Endvertex-deleted Subgraphs

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We consider two seemingly related problems. The first concerns pairs of graphs G and H containing endvertices (vertices of degree 1) and having the property that, although they are not isomorphic, they have the same collection of endvertex-deleted subgraphs. The second question concerns graphs G containing endvertices and having the property that, although no two endvertices are similar, any two endvertex-deleted subgraphs of G are isomorphic.

1. Preliminaries.

All graphs considered are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [9], the most notable exception being that here we use the terms vertex and edge instead of point and line respectively.

For any graph G, the sets of vertices and of edges will be denoted by $V(G)$ and $E(G)$ respectively. The *order* of G is $|V(G)|$. Two adjacent vertices are said to be *neighbours* and the set of neighbours of a vertex v in G is denoted by $N_G(v)$. As in [9], two vertices u and v of G are *similar* if there exists some automorphism α of G such that $\alpha(u) = v$; they are *removal-similar* if $G - u$ and $G - v$ are isomorphic, and they are *pseudosimilar* if they are removal-similar but not similar. We recall that an *endvertex* is a vertex whose degree equals 1.

For definitions and results related to the reconstruction problem the reader is referred to [3]. In particular we note here that a subgraph of G obtained by deleting a vertex v together with its incident edges is called a *vertex-deleted subgraph* (and, if v is an endvertex, an *endvertex-deleted subgraph*) of G. The collection of vertex-deleted (endvertex-deleted) subgraphs of G is called the *deck* (*endvertex-deck*) and is denoted by $\mathcal{D}(G)$ ($\mathcal{D}_1(G)$).

Finally, if Γ is a group of permutations acting on a set X, and $R \subseteq X$, then $\Gamma_{(R)}$ denotes the pointwise stabiliser of R under the action of Γ, and $\Gamma_{\{R\}}$ denotes the setwise stabiliser of R. The automorphism group of a graph G is denoted by Aut$G$.

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2. Endvertex-reconstruction

There are many variants of the reconstruction problem involving the reconstruction of a graph from a proper subcollection of its deck. We are concerned here with the question of whether a graph is reconstructible from its endvertex-deck. Trees form one notable class of graphs which are endvertex-reconstructible. In [2] Bondy had conjectured that all graphs with sufficiently many endvertices are endvertex-reconstructible. The problem was investigated by various authors [2, 8, 16] but the final *coup de grâce* was delivered by Bryant in [5] who showed that, for any k, there exist nonisomorphic graphs G and H with k endvertices each and with $\mathcal{D}_1(G) = \mathcal{D}_1(H)$.

But the story need not end here. Although a graph with an arbitrarily large number of endvertices need not be endvertex-reconstructible, if the proportion of endvertices is sufficiently large then it would be endvertex-reconstructible. For example, if G is a graph with minimum degree at least 2 and G' is obtained from G by attaching one endvertex to each vertex of G, then G' is endvertex-reconstructible. We shall show that, in this case, if the number of endvertices added to G is greater than $|V(G)|/2$ then the resulting graph G' is endvertex-reconstructible.

What follows involves the straightforward application of techniques which have now become standard in reconstruction since [19] and [21], with slight adaptations to fit the present situation. We shall be considering graphs with endvertices such that the neighbour of any endvertex has degree at least 3. First we require some definitions. Let G be a graph with minimum degree at least 2 and let $S = (v_1, v_2, \ldots, v_t)$ be an ordered set of distinct vertices of G. Let $\mathrm{s} = (a_1, a_2, \ldots, a_t)$ be an ordered set of nondecreasing positive integers. Then the graph obtained from G by attaching a_i endvertices to each vertex v_i is denoted by $G[S; \mathrm{s}]$. It is convenient to consider $G[S; \mathrm{s}]$ as the graph G with labels on its vertices: the vertices in $V(G) - S$ are given the label 0 while each vertex v_i in S is given the label a_i. Considered this way, an endvertex-deleted subgraph of $G[S; \mathrm{s}]$ is a labelled graph obtained from $G[S; \mathrm{s}]$ by reducing one of its positive labels by 1. Two labelled graphs are isomorphic if there is an automorphism of G which preserves labels.

