The chromatic index of a graph G depending on the subgraph induced by the vertices of maximum degree

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Abstract. The Δ-subgraph G_Δ of a simple graph G is the subgraph induced by the vertices of maximum degree of G. In this paper, we obtain some results about the construction of a graph G if the graph G is Class 2 and the structure of G_Δ is particularly simple.

1. Introduction

In this paper, we consider simple graphs (that is graphs which have no loops or multiple edges). An *edge-coloring* of a graph G is a map $\phi: E \to B$, where B is a set of colors and $E(G)$ is the set of edges of G, such that no two incident edges receive the same color. The *chromatic index*, $\chi'(G)$ of G is the least value of $|B|$ for which an edge-coloring of G exists. A well-known theorem of Vizing [5] states that, for a simple graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \qquad (1)$$

where $\Delta(G)$ denotes the maximum degree of G. Graphs for which $\chi'(G) = \Delta(G)$ are said to be *Class 1*, and otherwise they are *Class 2*.

A graph G is *critical* if it is Class 2, connected and for each edge e of G, $\chi'(G/e) < \chi'(G)$.

Let G_Δ be the subgraph of G induced by the vertices of maximum degree. A graph G is said to be *overfull* if it satisfies $|E(G)| \geq \Delta(G)\lfloor|V(G)|/2\rfloor + 1$. If there is equality here then G is called *just overfull*. Obviously, if G is overfull, then G is Class 2. A cycle C is a 2-regular connected graph. Let $N(G_\Delta)$ denote the set of vertices in $V(G) - V(G_\Delta)$ which are adjacent (in G) to vertices in G_Δ. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G.

In this paper, we prove the following conclusions:

Theorem 1. *Let G be a Class 2 connected graph, and let G_Δ be a family of vertex-disjoint cycles. Then*

(i) *G is critical,*
(ii) *$\delta(G) \geq \Delta(G) - 1$, unless G is an odd circuit,*
(iii) *$N(G_\Delta) = \{v | d(v) = \Delta(G) - 1, v \in V(G)\}$.*

Theorem 2. *Let G be a simple connected graph, G_Δ be a cycle, and $\Delta(G) \geq (|V(G)| + 3)/2$. Then G is Class 2 if and only if G is just overfull.*

The following conjecture was made by Chetwynd and Hilton:

Conjecture [3]: Let G be a simple graph with $\Delta(G) > |V(G)|/3$. Then G is Class 2 if and only if G contains an overfull subgraph H with $\Delta(G) = \Delta(H)$.

Theorem 3. *Let G be a simple Class 2 connected graph, let G_Δ be a cycle and let $\Delta(G) > |V(G)|/3$. If the above conjecture is true, then $\Delta(G) \geq (|V(G)| + 3)/2$.*

Theorem 4. *Let G be a simple connected graph. Let G_Δ be a family of disjoint cycles. And suppose that $\Delta(G) - 2 = |V(G)| - |V(G_\Delta)|, (\Delta(G) \geq 3)$. Then G is Class 2 if and only if G is just overfull.*

2. Some useful lemmas

The first of these is Vizing's Adjacency Lemma [6]. For $v \in V(G)$, let $d^*(v)$ be the number of vertices of $V(G_\Delta)$ to which v is adjacent in G.

Lemma 1. *Let G be a critical simple graph, let $uv = e \in E(G)$. Then $d^*(u) + d(v) \geq \Delta(G) + 1$ if $d(v) < \Delta(G)$; $d^*(u) + d(v) \geq \Delta(G) + 2$ if $d(v) = \Delta(G)$.*

The next lemma can be proved easily by Lemma 1:

Lemma 2. *If G is a simple graph and G_Δ is a forest, then G is Class 1.*

The next lemma is a consequence of Lemma 1:

Lemma 3. *For a graph G, let $e = uv \in E(G)$ be such that $d(v) \leq \Delta(G) - d^*(u)$. If $\Delta(G - e) = \Delta(G)$, then $\chi'(G - e) = \chi'(G)$.*

Proof: If G is Class 1, we know that $\Delta(G) = \chi'(G) \geq \chi'(G - e) \geq \Delta(G - e)$, then the lemma is done.

