On extremal nonsupereulerian graphs with clique number m

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Abstract. A graph G is supereulerian if it contains a spanning eulerian subgraph. Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph on $n > p + 6$ vertices containing no K_{m+1}. We prove that if

$$|E(G)| \geq \binom{n-p+1-k}{2} + (m-1)\binom{k+1}{2} + 2p - 4, \tag{1}$$

where $k = \lfloor \frac{n-p+1}{m} \rfloor$, then either G is supereulerian, or G can be contracted to a nonsupereulerian graph of order less than p, or equality holds in (1) and G can be contracted to $K_{2,p-2}$ (p is odd) by contracting a complete m-partite graph $T_{m,n-p+1}$ of order $n - p + 1$ in G. This is a generalization of the previous results in [3] and [5].

1. Introduction

We follow the notation of Bondy and Murty [1], except that graphs have no loops. For a graph G, the order of the maximum complete subgraph of G is called *clique number* of G and denoted by $cl(G)$. A graph is *eulerian* if it is connected and every vertex has even degree. A graph G is called *supereulerian* if it has a spanning eulerian subgraph H. A cycle C of G is called a *hamiltonian cycle* if $V(C) = V(G)$ and is called *dominating cycle* if $E(G - V(C)) = \emptyset$. A graph is *hamiltonian* if it contains a hamiltonian cycle. Obviously, hamiltonian graphs are special supereulerian graphs.

There is rich literature on the following extremal graph theory problems: for a given family $\mathcal{F}$ of graphs and for a natural number n, what is the maximum size of simple graphs of order n which are not in $\mathcal{F}$. For example, when $\mathcal{F}$ = {graphs with clique number at least m}, this is Turán's Theorem. In this note, we consider the family

$$\mathcal{F} = \{\text{supereulerian graphs with clique number } m\}.$$

In fact, our results are related to Turán's Theorem.

Let G be a graph, and let H be a connected subgraph of G. The contraction G/H is the graph obtained from G by contracting all edges of H, and by deleting any resulting loops. Even when G is simple, G/H may not be.

Here are some prior results related to our subject.

Theorem A. *(Ore [8] and Bondy [2]). Let G be a simple graph on n vertices. If*

$$|E(G)| \geq \binom{n-1}{2} + 1, \tag{2}$$

then exactly one of the following holds:

(a) *G is hamiltonian;*
(b) *Equality holds in (2), and $G \in \{K_1 \vee (K_1 + K_{n-2}), K_2 + K_3^c\}$ (where K_3^c is the complement of K_3).*

∎

Theorem B. *(Veldman [10]). Let G be a 2-connected simple graph of order n. If*

$$|E(G)| \geq \binom{n-4}{2} + 11,$$

then G has a dominating cycle. ∎

Theorem C. *(Cai [3]). Let G be 2-edge-connected simple graph on n vertices. If*

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{3}$$

then exactly one of the following holds:

(a) *G is supereulerian;*
(b) *$G = K_{2,5}$;*
(c) *Equality holds in (3), and either $G = Q_3 - v$ (the cube minus a vertex), or G contains a complete subgraph $H = K_{n-4}$ such that $G/H = K_{2,3}$.*

∎

Theorem D. *(Catlin and Chen [5]). Let G be a 3-edge-connected simple graph on n vertices. If*

$$|E(G)| \geq \binom{n-9}{2} + 16,$$

then G is supereulerian. ∎

In this paper, following closely the method of [5], we shall generalize Theorem C and Theorem D. In particular, we found that if a graph G is K_3-free or has small clique number then the lower bound of the inequalities in Theorem C and Theorem D can be improved.