If $\mathcal{D}_1(G[S_1; \mathrm{s}_1]) = \mathcal{D}_1(G[S_2; \mathrm{s}_2])$ and $|S_1| = |S_2|$, then $\mathrm{s}_1 = \mathrm{s}_2$. Therefore, in the sequel, S_1 and S_2 will always denote subsets of $V(G)$ with $|S_1| = |S_2|$ and we shall drop the reference to s_1 and s_2, denoting the two graphs by $G[S_1]$ and $G[S_2]$ respectively. If $S_3 \subseteq S_1$ then $G[S_3]$ is the labelled graph obtained by giving to each vertex in S_3 the same label as in $G[S_1]$ and giving all the other vertices the label 0.

The set of all isomorphisms from $G[S_1]$ to $G[S_2]$ will be denoted by $(G[S_1] \longrightarrow G[S_2])$. For $X \subseteq S_1$, $(G[S_1] \xrightarrow{X} G[S_2])$ will denote the set of automorphisms α of G such that

(i) if $u \in S_1 - X$ then the label of $\alpha(u)$ in $G[S_2]$ equals the label of u in $G[S_1]$,

and (ii) if $u \in X$ then the label of $\alpha(u)$ in $G[S_2]$ does not equal the label of u in $G[S_1]$.

The orders of these sets are denoted by

$$\left|G[S_1] \longrightarrow G[S_2]\right| \text{ and } \left|G[S_1] \xrightarrow{X} G[S_2]\right| \text{ respectively.}$$

The first lemma is the analogue of the well-known result commonly referred to as Kelly's Lemma.

Lemma 1. *Let $S_3 \subset S_1 \subseteq V(G)$ and let $\mathcal{D}_1(G[S_1]) = \mathcal{D}_1(G[S_2])$. Then $s(G[S_3], G[S_1]) = s(G[S_3], G[S_2])$, where $s(G[S_i], G[S_j])$ denotes the number of subgraphs of $G[S_j]$ isomorphic to $G[S_i]$.*

Proof. Let r be the total number of graphs isomorphic to $G[S_3]$ which appear as subgraphs of the graphs in $\mathcal{D}_1(G[S_1])$ (and hence $\mathcal{D}_1(G[S_2])$). Let p be the sum of the labels in $G[S_1]$ of all the vertices in $S_1 - S_3$. Clearly $p > 0$ since $|S_3| < |S_1|$ and the labels are positive. Then

$$r = p \cdot s(S(G[S_3], G[S_1]) = p \cdot s(S(G[S_3], G[S_2])$$

and the result follows since $p > 0$. □

Corollary 1. *If $\mathcal{D}_1(G[S_1]) = \mathcal{D}_1(G[S_2])$ and $S_3 \subset S_1$ then*

$$\left|G[S_3] \longrightarrow G[S_1]\right| = \left|G[S_3] \longrightarrow G[S_2]\right|.$$

Proof. This follows easily from Lemma 1 since,

$$\begin{aligned} \left|G[S_3] \longrightarrow G[S_1]\right| &= s(G[S_3], G[S_1]) \cdot \left|G[S_3] \longrightarrow G[S_3]\right| \\ &= s(G[S_3], G[S_2]) \cdot \left|G[S_3] \longrightarrow G[S_3]\right| \\ &= \left|G[S_3] \longrightarrow G[S_2]\right|. \end{aligned}$$

□

Lemma 2. *Let $X \subseteq S_1$. Then*

$$\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \sum_{Y \subseteq X} (-1)^{|Y|} \left|G[(S_1 - X) \cup Y] \longrightarrow G[S_2]\right|.$$

Proof. For $u \in X$ let A_u denote the set $(G[(S_1 - X) \cup \{u\}] \longrightarrow G[S_2])$. Note that

$$\cap_{i=1}^{r} A_{u_i} = (G[(S_1 - X) \cup_{i=1}^{r} \{u_i\}] \longrightarrow G[S_2]).$$

Since $\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \left|G[S_1 - X] \longrightarrow G[S_2]\right| - \left|\cup_{u \in X} A_u\right|$, the result follows by applying the inclusion-exclusion principle. □

The next theorem is the analogue of the Nash-Williams Lemma in edge-reconstruction.

