Some results about flag transitive diagram geometries using coset enumeration

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1 Introduction

In the following note we collect some results on diagram geometries admitting a flag transitive group of automorphisms. The geometries under consideration are BUEKENHOUT-TITS geometries, i. e. connected, residually connected geometries of finite rank, whose rank-2-residues are essentially finite classical generalized polygons, generalized digons and "circle-geometries" (see [2]).

We are interested in the question whether the hypothesis on the rank-2-residues and the flag transitive action of a group implies finiteness and makes it possible to give a full classification.

In particular, for all geometries given, also their universal 2-covers are determined.

Proposition (2.1) proves the non-existence of a certain "parabolic system" implicitly used in [13]. This result was independently obtained by A. A. IVANOV (personal communication).

In section 3 we determine the semibiplanes with block size ≤ 12 that admit a flag transitive automorphism group. Proposition (3.10) answers a question of JANKO/VAN TRUNG on the automorphism group of a certain semibiplane. This result could also be derived in a different way from [4], p. 399. In addition to the known examples, an interesting family of geometries with automorphism groups $Gl_2(q)/Z(Sl_2(q))$, q odd, as well as geometries with flag transitive group $2.L_3(4)$ resp. $U_3(3)$ arise.

The method of proof is always the following: we derive a presentation for the flag transitive group G from the action on the diagram geometry $\mathcal{G}$. By coset enumeration (we used CAYLEY V3.7) we get the order of G and hence the information needed to determine G and $\mathcal{G}$.

In some proofs and examples, we work in the chamber systems of the geometries; these are equivalent to the geometries, if certain conditions are satisfied; compare [1], [11].

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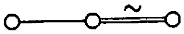
2 A non-existence theorem

The first result shows basically that a certain flag transitive geometry with the diagram  where the rank-2-residue with the diagram  is a triple cover of the $Sp_4(2)$-quadrangle, does not exist. In terms of chamber systems, it reads as follows.

(2.1) PROPOSITION. *There is no chamber system $\mathcal{C}(G; S; X_1, X_2, X_3)$ with $S \simeq D_8$, $X_i \simeq \Sigma_4$ $(i = 1, 2, 3)$, $\langle X_1, X_2 \rangle \simeq L_3(2)$, $\langle X_1, X_3 \rangle \simeq 3.A_6$, $X_2X_3 = X_3X_2$.*

PROOF: Assume $\mathcal{C}(G; S; X_1, X_2, X_3)$ is a chamber system with the above mentioned properties. We choose generators a, b, d_1, d_2, d_3 of G that satisfy a suitable set R of relations. By coset enumeration we show that

$$G = \langle a, b, d_1, d_2, d_3 \mid R \rangle \simeq A_7.$$

This is a contradiction to $\langle X_1, X_3 \rangle \simeq 3.A_6 \not\subset A_7$.

First we introduce the relations and the results of the coset enumeration. Let a, b, d_1, d_2, d_3 be generators and let $z := (ab)^2$. Let

$$\begin{aligned} R_S &= \{a^2, b^2, (ab)^4\} \\ R_1 &= \{d_1^3, d_1^a d_1, b^{d_1} z\} \\ R_j &= \{d_j^3, d_j^b d_j, z^{d_j} a\} \qquad \text{for} \quad j = 2, 3. \end{aligned}$$

An easy calculation shows that $\langle a, b, d_i \mid R_S \cup R_i \rangle \simeq \Sigma_4$ for $i = 1, 2, 3$. Coset enumeration yields the following presentations:

$$\langle a, b, d_1, d_3 \mid R_S \cup R_1 \cup R_3 \cup \{(d_1 d_3^{-1})^5\} \rangle \simeq 3.A_6 \qquad (1)$$
$$\langle a, b, d_1, d_2 \mid R_S \cup R_1 \cup R_2 \cup \{(d_1 d_2^{-1})^3\} \rangle \simeq L_3(2) \qquad (2)$$

and finally:

$$\begin{aligned} &\langle a, b, d_1, d_2, d_3 \mid R_S \cup R_1 \cup R_2 \cup R_3 \cup \{(d_1 d_3^{-1})^5, (d_1 d_2^{-1})^3, [d_2, d_3]\} \rangle \\ &\simeq \langle a, b, d_1, d_2, d_3 \mid R_S \cup R_1 \cup R_2 \cup R_3 \cup \{(d_1 d_3^{-1})^5, (d_1 d_2^{-1})^3, [d_2, d_3]a\} \rangle \\ &\simeq A_7 \qquad (3) \end{aligned}$$

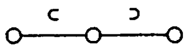
Now we prove that G satisfies the relations in (3). Let $P_{ij} = \langle X_i, X_j \rangle$ for $i,j \in \{1,2,3\}$, $i \neq j$. Let a,b be involutions of G such that $\langle a,b \rangle = S$, let $\langle z \rangle := Z(S)$. There is a $d_1 \in X_1$ such that a,b and d_1 satisfy $R_S \cup R_1$. For each $d_3 \in X_3$ with $d_3^3 = 1$ we have $d_3 \in N_G(\langle a,z \rangle)$, as $P_{13} \simeq 3.A_6$ has $O_2(P_{13}) = 1$. So we can obviously pick $d_3 \in X_3$, such that d_3 satisfies R_3. So we can pick $d_3 \in X_3$ satisfying R_3, and using the isomorphism (1), we may even choose d_3 such that a,b,d_1,d_3 generate P_{13} and obey to relations $R_S \cup R_1 \cup R_3 \cup \{(d_1 d_3^{-1})^5\}$.

Consider P_{12}. By easy counting one verifies that there are precisely two Frobenius groups F, F' of order 21 in P_{12} containing d_1. Since they are self-normalizing in P_{12}, we get $F^a = F'$ and $(F \cap X_2)^a = F' \cap X_2$ holds. Suppose $(F \cap X_2)^b = F' \cap X_2$. Then $F \cap X_2$ and $F' \cap X_2$ are z-invariant. That contradicts $z \in O_2(X_2)$. But now b, which leaves invariant two Sylow 3-subgroups of X_2, must fix one of $F \cap X_2$ or $F' \cap X_2$. Hence we may choose d_2 in, say, $F \cap X_2$, to satisfy R_2. By $F = \langle d_1, d_2 \rangle$, we know that $d_1 d_2^{-1}$ has order 3 or 7. But inside P_{12}, using the isomorphism (2), we see that the element $d_1 d_2^{-1}$ must have order 3.

Obviously, $P_{23} \simeq (A_4 \times 3):2$. An easy calculation in this group shows that $[d_2, d_3] \in \{1, a\}$. We end up with the relations of one of the groups in (3), whence G is a homomorphic image of A_7 by (3), a contradiction.

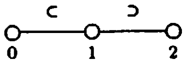
3 Semibiplanes

We now consider geometries with diagram



that admit a flag transitive automorphism group. HUGHES mentions in [6] that a semibiplane corresponds to a geometry with the above diagram.