2. Notation and Turán's Theorem

Let n and m be natural numbers, we define $t(m,n)$ as the following;

$$t(m,n) = \binom{n-k}{2} + (m-1)\binom{k+1}{2},$$

where $k = \lfloor \frac{n}{m} \rfloor$. It is easy to see that if $m = n$ or $m > n$ then $k = 1$ or $k = 0$, repectively, and so the right side of the equation above is equal to $\binom{n}{2}$. If $m = 2$ then

$$t(2,n) = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

Note that for $m > n$,

$$t(2,n) < t(3,n) < \cdots < t(n-1,n) < t(n,n) = t(m,n) = \binom{n}{2}. \tag{5}$$

One can see that $t(m,n)$ is related to the Turán numbers below.

For $m \leq n$, denote by $T_{m,n}$ the *complete m-partite graph* of order n with

$$\left\lfloor \frac{n}{m} \right\rfloor, \left\lfloor \frac{n+1}{m} \right\rfloor, \ldots, \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

vertices in the various independent classes. Note that $T_{m,n}$ is the unique complete m-partite graph of order n whose independent classes are as equal as possible and $T_{n,n} = K_n$. Let $k = \lfloor \frac{n}{m} \rfloor$, it is known that the size of $T_{m,n}$ is

$$|E(T_{m,n})| = t(m,n) = \binom{n-k}{2} + (m-1)\binom{k+1}{2}.$$

Theorem E. *(Turán [9]). Let m and n be natural numbers, $m \geq 2$. Then every graph of order n and size greater than $|E(T_{m,n})|$ contains a K_{m+1}. Furthermore, $T_{m,n}$ is the only graph of order n and size $|E(T_{m,n})|$ that does not contain a K_{m+1}.* ∎

Remark. Let G be a graph of order n with maximum size that does not contain a K_{m+1}. If $m > n$ then $|E(G)| = \binom{n}{2}$. If $m \leq n$ then by Theorem E $|E(G)| = |E(T_{m,n})|$. Thus, if G is a graph containing no K_{m+1} then $|E(G)| \leq t(m,n)$. For convenience, we define

$$H_{m,n} = \begin{cases} T_{m,n} & \text{if } m < n; \\ K_n & \text{if } m \geq n. \end{cases}$$

3. Catlin's Reduction Method

The following concept was given by Catlin [4].

For a graph G, let $O(G)$ denoted the set of vertices of odd degree in G. A graph G is called *collapsible* if for every even set $X \subseteq V(G)$ there is a spanning connected subgraph H_X of G, such that $O(H_X) = X$. The *trivial graph* K_1 is both supereulerian and collapsible. The cycles C_2 and C_3 are collapsible, but C_t is not if $t \geq 4$. In fact, if G is collapsible then G contains a spanning (u, v)-trail for any $u, v \in V(G)$. In particular, a collapsible graph is supereulerian.

In [4], Catlin showed that every graph G has a unique collection of disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. Define G' to be the graph obtained from G by contracting each H_i into a single vertex, $(1 \leq i \leq c)$. Since $V(G) = V(H_1) \cup \cdots \cup V(H_c)$, the graph G' has order c. We call the graph G' the *reduction* of G. Any graph G has a unique reduction G' [4]. A graph G is *reduced* if $G = G'$.

We shall make use of the following theorems:

Theorem F. *(Catlin [4]) Let G be a graph. Let G' be the reduction of G.*

(a) *Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.*
(b) *G is supereulerian if and only if G' is supereulerian.*
(c) *If G is a reduced graph of order n, then G is simple and K_3-free with $\delta(G) \leq 3$ and either $G \in \{K_1, K_2\}$ or*

$$|E(G)| \leq 2n - 4.$$

∎

Theorem G. *(Catlin, Han and Lai [6]). Let G be a connected reduced graph of order n. Then $|E(G)| = 2n - 4$ if and only if $G = K_{2,n-2}$.* ∎

4. Main Result and Consequences

The set of natural numbers is denoted by N. Let K be a graph. A graph G is called K-free if it contains no subgraph K.