Theorem 1. *Let $\mathcal{D}_1(G[S_1]) = \mathcal{D}_1(G[S_2])$, and let $X \subseteq S_1$. Then*

$$\left|G[S_1] \longrightarrow G[S_2]\right| = \left|G[S_1] \longrightarrow G[S_1]\right| + (-1)^{|X|}\left(\left|G[S_1] \xrightarrow{X} G[S_2]\right| - \left|G[S_1] \xrightarrow{X} G[S_1]\right|\right).$$

Proof. By Lemma 2,

$$\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \sum_{Y \subseteq X} (-1)^{|Y|} \left|G[(S_1 - X) \cup Y] \longrightarrow G[S_2]\right|$$

and

$$\left|G[S_1] \xrightarrow{X} G[S_1]\right| = \sum_{Y \subseteq X} (-1)^{|Y|} \left|G[(S_1 - X) \cup Y] \longrightarrow G[S_1]\right|.$$

Subtracting these two equations, all terms on the right hand side cancel (by Corollary 1) except for $Y = X$, giving the required result. □

Corollary 2. *Let* $\mathcal{D}_1(G[S_1]) = \mathcal{D}_1(G[S_2])$ *and suppose that* $G[S_1]$ *is not isomorphic to* $G[S_2]$. *Let* $X \subseteq S_1$.
Then (i) *if* $|X|$ *is odd, then* $\left|G[S_1] \xrightarrow{X} G[S_2]\right| > 0$,
and (ii) *if* $|X|$ *is even, then* $\left|G[S_1] \xrightarrow{X} G[S_1]\right| > 0$.

Proof. Since $G[S_1] \not\simeq G[S_2]$, $\left|G[S_1] \longrightarrow G[S_2]\right| = 0$. Therefore when $|X|$ is odd,

$$\left|G[S_1] \xrightarrow{X} G[S_2]\right| = \left|G[S_1] \longrightarrow G[S_1]\right| + \left|G[S_1] \xrightarrow{X} G[S_1]\right| > 0,$$

and, when $|X|$ is even,

$$\left|G[S_1] \xrightarrow{X} G[S_1]\right| = \left|G[S_1] \longrightarrow G[S_1]\right| + \left|G[S_1] \xrightarrow{X} G[S_2]\right| > 0.$$

□

Theorem 2. *Let* $R \subseteq S$ *be a set of vertices which are given the same label in* $G[S]$. *If either* $|R| > |V(G)|/2$ *or* $|R| > 1 + \log_2 |\text{Aut}G|$, *then* $G[S]$ *is endvertex-reconstructible.*

Proof. Suppose $G[S]$ is not endvertex-reconstructible and let $G[S']$ be an endvertex-reconstruction of $G[S]$, not isomorphic to $G[S]$. Then taking $X = R$ in Corollary 2 implies that there is a $T \subseteq V(G)$ disjoint from R such that $G[(S-R)\cup T] \simeq G[S]$, if $|S|$ is even, or $G[(S-R)\cup T] \simeq G[S']$, if $|S|$ is odd. But this is impossible if $|S| > |V(G)|/2$.