Throughout this section we fix the following notation: Let $\mathcal{G}$ be a connected geometry with diagram



that admits a flag transitive automorphism group G. Objects of type 0 resp. 1 resp. 2 are called points resp. lines resp. planes. $Res(a)$ resp. G_a stands for the residue resp. the stabilizer of an object $a \in \mathcal{G}$. If $\mathcal{G}$ is finite, flag transitivity implies that the number of planes through a point is a constant k, say. Let v be the number of points in $\mathcal{G}$.

After some introductory general lemmas we determine all such geometries with $k \leq 12$. The first lemma gives a condition for the objects of type 0 and 2 of $\mathcal{G}$ to form a semibiplane.

In the proof of the lemma we use the following notion:

(3.1) Definition. *A geometry $\mathcal{G}$ is said to satisfy the axiom (LL) if and only if the "shadow space" of $\mathcal{G}$ does not contain multiple objects.*

(3.2) Lemma. *Assume the stabilizer G_p resp. G_x of a point p resp. plane x acts primitively on the set of lines resp. points in $Res(p)$ resp. $Res(x)$. Then either all planes and all points are incident or the truncation of $\mathcal{G}$ to points and planes (= blocks) is a semibiplane on which G still acts flag transitively.*

Proof: Assume two planes x, y are incident to the same points. Let p be one of them. The 2-transitive action of G_p on planes in $Res(p)$ implies that all planes in $Res(p)$ are incident to the same points and by connectivity of $\mathcal{G}$ and flag transitivity of G all planes are incident to all points.

For the remainder of the proof, we assume that planes are determined by the points in their residue, hence by their shadows. We have to show that also lines are determined by their point shadows (i. e. $\mathcal{G}$ satisfies (LL)).

Assume there are two lines l, l' which are both incident to the points p, q. Since $G_l \subset G_{\{p,q\}}$ and any element g mapping l to l' is contained in $G_{\{p,q\}}$ but not in G_l, we obtain that G_l is properly contained in $G_{\{p,q\}}$ by flag transitivity of G. But $G_{\{p,q\}} = G_l(G_{\{p,q\}} \cap G_p)$ by the transitive action of G_l on $\{p, q\}$; hence $G_{\{p,q\}} \cap G_p$ is a subgroup of G_p properly containing $G_p \cap G_l$. Primitivity of G_p on lines in $Res(p)$ implies $G_p \cap G_{\{p,q\}} = G_p$, and G_p leaves invariant the set $\{p, q\}$. Since $G = \langle G_p, G_l \rangle$, there are only two points in $\mathcal{G}$, a contradiction. Hence if l, l' are incident to the same points p, q we have $l = l'$.

The same argument shows that $l = l'$ follows also, if l, l' are incident to the same planes x, y. We can now prove that the points and planes of $\mathcal{G}$ form a semibiplane.

Let p, q be two points incident with some plane x. Then there is a line l in $Res(x)$ incident to p and q, and the second plane y in $Res(l)$ is also incident to p and q. Let z be another plane incident to p and q; then again some line l' in $Res(z)$ is incident to p and q, and by the above, $l = l'$. Now

z is one of x or y. Hence any two points that are incident to a plane, are incident to exactly two planes.

The same argument shows that two planes that meet in at least one point, meet in exactly two points.

Of course G acts still flag transitively on the truncation to points/planes.

The conditions of the next lemma are satisfied in an infinite family of geometries with flag transitive groups $Gl_2(q)/\langle -id\rangle$, see (3.5).

(3.3) LEMMA. *Assume the stabilizer G_p of a point p acts as a Frobenius group $F_{q(q-1)}$ on the q planes in $Res(p)$, q an odd prime. Then the truncation of $\mathcal{G}$ to points/planes is a semisymmetric λ-design for some $\lambda \geq 2$, λ divides $k(k-1)$, $\mathcal{G}$ is finite and $1 + \frac{k(k-1)}{2} \leq v \leq 2^{k-1}(k-\lambda)/\binom{k-2}{\lambda-1}$.*

PROOF: Clearly $G_x \simeq G_p \simeq F_{q(q-1)}$ and therefore we have $G_{px} \simeq \mathbb{Z}_{q-1}$ generated by some element f of order $q-1$. Let $G_{pl} = \langle u\rangle$ and $G_{lx} = \langle v\rangle$ and let $a := uf^{\frac{q-1}{2}}$, $b := vf^{\frac{q-1}{2}}$. Then $G_p = \langle a, f\rangle$ resp. $G_x = \langle b, f\rangle$ and f acts via some generators $r, r^s \in GF(q)^*$ on $\langle a\rangle$ resp. $\langle b\rangle$. Clearly, the universal cover of $\mathcal{G}$ has a flag transitive automorphism group

$$H = \langle a, b, f \mid a^q, b^q, f^{q-1}, a^f a^{-r}, b^f b^{-r^s}, [f^{\frac{q-1}{2}}a, f^{\frac{q-1}{2}}b]\rangle$$

and H has an involutory automorphism τ such that $a^\tau = b$ and $f^\tau = f^{s^{-1}}$ hold. Clearly, τ induces a polarity on $\mathcal{G}$.

We show now that $\mathcal{G}$ is a semisymmetric design for some $\lambda \geq 2$. Assume two planes π, π' have the same point shadow, containing the point p, say. Then the equivalence relation $\sim$ on planes through p given by "$\pi \sim \pi^*$ iff the point shadows of π and π^* are the same" has classes of size greater that 1. By the 2-transitive action of G_p on planes through p, all planes through p have the same point shadow. By connectedness of $\mathcal{G}$, all planes have the same point shadow, and all points are incident to all planes. It follows $|G| = q^2(q-1)$, a contradiction.

Hence the truncation of $\mathcal{G}$ to points/planes yields a 1-design in the sense of [8]. By transitivity of G on the sets of two collinear points there is a constant $\lambda \geq 2$ such that two distinct points of $\mathcal{G}$ are on 0 or λ planes. Since τ induces a polarity on $\mathcal{G}$, two distinct planes of $\mathcal{G}$ are on 0 or λ points. Thus the truncation of $\mathcal{G}$ to points/planes is a semisymmetric design with parameter $\lambda \geq 2$.

The remaining parts of the Lemma follow from Theorem 7.14 of [8].

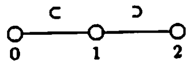
(3.4) Remark: For a semibiplane, i. e. $\lambda = 2$, we obtain $1 + \binom{k}{2} \leq v \leq 2^{k-1}$. By p. 204 of [8] there is a unique semibiplane with $v = 2^{k-1}$ for each $k > 2$. We refer to this plane as the hypercube $H(k)$ as in [16]. If, under certain hypotheses on G_p, we can derive a unique presentation for the group $\overline{G}$ lifted from G to the universal cover $\overline{\mathcal{G}}$ of $\mathcal{G}$, this geometry $\overline{\mathcal{G}}$ is uniquely determined, hence all possible geometries are projections of this particular $\overline{\mathcal{G}}$. Often it is clear, that $\overline{\mathcal{G}}$ is a semibiplane, and $\overline{\mathcal{G}}$ projects onto the corresponding hypercube. Then by the above, $\overline{\mathcal{G}}$ is isomorphic to this hypercube.