If G is Class 2, let H be any critical subgraph of G with $\Delta(H) = \Delta(G)$. We claim that $e \notin E(H)$. Otherwise, suppose that $e \in E(H)$. Since $d^*_H(u) \leq d^*(u)$, $d^*_H(v) \leq d(v)$, then by the Vizing Adjacency Lemma, we know that $d^*_H(u) \geq \Delta - (\Delta - d^*(u)) + 1 = d^*(u) + 1$, if $d^*(u) \geq 1$; and $d^*_H(u) \geq 2$, if $d^*(u) = 0$. This is a contradiction. Then $\chi'(G - e) = \chi'(G)$. ∎

A star-multigraph G contains a vertex v, called the *star centre*, with which each non-simple edge is incident. The following two conclusions proved in [2] are useful in this paper.

Lemma 4 [2]. *If G is a star multigraph, then $\chi'(G) \leq \Delta + 1$.*

Lemma 5 [2]. *Let G be a critical star-multigraph with star centre v^*, where $d(v^*) = \Delta(G)$. Let u and w be adjacent vertices, $w \neq v$. Then, $d^*(w) + d(u) \geq \Delta(G) + 1$ if $d(u) < \Delta(G)$; $d^*(w) \geq 2$ if $d(u) = \Delta(G)$.*

The proof of the next lemma is similar to the proof of Lemma 3 by using Lemma 4 and Lemma 5.

Lemma 6. *Let G be a star-multigraph with star center v^*, and let v^* be a vertex of maximum degree.*

(1) *Let $d^*(w) \leq 1$ for some $w \in V(G) - \{v^*\}$. If $\Delta(G) = \Delta(G - w)$, then $\chi'(G) = \chi'(G - w)$.*

(2) *Let edge $e = uv \in E(G)$, and $\{u, v\} \subset V(G) - \{v^*\}$, $d(v) \leq \Delta(G) - d^*(u)$, where $d^*(u) \geq 2$. Then, $\Delta(G) = \Delta(G - e) \Rightarrow \chi'(G) = \chi'(G - e)$.*

Lemma 7 [2]. *Let G be a star-multigraph with star center v^*, and let G have at most two vertices, v^* (possibly v_1), of high degree. If v^* and v_1 are joined by more than one edge, let there be a vertex w such that w is joined to v_1 but not to v^*. Let G not contain a subgraph on three vertices with $\Delta(G) + 1 \geq 3$ edges. Then G is Class 1.*

Finally we give a result due to Berge [1], generalizing a well-known theorem of Chvatal [5].

Lemma 8 [5]. *Let G be a simple graph of order n with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Let q be an integer, with $0 \leq q \leq n - 3$. If, for every k with $q < k < (n + q)/2$, the following condition holds:*

$$d_{k-q} \leq k \Rightarrow d_{n-k} \geq n - k + q.$$

Then, for each set F of independent edges with $|F| = q$, there exists a Hamiltonian circuit containing F.

3. Proof of Theorem 1

Let k denote the number of vertex-disjoint cycles of G_Δ.

Case 1. $k = 1$. Since G is Class 2, we claim that the vertices of $N(G_\Delta)$ have degree at least $\Delta - 1$. Otherwise, there exists a vertex v ($v \in N(G_\Delta)$) such that $d(v) \leq \Delta - 2$, edge $e = uv \in E(G)$, and $d(u) = \Delta(G)$. By using Lemma 3, the edge e is not critical, so $\chi'(G) = \chi'(G - e)$. But in graph $G - e$, $d_{G-e}(u) = \Delta - 1$, it is easy to see that $[G - e]_\Delta$ is a forest, it follows that $G - e$ is Class 1, this is a contradiction.

Now let $V_1 = V(G_\Delta) \cup N(G_\Delta)$ and $V_2 = V(G) - V_1$. We claim that $V_2 = \phi$. Otherwise, since G is connected, it follows that $[V_1, V_2] \neq \phi$, where $[V_1, V_2] = \{uv \in E(G) | u \in V_1, v \in V_2\}$. Let $e_1 = u_1 v_1 \in [V_1, V_2]$, then $d(u_1) \leq \Delta - 1$ and $d(v_1) \leq \Delta - 1$. Obviously, $d^*(v_1) = 0$. By Lemma 3, $\chi'(G) = \chi'(G - e_1)$. But $d_{G-e_1}(u_1) \leq \Delta - 2$, and $[G - e_1]_\Delta$ is a cycle, $u_1 \in N([G - e_1]_\Delta)$. By the above proof, $G - e$ is Class 1. This is a contradiction. Then $V_2 = \phi$. Therefore, $\delta(G) \geq \Delta - 1$, and $N(G_\Delta) = \{v | d(v) = \Delta(G) - 1\}$.