Also, if $G[S]$ is not endvertex-reconstructible then, by Corollary 2(ii), for every even subset X of R, $\left|G[S] \xrightarrow{X} G[S]\right| \geq 1$. There are $2^{|R|-1}$ even subsets of R and, since the sets $(G[S] \xrightarrow{X} G[S])$ are disjoint for different X, it follows that $|\text{Aut}G| \geq 2^{|R|-1}$. Therefore if $|R| > 1 + \log_2 |\text{Aut}G|$, then $G[S]$ is endvertex-reconstructible. □

Corollary 3. *Let* G *be a graph with minimum degree at least 2 and let* G' *be obtained from* G *by attaching one endvertex to each of* k *distinct vertices of* G. *If either* $k > |V(G)|/2$ *or* $k > 1 + \log_2 |\text{Aut}G|$, *then* G' *is endvertex-reconstructible.* □

(**Note:** The second condition of Corollary 3 can also be obtained as a special case of Corollary 2.4 of [1].)

Bryant's counterexamples to Bondy's conjecture are, in fact, graphs like G', that is, having endvertices no two of which have a common neighbour and none adjacent to a vertex of degree 2. In view of this and of the corollary, the natural question to ask is: If G' is as in Corollary 3, what is the largest value of $|k|\,/\,|V(G)|$ for which G' can be not endvertex-reconstructible?

3. Mutually pseudosimilar endvertices

The connection between pseudosimilarity and the reconstruction problem has become part of the folklore of graph theory since it was reported in [10] that a purported proof of the reconstruction conjecture failed because it assumed that removal-similar vertices are necessarily similar. Although this has drawn on pseudosimilarity the suspicion that it might possibly account for the eventual falsity of the conjecture, there does not seem to be much concrete evidence that the concept of pseudosimilarity might help in settling the problem one way or the other. Possible pointers in this direction could be the facts that a tree is reconstructible both from its endvertex-deck and its end-cutvertex-deck (an *end-cutvertex* is a vertex having only one neighbour with degree greater than 1) and that endvertices and end-cutvertices cannot be pseudosimilar in a tree (see [12], for example).

The following simple result gives a tenuous link between pseudosimilarity and reconstruction. Here, a vertex u in a graph G is said to be *replaceable* if, for some set A of neighbours of u, there is a set B of vertices of G not adjacent to u such that G is isomorphic to the graph obtained by removing all the edges in $\{ua : a \in A\}$ and replacing them by the edges in $\{ub : b \in B\}$; u is *irreplaceable* if it is not replaceable.

Theorem 3. *Let u be an irreplaceable vertex in a graph G. Then u cannot be pseudosimilar to any vertex to which it is not adjacent. Also, if $d(u) \geq 2$ and u has minimum degree in G then G is edge-reconstructible.*

Proof. Suppose first that v is a vertex of G not adjacent to u and that u and v are pseudosimilar. Note that, since u and v are not adjacent, $N_{G-v}(u) = N_G(u)$ and $N_{G-u}(v) = N_G(v)$. Let us denote $N_G(u)$ and $N_G(v)$ by $N(u)$ and $N(v)$ respectively. (Note that $|N(u)| = |N(v)|$, since u and v are pseudosimilar.) Now let α be an isomorphism from $G-u$ to $G-v$. If $\alpha(N(u)) = N(v)$, then α can be extended to an automorphism of G mapping u into v. Since u and v are pseudosimilar this is not possible. Therefore, letting $X = \alpha^{-1}(N(v))$, the sets $A = N(u) - X$ and $B = X - N(u)$ are not empty. But, since G is obtained from $G - v$ by adding a new vertex and joining it to $N(v)$, and since $\alpha(X) = N(v)$, then G is isomorphic to the graph obtained from $G-u$ by adding a new vertex and joining it to the vertices in X; that is, G is isomorphic to the graph obtained by removing, from G, all the edges in $\{ua : a \in A\}$ and replacing them by the edges in $\{ub : b \in B\}$, contradicting the fact that u is irreplaceable. Therefore u and v cannot be pseudosimilar.