(3.5) Example: We now give the infinite family of geometries $\mathcal{FF}(q)$ with Frobenius groups as a point stabilizers announced above.

Let $G = Gl_2(q)$ with q odd and $-$ be the natural homomorphism from G onto $\overline{G} = G/\langle -id\rangle$. Choose $x, y \in GF(q)$ such that $xy = -2$, λ a generator of $GF(q)^*$ and

$$a = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ y & -1 \end{pmatrix}.$$

Set $\overline{G}_0 = \langle \overline{a}, \overline{b}\rangle$, $\overline{G}_1 = \langle \overline{b}, \overline{c}\rangle$, $\overline{G}_2 = \langle \overline{a}, \overline{c}\rangle$. Obviously, $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c}\rangle$. Then $[\overline{b}, \overline{c}] = 1$ and thus the chamber system $(\overline{G}; 1; \langle\overline{c}\rangle, \langle\overline{a}\rangle, \langle\overline{b}\rangle)$ has diagram



with object stabilizers $\overline{g}G_i$ for $i = 0, 1, 2$.

(3.6) Lemma. *For each q, q odd prime power, the geometry $\mathcal{FF}(q)$ satisfies (LL), hence the truncation to points/planes yields a semibiplane. Quotients of $\mathcal{FF}(q)$ whose automorphism group is a proper quotient of $\overline{G}$ by a central subgroup of $\overline{G}$ do not satisfy (LL).*

Proof: Clearly, $\overline{G}$ acts transitively on lines and on 2-sets of collinear points. We choose the 2-set $\{p, p'\}$ of collinear points to be the point p fixed by $\overline{G}_0$, and its conjugate $p' = p^{\overline{c}}$, such that $\{p, p'\} \subset res(l)$ for the line l fixed by $\overline{G}_1$. We have to show that $\overline{G}_p \cap \overline{G}_{p'} = \langle \overline{b}\rangle$, then the number of lines and of 2-sets of collinear points is the same, and (LL) holds. But this follows from an easy calculation.

If we consider the quotient of the geometry $\mathcal{FF}(q)$ by a central subgroup Z of $Gl_2(q)/\langle -id\rangle$ of order $m > 1$, the stabilizer in $\overline{G}/Z$ of p and p' has $2m$ elements, hence (LL) is not satisfied.

It would be interesting to determine all geometries $\mathcal{G}$ with point stabilizer a Frobenius group $F_{q(q-1)}$, q odd, and to check whether $\mathcal{G}$ always has to be a quotient of $\mathcal{FF}(q)$ or the hypercube $H(q)$. We have checked this above for small values of q.

(3.7) REMARK. *For any object $a \in \mathcal{G}$ let K_a be the kernel of the action of G_a on $Res(a)$. Then $K_p = K_x = 1$ for all points p resp. planes x in $\mathcal{G}$.*

PROOF: Let p, x be an incident point/plane-pair and $q \neq p$ be an arbitrary point in $Res(x)$. Then there is a line l incident with both p and q. K_p fixes l, so it also fixes q. So K_p fixes every point in $Res(x)$ and thus also every line in $Res(x)$, i. e. $K_p \subset K_x$. A dual argument shows $K_x \subset K_p$. Connectivity of $\mathcal{G}$ implies $K_p = K_x = 1$.

Thus we assume without loss $K_p = K_x = 1$ for all points p resp. planes x in $\mathcal{G}$.

In the sequel, p, l, x is always a flag with stabilizer $B := G_{plx}$.

The next proposition treats the case $G_p \simeq A_n$ or $G_p \simeq \Sigma_n$ acting on the n planes of $Res(p)$, $n \geq 6$. Note that (3.9) shows that (3.8) does not hold for $k = 5$.

(3.8) PROPOSITION. *Assume that G_p acts transitively as A_n resp. Σ_n on the n planes of $Res(p)$, $n \geq 6$. Then one of the following holds*

(i) $\mathcal{G}$ has 2^{n-1} points and $G \simeq 2^{n-1}{:}A_n$ resp. $G \simeq 2^{n-1}{:}\Sigma_n$.

(ii) n even, $\mathcal{G}$ has 2^{n-2} points and $G \simeq 2^{n-2}{:}A_n$ resp. $G \simeq 2^{n-2}{:}\Sigma_n$.

The truncation to points/planes gives a semi-biplane.

PROOF: First we assume that $G_p \simeq A_n$. Then $G_{px} \simeq A_{n-1}$, $B \simeq A_{n-2}$ and $G_x \simeq A_n$.

It is well-known, see e. g. I.19.8 in [9], that we can pick $a_1, \ldots, a_{n-4} \in B$, $a_{n-3} \in G_{px}$ and $a_{n-2} \in G_p$ such that $G_p = \langle a_1, \ldots, a_{n-2} \rangle$ and the relations

$$\begin{aligned} R_1 \;:=\; & \{a_1^3\} \cup \{a_i^2 \,|\, i = 2, \ldots, n-2\} \cup \{(a_i a_{i+1})^3 \,|\, i = 1, \ldots, n-3\} \\ & \cup \{(a_i a_j)^2 \,|\, i, j = 1, \ldots, n-2, |i-j| \geq 2\} \end{aligned}$$

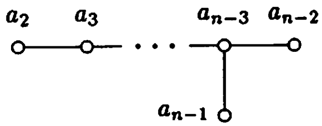
are satisfied. Similarly we can choose $a_{n-1} \in G_x$ such that $G_x = \langle G_{px}, a_{n-1} \rangle$ and the relations

$$R_2 := \{a_{n-1}^2, (a_{n-1} a_{n-3})^3\} \cup \{(a_{n-1} a_i)^2 \,|\, i = 1, \ldots, n-4\}$$

are satisfied. Moreover, $G_{pl} = N_{G_p}(B) \simeq G_{xl} = N_{G_x}(B) \simeq \Sigma_{n-2}$. It is easy to check that $G_{pl} \simeq \langle a_1, \ldots, a_{n-4}, a_{n-2} \rangle$ and $G_{xl} \simeq \langle a_1, \ldots, a_{n-4}, a_{n-1} \rangle$. As $G_l = G_{pl}G_{xl}$ contains G_{pl} of index 2, we get $G_l \simeq \Sigma_{n-2} \times \mathbb{Z}_2$ and thus $[a_{n-1}, a_{n-2}] \in G_l' \cap C_{G_l}(B) = 1$. Thus G is an epimorphic image of

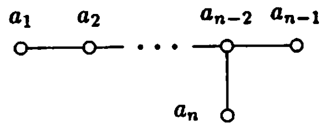
$$H := \langle a_1, \ldots, a_{n-1} \,|\, R_1 \cup R_2 \cup \{[a_{n-1}, a_{n-2}]\} \rangle.$$