Now we show that G is critical. If e is an edge which is adjacent to the vertex of maximum degree, we claim that $\chi'(G - e) < \chi'(G)$. Since $[G - e]_\Delta$ is a forest in this case, it follows that $\chi'(G-e) = \Delta(G) < \chi'(G)$. If e is an edge which is not adjacent to any vertex of maximum degree, we want to show that $G-e$ is Class 1. Otherwise, assume that $G-e$ is Class 2. In this case, there exist two vertices of degree $\Delta-2$ in $G-e$, and $[G-e]_\Delta$ is still a cycle. Since $G-e$ is Class 2 and is connected, it follows that $\delta(G-e) \geq \Delta-1$ by the previous proof. This is a contradiction. Therefore $G-e$ is Class 1.

Case 2. We use induction on k by assuming that the theorem is true if the number of cycles of G_Δ is at most $k-1$. Let $G_\Delta = C_1 \cup \cdots \cup C_k$, $(k>1)$. Without loss of generality, assume that $N(G_\Delta) \neq \phi$. Since G is Class 2, we claim that the vertex degree of $N(G_\Delta)$ is $\Delta-1$ in G. Otherwise let $v \in N(G)$ be such that $d(v) \leq \Delta-2$, and $e = uv \in E(G)$, $d(u) = \Delta(G)$. Without loss of generality, suppose that $u \in V(C_k)$. By Lemma 3, $\chi'(G) = \chi'(G-e)$. But in graph $G-e$, we know that $d_{G-e}(u) = \Delta-1$, then $[G-e]_\Delta = C_1 \cup \cdots \cup C_{k-1} \cup (C_k - u)$. Clearly, $C_k - u$ is a path. By using Lemma 3 again and again, it easily follows that $\chi'(G) = \chi'(G-e-E(C_k-u))$, and $[G-e-E(C_k-u)]_\Delta = C_1 \cup \cdots \cup C_{k-1}$. We claim that $G-e-E(C_k-u)$ is disconnected. Otherwise, by induction, we know that $\delta(G-e-E(C_k-u)) \geq \Delta-1$, which contradicts the structure of G_Δ and connectedness of G. Since $G-e-E(C_k-u)$ is disconnected, one of the components must be Class 2. Now let G_1 be a component of $G-e-E(C_k-u)$ such that G_1 is Class 2. Obviously, $[G_1]_\Delta$ is a family of disjoint cycles and the number of the disjoint cycles of $[G_1]_\Delta$, is less than k. By induction hypothesis applied to G_1, we know that $\delta(G_1) \geq \Delta-1$, which also contradicts the structure of G_Δ in G and connectedness of G.

Now let $V_1 = V(G_\Delta) \cup N(G_\Delta)$, and $V_2 = V(G) - V_1$. We claim that $V_2 = \phi$. Otherwise, since G is connected, then $[V_1, V_2] \neq \phi$. Let $e = uv \in [V_1, V_2]$, then $d(u) \leq \Delta-1$, and $d(v) \leq \Delta-1$, $d^*(v) = 0$. By Lemma 3, $\chi'(G) = \chi'(G-e)$, and $d_{G-e}(u) \leq \Delta-2$. By using the same argument, we have that $\chi'(G) = \chi'(G-V_2)$. By the above conclusion, we know that $G-V_2$ is disconnected. One of the components must be Class 2, say G'. By induction hypothesis applied to G', $\delta(G') \geq \Delta-1$. This contradicts the connectedness of G. Therefore $\delta(G) \geq \Delta-1$, and $N(G_\Delta) = \{v \mid d(v) = \Delta-1, v \in V(G)\}$ in this case.

We can also prove the fact that G is critical. The proof of criticality is similar to the proof of case 1.

This completes the proof. ∎

4. Proof of Theorem 2

It is obvious that if G is overfull, G is Class 2. Conversely, let G be Class 2, we want to show that G is just overfull. We consider two cases:

Case 1. $|V(G)|$ is even. In this case, we claim that G must be Class 1. Otherwise, assume that G is Class 2. Let $V(G_\Delta) = \{a_1, \ldots, a_r\}$ and $G = C = a_1 a_2 \ldots a_r a_1$, (a cycle). By Theorem 1, G is critical and $\delta(G) \geq \Delta(G) - 1$.

