertex cycle σ_ν such that

$$\frac{|\sigma_\nu|}{|\sigma_\omega|} \not\equiv 0 \pmod{p} ,$$

where p is the field characteristic of $\mathbb{F}_q$. Note that $\frac{|\sigma_\nu|}{|\sigma_\omega|}$ is always an integer. It indicates how many vertices of σ_ν are contained in a component of σ_ω.

Lemma 2 $\dim M_\gamma(G;\mathbb{F}_q) = \omega(\gamma) - \kappa(\gamma)$.

Proof. Let $f \in M_\gamma(G;\mathbb{F}_q)$. Since $f \in ke(\delta)$, we may imagine f as a function on the components of G by setting $f(H) = f(i)$ for some component H of G and vertex i of H.

Now let σ_ω be a component cycle of γ, and let σ_ν be an associated vertex cycle. Then, by Conditions (2) and (1) of Lemma 1,

$$0 = \sum_{x\in\sigma_\nu} f(x) = \frac{|\sigma_\nu|}{|\sigma_\omega|} \sum_{H\in\sigma_\omega} f(H) . \tag{4}$$

If $\frac{|\sigma_\nu|}{|\sigma_\omega|} \equiv 0 \pmod p$ for every associated vertex cycle σ_ν, then we have $q^{|\sigma_\omega|}$ possible choices for f on the vertices of components contained in σ_ω. But if $\frac{|\sigma_\nu|}{|\sigma_\omega|} \not\equiv 0 \pmod p$ for some associated vertex cycle σ_ν, then f can be chosen only in $q^{|\sigma_\omega|-1}$ ways on these vertices by Equation 4. The assertion follows by taking the product of these numbers over all component cycles σ_ω of γ. □

We will denote the number of vertex cycles of γ by $\nu(\gamma)$.

Lemma 3 $\dim C_\gamma(G;\mathbb{F}_q) = \omega(\gamma) - \kappa(\gamma) + \nu(\gamma)$.

Proof. Set $\mu'_\gamma = \mu_\gamma \mid C_\gamma(G;\mathbb{F}_q)$. Clearly $\dim(ke(\mu_\gamma)) = \nu(\gamma)$, since $ke(\mu_\gamma)$ consists of all $f \in C^0(G;\mathbb{F}_q)$ that are constant on the vertex cycles of γ. An easy calculation shows that $ke(\mu'_\gamma) = ke(\mu_\gamma)$. Now the proof can be completed by the dimension formula for μ'_γ and Lemma 2. □

Now let k be the number of components of G.

Lemma 4 $\dim N_\gamma(G;\mathbb{F}_q) = \omega(\gamma) - \kappa(\gamma) + \nu(\gamma) - k$.

Proof. Set $\delta' = \delta \mid C_\gamma(G;\mathbb{F}_q)$. Again it is easy to see that $ke(\delta') = ke(\delta)$. We conclude that $\dim(ke(\delta')) = k$, since $ke(\delta)$ consists of all $f \in C^0(G;\mathbb{F}_q)$ that are constant on the components of G. Now the proof can be completed by the dimension formula for δ' and Lemma 3. □

Theorem 1 *The number of non-equivalent cocycles of G over $\mathbb{F}_q$ is*

$$\frac{1}{|\Gamma| q^k} \sum_{\gamma \in \Gamma} q^{\omega(\gamma)-\kappa(\gamma)+\nu(\gamma)} .$$

Proof. It follows from Lemma 4 that the number of cocycles of G over $\mathbb{F}_q$ that are fixed by $\gamma \in \Gamma$ is

$$q^{\omega(\gamma)-\kappa(\gamma)+\nu(\gamma)-k} .$$

The theorem follows from this by *Burnside*'s lemma. □

4 Non-equivalent cocycles of complete graphs

As an application of Theorem 1, let $G = K_n$ be the complete graph with n vertices. Then we have $k = 1$ and $\Gamma = S_n$, the symmetric group on the n vertices. For every $\gamma \in S_n$ we have $\omega(\gamma) = 1$, hence $|\sigma_\omega| = 1$ for the only component cycle of γ. We conclude that

$$\kappa(\gamma) = \begin{cases} 0 & \text{if } |\sigma_\nu| \equiv 0 \pmod{p} \text{ for every vertex cycle } \sigma_\nu \text{ of } \gamma, \\ 1 & \text{otherwise} . \end{cases}$$

Now, by Theorem 1, the number of non-equivalent cocycles of K_n over $\mathbb{F}_q$ is

$$\frac{1}{n!} \sum q^{\nu(\gamma)} + \frac{1}{q \cdot n!} \sum q^{\nu(\gamma)}, \tag{5}$$

where the first sum extends over all $\gamma \in S_n$ such that $|\sigma_\nu| \equiv 0 \pmod{p}$ for every vertex cycle σ_ν of γ, and the second sum extends over the remaining permutations in S_n.