Let $u := a_{n-2}a_{n-1}$. We show now that the relations $R_1 \cup R_2 \cup \{[a_{n-1}, a_{n-2}]\}$ imply that $N := \langle u, u^{a_{n-3}}, u^{a_{n-3}a_{n-4}}, \ldots, u^{a_{n-3}\cdots a_1}, u^{a_{n-3}\cdots a_1^2} \rangle$ is an elementary abelian normal subgroup of order $\leq 2^{n-1}$. N is closed under a_1, since a_1 permutes $u^{a_{n-3}\cdots a_2}, u^{a_{n-3}\cdots a_1}, u^{a_{n-3}\cdots a_1^2}$ and centralizes u, $u^{a_{n-3}}$, $u^{a_{n-3}a_{n-4}}$, ..., $u^{a_{n-3}\cdots a_3}$, since $a_1^{a_j a_i} = a_1$ for all $i, j = 3, \ldots, n-1$. $\langle u, u^{a_{n-3}}, \ldots, u^{a_{n-3}\cdots a_2} \rangle$ is an elementary abelian normal subgroup in $\langle a_2, \ldots, a_{n-1} \rangle$, since $a_2, \ldots, a_{n-1}$ satisfy the diagram relations of



By $(a_1a_2)^3 = 1$ we get that a_2 interchanges $u^{a_{n-3}\cdots a_1}$ and $u^{a_{n-3}\cdots a_1^2}$. Since $a_1a_j = a_ja_1^{-1}$ for $j = 3, \ldots, n-1$, it follows $u^{a_{n-3}\cdots a_1^i a_j} = u^{a_{n-3}\cdots a_j a_1^{-i}}$ for $i = 1, 2$ and $j = 3, \ldots, n-1$. Thus N is also invariant under $a_2, \ldots, a_{n-1}$. It is only left to show that $[u, u^{a_{n-3}\cdots a_1}] = [u, u^{a_{n-3}\cdots a_1^2}] = 1$. This follows by $[a_1, u] = 1$. Obviously, the action of $\langle a_1, \ldots, a_{n-2} \rangle \simeq A_n$ on N is the action of A_n on the invariant hyperplane of the permutation module.

We now assume that $G_p \simeq \Sigma_n$. Then $G_x \simeq \Sigma_n$, $G_{px} \simeq \Sigma_{n-1}$ and $B \simeq \Sigma_{n-2}$. Thus there are involutions $a_1, \ldots, a_{n-3} \in B$, $a_{n-2} \in G_{px}$ and $a_{n-1} \in G_p$ that satisfy the diagram relations of the Coxeter diagram A_{n-1}. We can obviously pick an involution $a_n \in G_x$ such that also $a_1, \ldots, a_{n-2}, a_n$ satisfy the diagram relations of a diagram A_{n-1}, as indicated below. Moreover, $G_{pl} = N_{G_p}(B) \simeq G_{xl} = N_{G_x}(B) \simeq \Sigma_{n-2} \times \mathbb{Z}_2$. As G_l contains G_{pl} of index 2, it follows $G_l \simeq \Sigma_{n-2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to check that $\langle a_1, \ldots, a_{n-3}, a_{n-1} \rangle = N_{G_p}(B)$ and $\langle a_1, \ldots, a_{n-3}, a_n \rangle = N_{G_x}(B)$. Thus $[a_{n-1}, a_n] \in G_l' \cap C_{G_l}(B) = 1$. $a_1, \ldots, a_n$ satisfy the diagram relations of the diagram D_n



and G is a quotient of $W(D_n)$. The structure of $W(D_n)$ yields the result.

Now we start the classification of $\mathcal{G}$ for fixed k, $k \leq 12$. Semibiplanes with $k \leq 6$ are well-known ([16]). The classification for $k \leq 4$ is easy: for $k = 3$ we get only four points, for $k = 4$ we get either 7 points with group $L_3(2)$ or $H(3)$.

The next proposition treats the case $k = 5$.

(3.9) Proposition. *Assume that G_p acts transitively as $A_5 \simeq L_2(4)$ resp. Σ_5 resp. F_{20} on the five planes of $Res(p)$. Then*

(i) If $G_p \simeq A_5$, $\mathcal{G}$ has 16 points and $G \simeq 2^4 : A_5$ or $\mathcal{G}$ has 11 points and $G \simeq L_2(11)$.

(ii) If $G_p \simeq \Sigma_5$, $\mathcal{G}$ has 16 points and $G \simeq 2^4 : \Sigma_5$.

(iii) If $G_p \simeq F_{20}$, $\mathcal{G}$ has 12 points and $G \simeq Gl_2(5)/Z(Sl_2(5))$ or $\mathcal{G}$ has 6 points and $G \simeq \Sigma_5$ or $\mathcal{G}$ has 16 points and $G \simeq 2^4 : F_{20}$.

The truncation to points/planes gives the unique biplane on 11 points resp. a semibiplane on 12 points resp. the semibiplane $H(5)$. The geometry with $v = 6$ yields no semibiplanes.

Proof: We first assume that $G_p \simeq L_2(4)$. Then $G_x \simeq L_2(4)$, $G_{px} \simeq A_4$, $B \simeq \mathbb{Z}_3$. Obviously there are elements $d \in B$, $u \in G_{px}$ such that $d^3 = u^2 = (ud)^3 = 1$.

Moreover, $G_{pl} = N_{G_p}(B) \simeq G_{xl} = N_{G_x}(B) \simeq \Sigma_3$. So there are involutions $t_1 \in G_{pl}$, $t_2 \in G_{xl}$ such that $d^{t_i}d = (t_i u)^3 = 1$ for $i = 1,2$. As $G_l = G_{pl}G_{xl}$ containing G_{pl} of index 2, we have $G_l \simeq \mathbb{Z}_2 \times \Sigma_3$. So we have $[t_1, t_2] \in G_l' \cup C_{G_l}(B) = B$.

Let $H_i = \langle d, u, t_1, t_2 \,|\, d^3, u^2, t_1^2, t_2^2, (du)^3, d^{t_1}d, d^{t_2}d, (t_1u)^3, (t_2u)^3, [t_1, t_2]d^i \rangle$ for $i = 0, 1, 2$. Coset enumeration yields $H_0 \simeq 2^4 : A_5$, $H_1 \simeq H_2 \simeq L_2(11)$. As G is a quotient of one of the H_i, (i) holds.

Assume now $G_p \simeq \Sigma_5$. Then $G_p = BG_p'$, $G_x = BG_x'$. So $G = B\langle G_p', G_x' \rangle$ holds. Clearly, $L := \langle G_p', G_x' \rangle$ is also flag transitive on $\mathcal{G}$. Thus $L \simeq L_2(11)$ or $L \simeq 2^4 : A_5$. As there is no group containing $L_2(11)$ of index 2 and containing Σ_5, (ii) follows.