Theorem 1. *Let G be a 2-edge-connected simple K_3-free graph of order n and let $p \in N - \{1\}$. If*

$$|E(G)| \geq t(2, n-p+1) + 2p - 4, \tag{6}$$

then exactly one of the following holds:

(a) *The reduction of G has order less than p;*
(b) *Equality holds in (6) and G contains a subgraph $H = T_{2,n-p+1}$ such that the reduction of G is $G' = G/H = K_{2,p-2}$;*
(c) *G is a reduced graph of order n such that $p + 1 \leq n \leq p + 6$ and*

$$2n-4 \geq |E(G)| \geq \begin{cases} 2n-4 & if n = 6+p; \\ 2n-5 & if n = 5+p; \\ 2n-6 & if n = i+p, i \in \{2,3,4\}; \\ 2n-5 & if n = 1+p. \end{cases}$$

Proof: Let G' be the reduction of G and let $|V(G')| = c$. If $c = 1$ then G is collapsible and (a) of Theorem 1 holds. Suppose that $c > 1$. Since G is 2-edge-connected and by the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 2$. Let $V(G') = \{v_1, v_2, \ldots, v_c\}$, and let H_i denote the preimage of v_i $(1 \leq i \leq c)$. Suppose that G has the maximum size among all K_3-free graphs which have the reduction G'. Then at most one H_i is a nontrivial K_3-free subgraph of G with order $n-c+1$. Therefore, by the remark following Theorem E and by Theorem F(c),

$$|E(G)| \leq |E(H_i)| + |E(G')| \leq t(m, n-c+1) + 2c-4, \tag{7}$$

with equality only if G has a complete bipartite graph H_i of order $n-c+1$, and its reduction graph G' has size $2c-4$. By (6) and (7)

$$t(2, n-p+1) + 2p-4 \leq |E(G)| \leq t(2, n-c+1) + 2c-4 \tag{8}$$

$$t(2, n-p+1) + 2p \leq t(2, n-c+1) + 2c. \tag{9}$$

Note that if $c < p$ then (a) of Theorem 1 holds. If $c = p$ then equality holds in (8). Therefore, $|E(G')| = 2c-4 = 2p-4$. By Theorem G, $G' = K_{2,p-2}$. Thus (b) of Theorem 1 holds.

In the following we consider the case

$$c > p. \tag{10}$$

By (4) and (9)

$$\frac{(n-p+1)^2 - 1}{4} + 2p \leq \frac{(n-c+1)^2}{4} + 2c.$$

Therefore,

$$(c-p)(2n-p-c+2) \leq 8(c-p) + 1,$$

$$2n \leq 6+p+c+\frac{1}{c-p}. \tag{11}$$

Case 1. $c = n$. Then G is a reduced graph. By (11)

$$n \leq 6+p+\frac{1}{n-p}. \tag{12}$$

If $n = p+1$ then (c) of Theorem 1 holds. If $n > p+1$ then it follows from (12) that $n \leq 6+p$. By routine computation, one can see that (c) of Theorem 1 holds.

Case 2. $c < n$. Since G is K_3-free, G has no nontrivial collapsible subgragh of order less than 6. Hence,

$$c + 5 \leq n. \tag{13}$$

By (11) and (13), we obtain

$$c+5 \leq n \leq 1+p+\frac{1}{c-p} \leq 2+p \leq 1+c,$$

a contradiction. The proof is complete. ∎

An immediate consequence of Theorem 1 is the following.

Corollary 2. *Let G be a 2-edge-connected simple K_3-free graph of order n and let $p \in N - \{1\}$. If $|E(G)| \geq t(2, n-p+1) + 2p - 1$, then the reduction of G has order less than p.* ∎

Lemma 3. *Let a, b, and m be integers with $a \geq 2, b \geq 3, m \geq 3$. Then*

$$t(m, a+b-1) \geq t(2, a) + t(m, b) + \varepsilon(m, a, b),$$

where

$$\varepsilon(m,a,b) = \begin{cases} 1 & \text{if } a = 2 \text{ and } b = m-3, \\ 2 & \text{if } a = 2 \text{ and } \max\{b, m\} > 3, \\ 3 & \text{if } a > 2. \end{cases}$$