For $\gamma \in S_n$ define $\nu_r(\gamma)$ to be the number of vertex cycles of γ of length r, $r = 1, \ldots, n$. The *cycle index* of S_n [9] is the polynomial

$$Z(S_n; \mathrm{s}) = \frac{1}{n!} \sum_{\gamma \in S_n} s_1^{\nu_1(\gamma)} \ldots s_n^{\nu_n(\gamma)} ,$$

where $\mathrm{s} = (s_1, s_2, s_3, \ldots)$. Set $\mathbf{1} = (1, 1, 1, \ldots)$ and for $r \in \mathbb{N}$ define $\mathbf{1}[r] = (x_1, x_2, x_3, \ldots)$ by setting

$$x_i = \begin{cases} 1 & \text{if } r \text{ is a divisor of } i , \\ 0 & \text{otherwise} . \end{cases}$$

$q\backslash n$	2	3	4	5	6	7	8
2	2	2	3	3	4	4	5
3	2	4	5	7	10	12	15
4	4	5	11	14	24	30	45
5	3	7	14	26	42	66	99
7	4	12	30	66	132	246	429
8	8	15	50	99	232	429	835
9	5	21	55	143	339	715	1430
11	6	26	91	273	728	1768	3978
13	7	30	140	476	1428	3876	9690
16	16	51	276	969	3504	10659	30954
17	9	57	285	1197	4389	14296	43263
19	10	70	385	1771	7084	25300	82225
23	12	100	650	3510	16380	67860	254475
25	13	117	819	4755	23751	105183	420732

Table 1

Then it follows from Expression 5 by a short calculation that the number of non-equivalent cocycles of K_n over $\mathbb{F}_q$ is

$$\frac{1}{q}(Z(S_n; q \cdot 1) + (q-1)Z(S_n; q \cdot 1[p])) \ ,$$

where, as usual, the number p is the field characteristic of $\mathbb{F}_q$. From this formula we obtained Table 1. The cycle indices of small order symmetric groups are tabulated in [5].

5 Cohomology

In the language of cohomology, the mapping $\delta: \mathcal{C}^0(G; \mathbb{F}_q) \to \mathcal{C}^1(G; \mathbb{F}_q)$ defined by Equation 1 is a coboundary operator that gives rise to an exact sequence

$$0 \to \mathcal{H}^0(G; \mathbb{F}_q) \xrightarrow{\delta^0} \mathcal{C}^0(G; \mathbb{F}_q) \xrightarrow{\delta} \mathcal{C}^1(G; \mathbb{F}_q) \xrightarrow{\delta^1} \mathcal{H}^1(G; \mathbb{F}_q) \to 0. \quad (6)$$

The space $\mathcal{H}^0(G;I\!F_q) = ke(\delta)$ is the 0-*cohomology space* of G, and the space $\mathcal{H}^1(G;I\!F_q) = \mathcal{C}^1(G;I\!F_q)/im(\delta)$ is the 1-*cohomology space* of G. The members of $\mathcal{H}^1(G;I\!F_q)$ are sometimes called *switching equivalent classes.* They have been enumerated up to isomorphism for $q = 2$ in [11] and later on for arbitrary prime powes q in [7].

Now consider $I\!F_q$ as a finite vector space over its prime field, $I\!F_p^r$, say. Then the action of $GL_r(I\!F_p)$ on $\mathcal{H}^1(G;I\!F_p^r)$ via left multiplication describes isomorphism of regular graph covering projections that stem from ordinary voltage assignments with voltage group $I\!F_q$, hence there is a link to topological graph theory. The corresponding isomorphism classes are counted in [8]; for an introduction to this theory we refer the reader e.g. to [6] or to the famous textbook [3].

6 A related problem

As already remarked in the last section, the mapping $\delta : \mathcal{C}^0(G;I\!F_q) \to \mathcal{C}^1(G;I\!F_q)$ defined by Equation 1 is a coboundary operator. We define a boundary operator

$$\partial : \mathcal{C}^1(G;I\!F_q) \to \mathcal{C}^0(G;I\!F_q)$$

by setting

$$\partial(F)(i) = \sum_{(i,j)\in A} F(i,j) .$$

The kernel of ∂ is the well known *cycle space* of G over $I\!F_q$. Cycle space and cocycle space are orthogonal subspaces of $\mathcal{C}^1(G;I\!F_q)$. As on the cocycles, the automorphism group Γ of G acts on the cycle space of G over $I\!F_q$. Call two cycles of G over $I\!F_q$ *equivalent*, if they are in the same orbit of this action. Similar as in the preceding sections we pose the problem to count the non-equivalent cycles of G over $I\!F_q$. If $q = 2$, this problem is the enumeration problem for non-equivalent even subgraphs of G, i.e. graphs with only even degrees. *Robinson* [10] solved this problem for complete graphs; but, as far as we know, the general case of arbitrary finite fields and graphs is still unsolved.

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