Assume now $G_p \simeq F_{20}$. As above it follows $G_{px} \simeq \mathbb{Z}_4$, $B = 1$, $G_{pl} \simeq G_{lx} \simeq \mathbb{Z}_2$ and $G_l = G_{pl}G_{xl} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let t_x be the Involution in G_{lx} and $f \in G_{px}$ of order 4. Then it follows $1 \neq a := t_x f^2 \in O_5(G_x)$. Either $a^f = a^2$ or $a^f = a^3$ holds. We can assume $a^f = a^2$, since we can replace f by f^{-1} in the second case. Let t_p be the Involution in G_{pl}. As above we obtain $1 \neq b := t_p f^2 \in O_5(G_p)$. Our choice of f is already fix, so we have to deal with the two cases $b^f = b^2$ and $b^f = b^3$. As $G_l = \langle t_p \rangle \times \langle t_x \rangle$, it follows in any case $[af^2, bf^2] = 1$.

Let $H_i = \langle a, b, f \mid a^5, b^5, f^4, a^f a^3, b^f b^i, [af^2, bf^2] \rangle$ for $i = 2, 3$. CAYLEY yields $H_3 \simeq 2^4 : F_{20}$ and $H_2 \simeq Gl_2(5)/Z(Sl_2(5))$.

For $k = 6$ there are well-known semibiplanes with 16, 18, and 32 points which have a flag transitive automorphism group ([10]).

The next proposition deals with this case $k = 6$.

(3.10) PROPOSITION. *Assume that G_p acts transitively as $A_5 \simeq PSl_2(5)$ resp. $\Sigma_5 \simeq PGl_2(5)$ on the six planes of $Res(p)$. Then one of the following holds:*

(i) $\mathcal{G}$ has 18 points and $G \simeq 3.A_6$ resp. $G \simeq 3.\Sigma_6$.

(ii) $\mathcal{G}$ has 6 points and $G \simeq A_6$ resp. $G \simeq \Sigma_6$.

(iii) $\mathcal{G}$ has 32 points and $G \simeq 2^5 : A_5$ resp. $G \simeq 2^5 : \Sigma_5$.

(iv) $\mathcal{G}$ has 16 points and $G \simeq 2^4 : A_5$ resp. $G \simeq 2^4 : \Sigma_5$.

The truncation to points/planes in (i) resp. (iii) resp. (iv) yields the semibiplanes $S_a(18)$ resp. $H(6)$ resp. one of the three biplanes on 16 points in the notation of Proposition 16 of [16].

PROOF: We first assume that $G_p \simeq A_5$. Then $G_{px} \simeq D_{10}$ and $B \simeq \mathbb{Z}_2$. As $|G_{px} : B| = 5$, the representation of G_x on the points of $Res(x)$ is also of degree six. So G_x is isomorphic to a subgroup of Σ_6 and $G_{px} \simeq D_{10}$ implies $G_x \simeq A_5$.

We pick $f \in G_{xp}$, $t \in B$ such that the relations $R_0 := \{f^5, t^2, f^t f\}$ are satisfied. Then $N_{G_p}(B) \simeq 2^2 \simeq N_{G_x}(B)$ and an easy calculation in A_5 shows that we can pick involutions s_1, s_2 such that $\langle s_1, t \rangle \simeq N_{G_p}(B)$, $\langle s_2, t \rangle \simeq N_{G_x}(B)$ and $s_1 f$, $s_2 f$ of order 3. Moreover, with the help of CAYLEY one can verify that $\langle f, t, s \mid R_0 \cup \{s^2, [t, s], (fs)^3\} \rangle$ is a presentation of $Sl(2,5)$ and that the center is generated by the element $tf^2 s f^{-2} s f^2 s$. Thus $G_x \simeq G_p \simeq \langle f, t, s_i \mid R_0 \cup R_i \rangle$ with $R_i = \{s_i^2, [t, s_i], (fs_i)^3, tf^2 s_i f^{-2} s_i f^2 s_i\}$ for $i = 1, 2$.

Moreover, $N_{G_p}(B) = G_{pl} \simeq N_{G_x}(B) = G_{lx}$ and $G_l = G_{pl} G_{xl}$ containing G_{pl} of index 2. Hence $\langle s_1, s_2, t \rangle$ is a group of order 8 either dihedral or elementary abelian. Thus either $[s_1, s_2]t$ or $[s_1, s_2]$ holds in G.

Let $H_i = \langle f, t, s_1, s_2 \mid R_0 \cup R_1 \cup R_2 \cup \{[s_1, s_2]t^i\} \rangle$ for $i = 0, 1$. By coset enumeration we obtain $H_0 \simeq 2^5 : A_5$ and $H_1 \simeq 3.A_6$. We get $Z(H_0) \simeq \mathbb{Z}_2$, since H_i is a perfect group for $i = 0, 1$. As $G = \langle G_x, G_p \rangle$ it follows that G is an epimorphic image of one of these groups. Thus either $G \simeq 2^5 : A_5$

and $v = 32$ or $G \simeq 2^4 : A_5$ and $v = 16$. If G is a quotient of H_1, we obtain $G \simeq 3.A_6$ and $v = 18$ or $G \simeq A_6$ and $v = 6$.

Now we assume that $G_p \simeq \Sigma_5$. Then as above $G_p \simeq G_x \simeq \Sigma_5$, $G_{px} \simeq F_{20}$ and $B \simeq \mathbb{Z}_4$.

Again we can choose f of order 5 in G_{px}, and t of order 4 in B, whence $G_{px} = \langle f, t \rangle$; now $N_{G_p}(B) \simeq N_{G_x}(B) \simeq D_8$. This time, we can pick involutions s_1, s_2 in $N_{G_p}(B) - B$ resp. $N_{G_x}(B) - B$ such that $s_i f$ has order 3 for $i = 1, 2$. Coset enumeration yields $G_p = \langle f, t, s_1 \mid R_1 \rangle$ and $G_x = \langle f, t, s_2 \mid R_2 \rangle$ with $R_i = \{f^5, t^4, f^t f, s_i^2, (s_i f)^3, t_i^s t\}$ for $i = 1, 2$.

Moreover, $G_l = G_{pl} G_{xl}$ is a product of two D_8's meeting in B. Hence again $G_l = \langle s_1, s_2, t \rangle$ and $[s_1, s_2] \in \langle t \rangle$.

Furthermore, $1 = [s_1, s_2^2] = [s_1, s_2][s_1, s_2, s_2][s_1, s_2] = [s_1, s_2, s_2]$ and similarly $1 = [s_2, s_1, s_1]$. Thus $[s_1, s_2] \in \langle t^2 \rangle$. Let $H_i = \langle f, t, s_1, s_2 \mid R_1 \cup R_2 \cup [s_1, s_2] t^{2i} \rangle$ for $i = 0, 1$. Coset enumeration shows that $H_1 \simeq 3.\Sigma_6$ and $H_0 \simeq 2^5 : \Sigma_5$. Obviously G is an epimorphic image of H_i for $i = 0$ or $i = 2$.