Proof: Let G_1 and G_2 be graphs such that $G_1 \cong T_{2,a}$, $G_2 \cong T_{m,b}$ and $|V(G_1) \cap V(G_2)| = 1$. Then $|V(G_1) \cup V(G_2)| = a+b-1$ and $G_1 \cup G_2$ is K_m-free. It is easily seen that $\varepsilon(m,a,b)$ edges can be added to $G_1 \cup G_2$ in such a way that the resulting graph G is still K_m-free. Hence by Theorem E,

$$\begin{aligned} t(m, a+b-1) \geq |E(G)| &= |E(G_1)| + |E(G_2)| + \varepsilon(m,a,b) \\ &= t(2,a) + t(m,b) + \varepsilon(m,a,b). \end{aligned}$$

∎

Theorem 4. *Let n, m and p be natural numbers, $m, p \geq 2$. Let G be a 2-edge-connected simple graph of order n with $cl(G) = m$. If*

$$|E(G)| \geq t(m, n-p+1) + 2p - 4, \tag{14}$$

then exactly one of the following holds:

(a) *The reduction of G has order less than p;*

(b) *Equality holds in (14),* $p \geq 4$ *and* G *contains a subgraph* $H = H_{m,n-p+1}$ *such that the reduction of* G *is* $G' = G/H = K_{2,p-2}$*;*

(c) $cl(G) = 3, n = p+3, p \geq 3$ *and* G *contains a subgraph* $H = K_3$ *such that* $G' = G/H = K_{2,p-1}$*;*

(d) G *is a reduced graph with order* n *such that* $n \geq 4$ *and* $p+1 \leq n \leq p+6$ *and*

$$2n-4 \geq |E(G)| \geq \begin{cases} 2n-4 & \text{if } n = 6+p; \\ 2n-5 & \text{if } n = 5+p; \\ 2n-6 & \text{if } n = i+p, i \in \{2,3,4\}; \\ 2n-5 & \text{if } n = 1+p. \end{cases}$$

Proof: Assume the conditions of Theorem 4 are satisfied. If $m = 2$, then we are done by Theorem 1. Hence assume $m \geq 3$.

Let G_1 be the K_3-free graph obtained from G by repeatedly contracting triangles until none remains. Set $n_1 = |V(G_1)|$. Let G' be the reduction of G and G_1, and set $c = |V(G')|$. Similar to the argument of the first paragraph in the proof of Theorem 1 before (7), now we have

$$|E(G)| \leq t(m, n-c+1) + 2c - 4, \tag{15}$$

with equality only if G has a complete m-partite subgraph H of order $n-c+1$, and its reduction G' has size $2c-4$.

Note that if $c < p$ then (a) of Theorem 4 holds. If $c = p$ then by (14) and (15), it is easy to see that (b) of Theorem 4 holds.

Now we may assume $c > p$. Since $m \geq 3$, we have $n \geq n_1 + 2$ and hence $n - n_1 + 1 \geq 3$. Furthermore, $n_1 - p + 1 \geq 2$, since $n_1 \geq c \geq p+1$. By Theorem E and Lamma 3 (with $a = n_1 - p + 1$ and $b = n - n_1 + 1$),

$$\begin{aligned} |E(G_1)| &\geq |E(G)| - t(m, n-n_1+1) \\ &\geq t(m, n-p+1) + 2p - 4 - t(m, n-n_1+1) \\ &\geq t(2, n_1-p+1) + 2p - 4 + \varepsilon(m, n_1-p+1, n-n_1+1). \end{aligned} \tag{16}$$

Set $\varepsilon = \varepsilon(m, n_1-p+1, n-n_1+1)$. Since $\varepsilon > 0$, G_1 is reduced by Theorem 1, i.e., $n_1 = c$. If $\varepsilon = 3$, then we are done by Corollary 2. Now assume $\varepsilon = 1$. Then $m = 3, n_1 - p + 1 = 2$ and $n - n_1 + 1 = 3$. By (16),

$$|E(G_1)| \geq t(2,2) + 2p - 3 = 2p - 2 = 2n_1 - 4.$$

By Theorem G, $G' = G_1 = K_{2n_1-2} = K_{2,p-1}$, whence (c) of Theorem 4 holds.