Because $|V(G)|$ is even, we have that $\Delta(G) \geq (|V(G)|+4)/2$. Consider

$$\delta(G-\{a_1,a_2\}) \geq \Delta - 3 \tag{2}$$
$$= (|v(G)|+4)/2 - 3 \tag{3}$$
$$= (|v(G)|-2)/2 \tag{4.}$$

By Dirac's theorem, $G-\{a_1,a_2\}$ has a Hamilton cycle. Then G has a 1-factor F containing the edge a_1a_2. Now $\Delta(G-F) = \Delta(G) - 1$. Obviously $[G-F]_{\Delta-1}$ is a forest, therefore $G-F$ is Class 1. Since $\chi'(G) \leq \chi'(G-F) + 1 = \Delta(G)$, this is a contradiction.

Case 2. $|V(G)|$ is odd. In this case, we claim that G must be just overfull if G is Class 2. Suppose that G is Class 2. Let $G_\Delta = C = a_1 \dots a_r a_1$. By Theorem 1, G is critical and $\delta(G) \geq \Delta(G) - 1$. Thus

$$2|E(G)| = r\Delta + (|V|-r)(\Delta-1) \tag{5}$$
$$= (|V|-1) + (\Delta - |V| + r) \tag{6.}$$

Then $|E(G)| = \Delta\lfloor |V|/2 \rfloor + 1$ if and only if $\Delta(G) - 2 = |V(G)| - r$. We want to prove the fact that $\Delta - 2 = |V(G)| - r$. By contradiction, suppose that $\Delta - 2 < |V| - r$. There is a vertex $u \in V(G) - V(G_\Delta)$, such that $a_1 u \notin E(G)$. Let $e = a_1a_2 \in E(G_\Delta)$. We claim that $G-u$ has a 1-factor F containing the edge a_1a_2. Consider two subcases:

Subcase 1. $\Delta(G) \geq (|V|+4)/2$. Since

$$\delta(G-\{a_1,a_2\}) \geq \delta(G) - 2 \tag{(7)}$$
$$= \Delta(G) - 1 - 2 \tag{8}$$
$$\geq (|V|-2)/2 \tag{9}$$

By Dirac's theorem, $G-\{a_1,a_2\}$ has a Hamilton cycle. Since $|V|-2$ is odd, then $G-u$ has a 1-factor F containing the edge a_1a_2, as required.

Subcase 2. $\Delta(G) = (|V(G)|+3)/2$. By Theorem 1, the degree sequence of G is $(\Delta-1,\dots,\Delta-1,\Delta,\dots,\Delta)$, where the number of vertices of degree $\Delta-1$ is $|V|-r$. The degree sequences of $G-u$ is $(\Delta-2,\dots,\Delta-2,\Delta-1,\dots,\Delta)$, where the number of vertices of degree $\Delta-2$ is at most $\Delta-3$ because $d^*(u) \geq 2$. Now we use Lemma 8. Taking $n = |V|-1$, $q = 1$, we must verify that, for every k satisfying $1 < k < |V|/2$, we have

$$d_{k-1} \leq k \Rightarrow d_{n-k} \geq |V| - k.$$

Obviously, when $k < \Delta - 2$, there is no such k satisfying $d_{k-1} \geq k$. When $k = (|V|-1)/2$, it is easy to see that $k = \Delta - 2$, and if $d_{k-1} = d_{\Delta-3} \geq \Delta - 2$, we have that

$$
\begin{aligned}
d_{n-k} &= d_{|V|-1-(\Delta-2)} && (10)\\
&= d_{|V|-\Delta+1} && (11)\\
&= d_{\Delta-2} && (12)\\
&= \Delta - 1 && (13)\\
&\geq |V| - (\Delta - 2) && (14)\\
&= n - k + q && (15.)
\end{aligned}
$$

Therefore the condition of Lemma 8 is satisfied. Then $G-u$ has a Hamilton cycle containing the edge a_1a_2. It follows that $G-u$ has a 1-factor F containing the edge a_1a_2, as required.