G_p contains a subgroup K isomorphic to A_5 acting transitively on $Res(p)$. $L = \langle K^G \rangle$ is also flag transitive and so L is isomorphic to one of the groups determined above. So one of the following cases holds:

(i) $G \simeq 3.\Sigma_6$ and $v = 18$.

(ii) $G \simeq \Sigma_6$ and $v = 6$.

(iii) $G \simeq 2^5 : \Sigma_5$ and $v = 32$.

(iv) $G \simeq 2^4 : \Sigma_5$ and $v = 16$.

In (i), (iii) and (iv) the truncation to points/planes gives a semibiplane by the Lemma.

For $k = 7, 8, 9$ there are several doubly transitive groups to be checked as point stabilizers. However, it turns out that in most cases the universal cover of $\mathcal{G}$ is the hypercube $H(k)$ or a geometry with automorphism group $Gl_2(q)/Z(Sl_2(q))$, $q = 7, 9$, as described in (3.5). We omit details. The next proposition deals with an exception.

(3.11) Proposition. *Assume that G_p acts transitively as $L_3(2)$ on the 7 planes of $Res(p)$. Then G is a quotient of $2^6 : L_3(2)$ or $U_3(3)$.*

Proof: As above we get $G_{px} \simeq \Sigma_4$, $G_x \simeq L_3(2)$ and $B \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

We identify G_p resp. G_x acting on $Res(p)$ resp. $Res(x)$ with $L_3(2)$ in the action on the projective plane π of order 2, whose points and lines are denoted by capital letters. U_P resp. U_L stands for the unipotent radical of the stabilizer of a point P resp. a line L. Without loss we can identify G_{px} with the stabilizer of a point in π, say P. G_{pl} is the stabilizer of a set of two points $\{P, P'\}$ and leaves the line $L = PP'$ invariant. Thus G_{pl} fixes P'', the third point on L. Let $L'' \neq L$ a line through P''.

Then we can pick elements $a, b, z, d, s \in G_p$ in the following way: $\langle a, d\rangle \simeq \Sigma_3$ is the stabilizer in G_{px} of the line L'', and $\langle a\rangle := \langle a, d\rangle \cap B \neq 1$. This choice is possible, since $\langle a, d\rangle \cap U_P = 1$. Choose $b \in U_P - B$. As $a \in U_L = B$, $\langle a, b\rangle \simeq D_8$ is the stabilizer of the flag (P, L). Then $\langle a, b\rangle \simeq D_8$ and without loss a, b, d, z satisfy the relations $R_{abdz} := \{a^2, b^2, d^3, z^2, d^a d, (ab)^2 z, z^d b\}$. Now G_{pl} normalizes $\{P, P'\}$, hence fixes U_L and P'', hence a, the involution in the center of the flag stabilizer of (L, P''). Pick an involution $s \in G_{pl}$. As $s \notin U_L$, we have $s \in U_{\tilde{P}}$ for the fixed point of s on L. Hence $s \in U_{P''}$, and $\langle s, d\rangle \simeq A_4$. Thus there is an element $s \in G_{pl}$ that satisfies the relations $R_s := \{s^2, [z, s]a, (ds)^3\}$. By coset enumeration, $\langle a, b, d, z \mid R_{abdz} \cup R_s\rangle$ is a presentation for G_p.

Now consider G_x. As $G_{lx} \simeq D_8$ normalizes $\langle a, z\rangle$ and $G_l = G_{pl}G_{xl}$, $Z(G_{pl}) = Z(G_{lx}) = \langle a\rangle$. Hence we can choose an involution t that centralizes a. Again it holds $\langle d, t\rangle \simeq A_4$ and thus t satisfies $R_t := \{t^2, [z, t]a, (dt)^3\}$.

Now $G_l = \langle a, z, s, t\rangle$ and $[s, t] \in G_l' \subset \langle z, a\rangle$ holds. Moreover, $1 = [t, s^2] = [t, s][t, s, s][t, s] = [t, s, s]$ and similarly $1 = [s, t, t]$. Thus $[s, t] \in \langle z\rangle$.

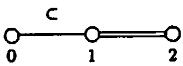
Let $H_i := \langle a, b, z, d, s, t \mid R_{abzd} \cup R_s \cup R_t \cup \{[s, t]z^i\}\rangle$ for $i \in \{0, 1\}$. Then G is an epimorphic image of one of the H_i.

Coset enumeration shows $H_0 \simeq 2^6{:}L_3(2)$ and $H_1 \simeq U_3(3)$.

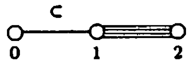
(3.12) PROPOSITION. *Assume that G_p acts transitively as $A_6 \simeq L_2(9)$ on the 10 planes of $Res(p)$. Then G is a quotient of either $2^9{:}A_6$ or $2.L_3(4)$.*

PROOF: $G_{p,x} \simeq 3^2 : 4$ is isomorphic to the normalizer of a 3-Sylow subgroup in A_6. There are obviously elements $d \in B$, $e, f \in G_{p,x}$ such that the relations $R_0 := \{d^3, e^3, f^4, e^f d^{-1}, d^f e\}$ are satisfied. Moreover, $G_{pl} = N_{G_p}(B) \simeq G_{xl} = N_{G_x}(B) \simeq D_8$. A simple calculation in A_6 shows that there are elements a_i, $i = 1, 2$, with $a_1 \in N_{G_p}(B)$ and $a_2 \in N_{G_x}(B)$ that satisfy the relations $R_i := \{a_i^2, (fa_i)^2, (da_ie)^3, (ea_id)^3, a_ida_ie^2a_idf\}$ for $i = 1, 2$. Moreover, $G_l = G_{pl}G_{xl}$ containing G_{pl} of index 2. Therefore $|G_l| = 16$ and as in the previous proof we get $[a_1, a_2] \in Z(G_l) = \langle f^2\rangle$. Let $H_i = \langle a_1, a_2, f, d, e \mid R_0 \cup R_1 \cup R_2 \cup \{[a_1, a_2]f^{2i}\}\rangle$ for $i = 0, 1$. Coset enumeration yields $H_0 \simeq 2^9{:}A_6$, $H_1 \simeq 2.L_3(4)$.

The collinearity graphs of the geometries of (3.10) resp. (3.12) coincide with the collinearity graphs of geometries with diagram



resp.



which are shown by BUEKENHOUT resp. WEISS [15] to have the same universal covers respectively. The geometries have the same points and lines but different objects of type 2, cliques of size 6 resp. 10 instead of 4.

Under the hypothesis that G_p acts as Σ_6 or M_{10} or $PGl_2(9)$ or $Aut(A_6)$ on the 10 planes of $Res(p)$ we obtain the same geometries as above and G is a quotient of a subgroup of $2^9\!:\!Aut(A_6)$ or $2.Aut(L_3(4))$.

(3.13) PROPOSITION. *Assume that G_p acts transitively as $L_2(11)$ on the 11 planes of $Res(p)$. Then one of the following holds:*

(i) $\mathcal{G}$ has 144 points and $G \simeq M_{12}$.