Finally, assume $\varepsilon = 2$. Then $n_1 - p + 1 = 2$, so by (16),

$$|E(G_1)| \geq t(2,2) + 2p - 2 = 2p - 1 = 2n_1 - 3,$$

contradicting Theorem F(c). ∎

Corollary 5. *(Catlin and Chen [5]). Let G be a 2-edge-connected simple graph of order n and let $p \in N - \{1\}$. If*

$$|E(G)| \geq \binom{n-p+1}{2} + 2p - 4, \tag{17}$$

then exactly one of these holds:

(a) *The reduction of G has order less than p;*

(b) *Equality holds in (17), G has a complete subgraph H of order $n-p+1$, and the reduction of G is $G' = G/H = K_{2,p-2}$.*

(c) *G is a reduced graph such that either*

$$|E(G)| \in \{2n-4, 2n-5\} \text{ and } n \in \{p+1, p+2\}$$

or

$$|E(G)| = 2n-4 \text{ and } n = p+3.$$

Proof: Choose m in Theorem 4 so that $m \geq n-p+1$. Then (5) and (14) together imply (17). Note that $m \geq n-p+1$ implies that $H_{m,n-p+1} = K_{n-p+1}$. Now Corollary 5 is an immediate consequence of Theorem 4. ∎

Remark. The case $p = 5$ of Corollary 5 is Theorem D which is a main result of Cai [3]. The case $p = 10$ of Corollary 5 for 3-edge-connected graph is Theorem E (Catlin and Chen [5]), which was a conjecture of Cai [3]. In the following we give some more results which improve the lower bounds of the inequalities in Theorem C and Theorem D.

Corollary 6. *Let G be a 2-edge-connected simple K_3-free graph of order n. If $n \geq 12$ and*

$$|E(G)| \geq t(2, n-4) + 6, \tag{18}$$

then exactly one of the following holds:

(a) *G is supereulerian;*

(b) *Equality holds in (18) and G contains a $H = T_{2,n-4}$ such that the reduction of G is $G' = G/H = K_{2,3}$.*

Proof: Set $p = 5$ in of Theorem 1. Since $n \geq 12 = p+7$, (c) of Theorem 1 is impossible. Corollary 6 now follows from Theorem 1, and the fact that any 2-edge-connected simple graph on $c \leq 4$ vertices is supereulerian. ∎

Corollary 7. *Let G be a 3-edge-connected simple K_3-free graph on n vertices. If $n \geq 16$ and*

$$|E(G)| \geq t(2, n-9) + 16, \tag{19}$$

then G is collapsible.

Proof: Since G is 3-edge-connected, the reduction of G is either 3-edge-connected or trivial. It is known that the Petersen graph is the only 3-edge-connected reduced graph of order at most 11 [7]. Combination of these facts with Theorem F(a) and the case $p = 10$ and $n \geq 16$ of Theorem 1 yields the desired result. ∎

Remark. Let G be the simple graph obtained from the Petersen graph and the complete bipartite graph $T_{2,n-9} = K_{\lfloor (n-9)/2 \rfloor, \lceil (n-9)/2 \rceil}$ with $n - 9 \geq 6$ by identifying one vertex from each graph. Then G is a 3-edge-connected graph of order $n = (n-9) + 10 - 1$. The size of G is

$$|E(G)| = t(m, n-9) + 15.$$

Since the reduction of G is the Petersen graph, G is not collapsible. Hence, (19) is sharp.

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