Note that $\Delta(G-F) = \Delta(G) - 1$, we want to show that $G-F$ is Class 1. Clearly, the set of maximum degree vertices of $G-F$ is $\{a_1, a_2, \ldots, a_r, u\}$. Let $G_0 = G - F$. And let $[G-F]_{\Delta-1} = G_1$, then $d_{G_1}(a_1) = 1$, $d_{G_1}(a_2) \leq 2$, $d_{G_1}(a_i) \leq 3$. for $3 \leq i \leq r$. We claim that

$(N_{G_0}(a_1) \cap N_{G_0}(a_i)) \cap (V(G) - (V(G_\Delta)) \neq \phi$, for $2 \leq i \leq r \ldots (*)$

This follows if $|V(G_\Delta)| \geq 4$, since

$$
\begin{aligned}
(\Delta - 3) + (\Delta - 3) &= 2\Delta - 6 && (16)\\
&\geq |V| - 3 && (17)\\
&> |V| - r && (18.)
\end{aligned}
$$

So $(*)$ is true in this case. If $|V(G_\Delta)| = 3$, we know that $\Delta \geq 2|V|/3$ since G is critical. Then $(*)$ is again true.

It follows that we can find a family of edges $a_ix_i (2 \leq i \leq r)$, where $x_i \in N_{G_0}(a_1) \cap (V(G) - V(G_\Delta))$, and $a_ix_i \in E(G) - F$, for $2 \leq i \leq r$. Let $G_2 = G - F - a_1$. Then $\chi'(G_0) = \chi'(G_2)$. In graph G_2, the vertex degree of $N_{G_0}(a_1) \cap (V - V(G_\Delta))$ is at most $\Delta - 3$. Note that $\Delta - 3 = \Delta(G-F) - 2$, and $d_{G_1}(a_2) \leq 2$, the edge a_2x_2 is not critical in G_2 by Lemma 3. Then $\chi'(G_2) = \chi'(G_2 - a_2x_2)$.

Denote $G_3 = G_2 - a_2x_2$. In G_3, the vertex a_3 is adjacent to at most two maximum degree vertices. The edge a_3x_3 is not critical in G_3 (by Lemma 3 again). Then

$$\chi'(G_2) = \chi'(G_2 - a_2x_2 - a_3x_3).$$

By the same argument, we know that

$$
\begin{aligned}
\chi'(G-F) &= \chi'(G - F - a_1) && (19)\\
&= \chi'(G - F - a_1 - a_2x_2 - \cdots - a_rx_r) && (20)\\
&= \Delta(G) - 1. && (21)
\end{aligned}
$$

Then, $\chi'(G) \leq \chi'(G-F) + 1 = \Delta(G)$, a contradiction. This completes the proof. ∎

5. Proof of Theorem 3

Let G be a Class 2 connected graph. Then G is critical by Theorem 1. Assume the conjecture is true, then G is overfull. That is, $|E(G)| > \Delta\lfloor|V|/2\rfloor$. By Theorem 1,

$$2|E(G)| = r\Delta + (|V| - r)(\Delta - 1) \tag{22}$$

$$= \Delta(|V| - 1) + (\Delta - |V| + r) \tag{23 .)}$$

Therefore, $\Delta - 2 \geq |V| - r$ (where $r = |V(G_\Delta)|$). Because there are at most $\Delta - 2$ edges between a maximum degree vertex and the vertices of $V(G) - V(G_\Delta)$, we have $\Delta - 2 = |V| - r$. Thus $d^*(u) = r$, for $u \in V(G) - V(G_\Delta)$. Then $\Delta - 1 - r \geq 0$. Since $\Delta - 2 + \Delta - 1 \geq |V|$, it follows that $2\Delta - 3 \geq |V|$. That is, $\Delta(G) \geq (|V| + 3)/2$.

This completes the proof. ∎

6. Proof of Theorem 4

We want to show that if G is Class 2, then G is just overfull. Now assume that G is Class 2, therefore G is critical and $\delta(G) \geq \Delta(G) - 1$ by Theorem 1. If $|V(G)|$ is odd, then

$$2|E(G)| = r\Delta + (|V| - r)(\Delta - 1) \tag{24}$$

$$= (|V| - 1) + (\Delta - |V| + r) \tag{25}$$

$$= \Delta(|V| - 1) + 2 \tag{26}$$

Then, G is just overfull.