(ii) G is a quotient of $2^{10}\!:\!L_2(11)$.

The truncations to points/planes give semibiplanes.

PROOF: As above we get $G_{px} \simeq A_5$, $B \simeq \Sigma_3$. We can choose $a, b \in B$ and $c \in G_{px}$ such that $a^2 = b^2 = (ab)^3 = 1$ and $[a,c] = c^2 = 1$. An easy calculation in A_5 shows that $(bc)^5 = (abc)^5 = 1$ and $\langle a,b,c\rangle = G_{px}$ hold. Let $R_{abc} := \{a^2, b^2, c^2, (ab)^3, [a,c], (bc)^5, (abc)^5\}$.

As $G_{pl} = N_{G_p}(B) = C_{G_p}(B) \simeq D_{12}$, we can pick $d_1 \in G_p$ such that $d_1^2 = [d_1, a] = [d_1, b] = 1$. It follows from the structure of $L_2(11)$ that either $(cd_1)^3 = 1$ or $a(cd_1)^3 = 1$ hold. Wlog we can assume $(cd_1)^3 = 1$, since we can replace a, b, c, d_1 by $a' = a, b' = b^c, c' = c, d_1' = d_1^c$ in the second case.

Similarly we can pick $d_2 \in G_x$ such that $d_2^2 = [d_2, a] = [d_2, b] = 1$. Now we have to consider the two cases $(cd_2)^3 = 1$ and $a(cd_2)^3 = 1$. As $G_{pl} = \langle a, b, d_1\rangle$, $G_{xl} = \langle a, b, d_2\rangle$ and $G_l = G_{pl}G_{xl} \simeq \Sigma_3 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, we obtain $[d_1, d_2] \in B \cap C(B) = 1$. Let $R_i := \{d_i^2, [d_i, a], [d_i, b]\}$ for $i = 1,2$ and $H_i = \langle a, b, c, d_1, d_2 \,|\, R_{abc} \cup R_1 \cup R_2 \cup \{[d_1, d_2], (cd_1)^3, (cd_2)^3 a^i\}\rangle$ for $i = 0, 1$.

Coset enumeration yields $H_1 \simeq M_{12}$ and $H_0 \simeq 2^{10} : L_2(11)$. It should be mentioned that $\langle a, b, c, d_1 \mid R_{abc} \cup R_1\rangle$ is a well-known presentation for $L_2(11)$, see e. g. [5].

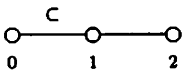
The case $k = 12$ yields no interesting new result. Checking $L_2(11)$, M_{11} and M_{12} as point stabilizers showed that the universal cover of $\mathcal{G}$ is the hypercube $H(12)$.

4 The diagram 

This diagram corresponds to a geometry of the Higman-Sims group [2] where $Res(0,3)$ is a projective plane of order 4. Here we will look at geometries with this diagram where $Res(0,3)$ is a projective plane of order 2. This is the only other possible order q of a projective plane occuring in this case, as merely projective planes of order 2,4, and 10 can have a one-point extension (Lemma 4.1 in [8]). In fact, it is known that geometries with this diagram satisfying the intersection property correspond to semisymmetric 3-designs, and such geometries are known ([7], [12], [14]). We are interested in geometries with flag transitive automorphism groups — for $q = 4$ this property implies the intersection property. For $q = 2$, this is not obvious and we give a "natural" proof of the corresponding classification without referring to results on semisymmetric 3-designs.

The geometry treated in the following proposition is the special case $n = 3$ in 59 of [3]. The presentations in the proof are needed for the construction of the defining relations of the automorphism group of the geometry with the above diagram. The result itself can be obtained as an easy consequence of the subgroup structure of Σ_8 without using coset enumeration at all.

(4.1) PROPOSITION. *Let $\mathcal{G}$ be a geometry with diagram*



such that $Res(0)$ is a projective plane of order 2. Let G be a flag transitive automorphism group G. Then one of the following holds

(i) $G_0 \simeq F_{21}$ and either $G \simeq 2^3F_{21}$ or $G \simeq L_3(2)$.

(ii) $G_0 \simeq L_3(2)$ and $G \simeq 2^3L_3(2)$.

PROOF: Assume the same notation as in the previous proofs. $\mathcal{G}$ is obviously a one-point-extension of a projective plane of order 2. So there are exactly 8 points, i. e. objects of type 0, in G, hence $K_0 = 1$. G_0 acts as a flag transitive automorphism group of a projective plane of order 2, therefore $G \simeq F_{21}$, $B = 1$ or $G \simeq L_3(2)$, $B \simeq D_8$. Obviously, $G_2/K_2 \simeq A_4$ or $G_2/K_2 \simeq \Sigma_4$. As $K_2 \subset B$, it follows $K_2 = 1$ in case $G_0 \simeq F_{21}$ and $K_2 \simeq 2^2$ for $G_0 \simeq L_3(2)$.

We first assume $G_0 \simeq F_{21}$. Then $G_0 \cap G_1 = \langle e \rangle$, $G_0 \cap G_2 = \langle d \rangle$ with $d^3 = e^3 = 1$, as $|\, G_0 \cap G_2 : B \,| = |\, G_0 \cap G_1 : B \,| = 3$. A simple calculation in

F_{21} shows that we can choose d, e such that $(de)^7 = ded^2e^2 = 1$. Obviously, $G_2 \simeq A_4$, $|G_1 \cap G_2| = 2$. Let $\langle u \rangle = G_1 \cap G_2$. Then $(ud)^3 = 1$. As $|G_1 : G_1 \cap G_0| = 2$, $|G_1| = 6$. If G_1 is abelian, it holds $[u, e] = 1$, otherwise $G \simeq \Sigma_3$ and $(ue)^2 = 1$. Let

$$\begin{aligned} R(u,d,e) &:= \{u^2, e^3, d^3, (ud)^3, (de)^7, ded^2e^2, [u,e]\} \\ \tilde{R}(u,d,e) &:= \{u^2, e^3, d^3, (ud)^3, (de)^7, ded^2e^2, (ue)^2\}. \end{aligned}$$

The terms $R(x, y, z)$ or $\tilde{R}(x, y, z)$ will be used for sets of relations of x, y, z such that x resp. y resp. z take the roles of u resp. d resp. e in the expressions above. Coset enumeration yields $\langle u, e, d \mid R(u, d, e) \rangle \simeq 2^3 : F_{21}$ and $\langle u, e, d \mid \tilde{R}(u, d, e) \rangle \simeq L_3(2)$.

Let now be $G_0 \simeq L_3(2)$. Then $B \simeq D_8$, $G_0 \cap G_1 \simeq \Sigma_4 \simeq G_0 \cap G_2$. There are $a, b \in B$ with $a^2 = b^2 = (ab)^4 = 1$. Let $z := (ab)^2$. Now we use the presentation of $L_3(2)$ from the proof of result 1. We can find $d_1 \in G_0 \cap G_1$, $d_2 \in G_0 \cap G_2$ such that $d_1^3 = d_2^3 = d_1^a d_1 = d_2^a d_2 = b^{d_1} z = z^{d_2} a = (d_1 d_2^{-1})^3 = 1$ As $|G_1 : G_0 \cap G_1| = 2$, we obtain $G_1 \simeq \Sigma_4 \times 2$. Let $\langle u \rangle = Z(G_1)$.