Now suppose that $d(u) \geq 2$ and that G is not edge-reconstructible. It then follows from the Nash-Williams Lemma in edge-reconstruction (see

[3]) that any set containing an even number of edges of G can be replaced by non-edges to give a graph still isomorphic to G. That is, given two neighbours a_1, a_2 of u, there are two vertices b_1, b_2 not adjacent to u such that $G \simeq G - ua_1 - ua_2 + ub_1 + ub_2$ (since u has minimum degree in G), again contradicting the fact that u is irreplaceable. Hence G is edge-reconstructible. □

But rather than because of any possible link with the reconstruction problem, pseudosimilarity has been studied mainly for its own independent interest. One question which has been considered [7, 11, 13, 15, 17] is the construction of graphs with large sets of mutually pseudosimilar vertices (the vertices in a set $S \subset V(G)$ are mutually pseudosimilar if any two of them are removal-similar but no two are similar).

In [17] a construction of [15] was extended to give a family of graphs having 3^t mutually pseudosimilar pseudosimilar endvertices for $t \geq 1$. This construction uses as its basis a graph containing exactly three endvertices all of which are mutually pseudosimilar, and it yields a sequence of graphs each having $k = 3^t$ mutually pseudosimilar endvertices and a total of $O(k^{\log 7/\log 3})$ vertices. This suggests two problems. The first one is to find, for $r \geq 4$, a graph all of whose r endvertices are mutually pseudosimilar. The second is to use this graph as the basis of a construction of a sequence of graphs having $k = r^t$ mutually pseudosimilar endvertices and $O(k^{1+\epsilon})$ vertices, with ϵ as small as possible. We now consider these problems for $r = 4$.

First, suppose that G' is a graph with r endvertices, all of which are mutually pseudosimilar. Let R be the set of neighbours of the endvertices of G' (note that no two endvertices can have a common neighbour, therefore $|R| = r$), and let G be obtained from G' by removing all its endvertices. Let $\Gamma = \mathrm{Aut}G$. Then,

(i) $\Gamma_{\{R\}} = \Gamma_{(R)}$,

and (ii) for any two $(|R| - 1)$-subsets A, B of R, there is a permutation α in Γ such that $\alpha(A) = B$.

The converse of this is also true, that is, if Γ is a group of permutations acting on some set X and, for some $R \subseteq X$, the above two conditions hold, then one can construct a graph G with minimum degree at least 2 and to which $|R|$ endvertices can be attached such that all are mutually pseudosimilar. This is possible because of the following result from [4], a short proof of which can be found in [20; Prob. 12.21].

Theorem 4. *Let Γ be a permutation group acting on a set X. Then there exists a graph G such that $X \subseteq V(G)$, X is invariant under the action of* Aut$G$ *and the restriction of* Aut$G$ *to X gives a permutation group equivalent to Γ.*

(**Note.** As observed in [5], one can always carry out the construction of G in such a way that it has minimum degree at least 2.)

Therefore all we have to do is to find a permutation group satisfying conditions (i) and (ii) above. For $r = |R| = 4$, we can let Γ be the group of affine transformations of the field $GF(8)$. Although this group is not 3-transitive, it is 3-homogeneous [18] (that is, any two 3-sets are related under the action of Γ on the set of 3-subsets of $GF(8)$). Therefore all we need is a 4-set R such that $\Gamma_{\{R\}} = \Gamma_{(R)}$. If we represent $GF(8)$ as $Z_2[x]/p(x)$, where $p(x)$ is the primitive, irreducible (over Z_2) polynomial $x^3 + x + 1$, and if we let $R = \{0, 1, x, x^2\}$, then one can check that the only permutation in Γ which fixes R setwise is the identity, therefore certainly $\Gamma_{\{R\}} = \Gamma_{(R)}$. We can then obtain the graph G of Theorem 4 with X being the set $GF(8)$. If the graph G' is then obtained from G by attaching an endvertex to every vertex in R, then G' would be a graph with four endvertices, all of which are mutually pseudosimilar.