If $|V(G)|$ is even, we claim that G is Class 1 in this case. Otherwise, assume that G is Class 2. We claim that $\Delta(G) - 2 \equiv O \pmod 2$. In fact, let $G_{\Delta-1}$ denote the graph induced by the vertices of degree $\Delta - 1$ in G. Then the number of the edges between G_Δ, and $G_{\Delta-1}$ must be

$$r(\Delta - 2) = (|V| - r)(\Delta - 1) - \sum_{u \in V(G_{\Delta-1})} d_{G_{\Delta-1}}(u). \tag{27}$$

Note that, if $\Delta - 2 \equiv 1 \pmod 2$, then $\Delta - 2 = |V| - r \equiv 1 \pmod 2$, where $r = |V(G_\Delta)|$. Because $|V(G)| \equiv 0 \pmod 2$, we know that $r \equiv 1 \pmod 2$, issuch that

(a) $\ldots r(\Delta - 2) \equiv 1 \pmod 2$.

(b) $\ldots(\Delta - 1) \equiv 0 \pmod 2$.

(c) $\cdots \sum_{u \in V(G_{\Delta-1})} d_{G_{\Delta-1}}(u) \equiv 0 \pmod 2$

Therefore (a) $\not\equiv$ (b) - (c). This contradiction shows that $\Delta - 2$ is even.

Now we construct a new star-multigraph from G as follows: $V(G^*) = V(G) \cup \{u^*, v^*\}$; let $V(G_{\Delta-1}) = V_1 \cup V_2$; $V_1 \cap V_2 = \phi$ and $|V_1| = |V_2| = (\Delta -$

$2)/2$, let $E_1 = \{u^*v | v \in V_1\}$, and $E_2 = \{v^*v | v \in V_2\}$, let E_3 denote $(\Delta + 2)/2$ multiple edges between u^* and v^* and $E(G^*) = E(G) \cup E_1 \cup E_2 \cup E_3$. Obviously, G^* is a Δ-regular star-multigraph. G_Δ is a collection of disjoint cycles, say $G_\Delta = C_1 \cup \cdots \cup C_k$, $k \geq 1$. Let $C_1 = a_1 a_2 \ldots a_m a_1$ (where $m \geq 3$). The set of maximum degree vertices of $G^* - a_1$ is contained in the set $\{a_3, \ldots, a_m, V(C_2), \ldots, V(C_k), u^*, v^*\}$. By Lemma 6,

$$\chi'(G^* - a_1) = \chi'(G^* - V(C_1)).$$

The vertex of $G_{\Delta-1}$ has degree at most $\Delta - 3$ in $G^* - V(C_1)$. Let $V(G_\Delta) - V(C_1) = \{a_{m+1}, \ldots, a_r\}$, since $\Delta - 2 = |V| - r$, we can find a family of edges, say $a_i x_i$, $m + 1 \leq i \leq r$, such that $x_i \in V(G_{\Delta-1})$. Because $d^*(a_i) \leq 2$, the edge $a_i x_i$ is not critical in $G^* - a_1$ by Lemma 6. Then

$$\chi'(G^* - a_1) = \chi'(G^* - V(C_1)) \tag{28}$$

$$= \chi'(G^* - V(C_1) - \bigcup_{i=m+1}^{r} a_i x_i) \tag{29}$$

$$= \Delta(G) \tag{30.}$$

As the graph $G^* - V(C_1) - \cup_{i=m+1}^{r} a_i x_i$ has only two vertices with Δ degree, the conclusion is true by Lemma 7.

Now we prove that G is Class 1. Suppose that $G^* - a_1$ is colored by $\Delta(G^*)$ colors. The graph $G^* - a_1$ has $\Delta(G^*)$ vertices of degree $\Delta(G^*) - 1$ and $|V(G - a_1)|$ is odd. Therefore each color is missing from exactly one vertex and each vertex of degree $\Delta(G) - 1$ has exactly one color missing from it. Therefore, a_1 and the edges on a_1 can be restored, with each edge $a_1 v(v \in V(G^* - a_1), d_{G^*-a_1}(v) = \Delta - 1)$ having the color previously missing at v. Therefore, $\chi'(G^*) = \Delta$. Then $\chi'(G) = \Delta$. A contradiction. This completes the proof. ∎

The authors wish to thank the referees for their helpful suggestions.

Note added in Proof

The authors have generalized the results in this paper. Theire results are in "The chromatic index of graphs where core has a maximum degree two" to appear in Discrete Math., and "A sufficient condition for a regular graph to by Class I", submitted.

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