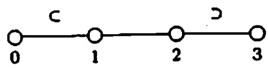
We show now $(ud_2)^3 = 1$. As $u \notin G_0 \cap G_1 \simeq \Sigma_4$, it follows $u \notin K_2$. $|G_1 : G_1 \cap G_2| = 3$, hence $u \in G_2 \cap G_1 \simeq D_8 \times 2$. If u is not contained in $O_2(G_2)$, u acts nontrivially on both four groups that are covered by $O_2(G_2)$. This contradicts $u \in G_1 \cap G_2$. Thus $u \in O_2(G_2)$ and $\langle u, u^{d_2}, u^{d_2^2} \rangle$ is a d_2-invariant submodule of $O_2(G_2)$. As there is no 3-dimensional submodule of $O_2(G_2)$ and $uu^{d_2^2}u^{d_2}$ is invariant under d_2, we obtain $(ud_2)^3 = uu^{d_2^2}u^{d_2} = 1$. Let

$$\begin{aligned} S(u, d_1, d_2) := \{&a^2, b^2, (ab)^4, d_1^3, d_2^3, u^2, d_1^a d_1, d_2^b d_2, \\ &b^{d_1} z, z^{d_2} a, (d_1 d_3^{-1})^3, (ud_2)^3\}. \end{aligned}$$

The term $S(x, y, z)$ will be used below in a similar fashion as the expressions $R(x, y, z)$ and $\tilde{R}(x, y, z)$ defined above. Coset enumeration yields $\langle a, b, u, d_1, d_2 \mid S(u, d_1, d_2) \rangle \simeq 2^3 : L_3(2)$.

We now consider the rank-4 cases of diagram 60 and 60′ of [3].

(4.2) Proposition. *Let $\mathcal{G}$ be a geometry with diagram*



and flag transitive automorphism group G. Let $Res(0,3)$ be a projective plane of order 2. Then G is an epimorphic image of one of the following groups:

(i) $2^7{:}F_{21}$, $2^4{:}L_3(2)$, A_7, *if G acts as F_{21} on $Res(0,3)$*

(ii) $2^7{:}L_3(2)$, A_8, *if G acts as $L_3(2)$ on $Res(0,3)$.*

PROOF: $K_0 = K_3 = 1$ as above.

Assume that G acts as F_{21} on $Res(0,3)$. This implies $G_{1,2,3} = \langle u \rangle$, $G_{0,1,2} = \langle v \rangle$ with u, v involutions. Then $|\langle u, v \rangle| = 4$ and so $[u, v] = 1$. Generating elements d, e for $G_{1,2} \simeq F_{21}$ are chosen as in the previous proof. There are three cases that can arise for the universal cover of G depending on the structure of G_0 and G_3. Let $G_0 \simeq G_3 \simeq 2^3{:}F_{21}$. From the previous result it follows that wlog u, v, d, e satisfy the relations $R(u, d, e)$ and $R(v, e, d)$. Note that the roles of d and e are interchanged in the second set. Coset enumeration yields $\langle u, v, d, e \mid R(u, d, e) \cup R(v, e, d) \cup \{[u, v]\}\rangle \simeq 2^7{:}F_{21}$. Next we treat the case that one of G_0 and G_3 is of type $2^3{:}F_{21}$ and the other of type $L_3(2)$. Assume $G_0 \simeq 2^3{:}F_{21}$. Then u, v, d, e satisfy wlog the relations $R(u, d, e)$ and $\tilde{R}(v, e, d)$. We obtain $\langle u, v, d, e \mid R(u, d, e) \cup \tilde{R}(v, e, d) \cup \{[u, v]\}\rangle \simeq 2^4{:}L_3(2)$. We now consider $G_0 \simeq G_3 \simeq L_3(2)$. Then, by coset enumeration, G is an epimorphic image of $\langle u, v, d, e \mid \tilde{R}(u, d, e) \cup \tilde{R}(v, e, d) \cup \{[u, v]\}\rangle \simeq A_7$.

Now let G act as $L_3(2)$ on $Res(0,3)$. Then $G_{1,2,3} \simeq D_8{\times}2$, $G_{0,1,2} \simeq D_8{\times}2$ and $|G_{0,1,2}G_{1,2,3}| = 32$. As in the previous result we get $G_{1,3} \simeq G_{0,2} \simeq \Sigma_4 \times 2$. Let $\langle u \rangle = Z(G_{1,3})$ and $\langle v \rangle = Z(G_{0,2})$. We choose similar presentations for $G_{0,3} \simeq L_3(2)$ as above. Then it follows that either $[u, v] = 1$ or $[u, v] = z$, as $G_{1,2}/\langle z \rangle$ is abelian. Wlog we can assume that a, b, d_1, d_2, u, v satisfy the relations $S(u, d_1, d_2)$ and $S(v, d_2, d_1)$. Coset enumeration yields

$$\begin{aligned} \langle a, b, d_1, d_2, u, v \mid S(u, d_1, d_2) \cup S(v, d_2, d_1) \cup \{[u, v]\}\rangle &\simeq 2^7{:}L_3(2) \\ \langle a, b, d_1, d_2, u, v \mid S(u, d_1, d_2) \cup S(v, d_2, d_1) \cup \{[u, v]z\}\rangle &\simeq A_8. \end{aligned}$$

This proves the result.

The two geometries occuring in (4.2) are well-known bi-affine spaces ([14]). An easy description of them is as follows.

(4.3) EXAMPLES: We will now give descriptions of the geometries of the previous result.

Let V be a 4-dimensional vector space over GF(2).

1. Objects of type i for $i = 0, 1, 2, 3$ are all i-dimensional affine subspaces that do not contain 0. Incidence is defined by containment.

This provides a geometry $\mathcal{G}_{15}$ on 15 points, i. e. objects of type 0. Obviously, $Gl_4(2) \simeq A_8$ acts flag transitively on $\mathcal{G}_{15}$ and A_7 is a flag transitive subgroup.

2. Let $0 \neq v \in V$. Let $\mathcal{A}_v$ be the set of all translates of linear subspaces of V that contain $\{0, v\}$. The objects of type i of $\mathcal{G}_{16}$ are all affine i-dimensional subspaces of V except elements of $\mathcal{A}_v$. This leads to a geometry $\mathcal{G}_{16}$ with 16 points. Obviously, an affine group $2^4 : 2^3 L_3(2)$ containing the translations of V and the point stabilizer of v in $Gl(V)$ acts on $\mathcal{G}_{16}$. $2^7 : F_{21}$ and $2^4 L_3(2)$ act as flag transitive subgroups.

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