The construction now proceeds using G as a basis. Let $G_1 = G'$ and let H_1 be G_1 less one of its endvertices. Having constructed G_t, let H_t be G_t less one of its pseudosimilar endvertices. Then, starting with G, G_{t+1} is obtained by attaching a copy of G_t to each vertex in $R \subset V(G)$ and a copy of H_t to each of the other vertices in $X - R = GF(8) - R \subset V(G)$. (By attaching a copy of G_t (or H_t) to a vertex v of G we mean joining v to every vertex of G_t (or H_t) which is not an endvertex.)

Each graph G_t so obtained has 4^t mutually pseudosimilar endvertices and $O(8^t)$ vertices. Therefore, if $k = 4^t$ is the number of pseudosimilar endvertices, then the total number of vertices in G_t is $O(k^{\log 8/\log 4}) = O(k^{3/2})$. Therefore this construction does better at "packing" pseudosimilar endvertices than both the one in [17] mentioned above, and the one in [15] which produced graphs with $k = 2^t$ mutually pseudosimilar endvertices and order $O(k^{\log 3/\log 2})$.

The most interesting feature about the above procedure is perhaps the construction of the graph G via a group of permutations Γ satisfying conditions (i) and (ii). We have here exploited the fact that the group used is 3-homogeneous. One cannot, however, hope to do this in general since, as shown in [18], if a group of permutations acting on a set X is s-homogeneous with $5 \leq s \leq |X|/2$, then Γ is s-transitive, and since the only s-transitive groups with $s \geq 6$ are the symmetric and the alternating groups. We therefore single out the following problem.

Problem. *For $r \geq 5$, find a group of permutations Γ acting transitively on a set X such that X contains an r-subset R with the following properties: (i) $\Gamma_{\{R\}} = \Gamma_{(R)}$; and (ii) if A, B are two $(r-1)$-subsets of R, then there is a permutation α in Γ such that $\alpha(A) = B$.*

(**Note.** The requirement that Γ be transitive imposes no extra restriction because if Γ and R satisfy (i) and (ii) then all the elements of R are in the same orbit of Γ acting on X. Therefore, if Γ is not transitive one can always restrict its action to the orbit containing the elements of R.)

As we saw in the previous discussion, this problem is equivalent to finding a graph G containing r endvertices all of which are mutually pseudosimilar. Peter Cameron [6] has constructed examples of permutation groups with the above properties. A slightly simplified version of his construction runs as follows. Let $X = F^{r-1}$, the vector space of dimension $r-1$ over the finite field F, and let Γ be the group of all linear automorphisms of X. Let $B = \{e_1, e_2, \ldots, e_{r-1}\}$ be a basis of X, $f = \sum a_i e_i$ an element of X and $R = B \cup \{f\}$. Suppose the following conditions on the a_i hold:

(1) $a_i \neq 0$, $1 \leq i \leq r-1$;

(2) $a_i \neq a_j, i \neq j$;

(3) $a_i a_j \neq 1$, $1 \leq i, j \leq r-1$;

(4) $a_i + a_j a_k \neq 0$, $1 \leq i, j, k \leq r-1$.

Then Γ and R have the required properties (i) and (ii). For since none of the a_i is zero, any two $r-1$-subsets of R are bases of X, therefore similar under the action of Γ. Hence condition (i) holds. Also, if, for some $\alpha \in \Gamma$ not equal to the identity, $\alpha(R) = R$ then, since the a_i are distinct, we cannot have $\alpha(B) = B$ and $\alpha(f) = f$; therefore, for some j, $\alpha(e_j) = f$, $\alpha(f) = e_{\pi(j)}$ and $\alpha(e_i) = e_{\pi(i)}$ for $i \neq j$, where π is a permutation of $\{1, 2, \ldots, r-1\}$. But then, since $f = \sum a_i e_i$, we have

$$e_{\pi(j)} = \sum_{i \neq j} a_i e_{\pi(i)} + a_j f.$$

This gives $a_i + a_j a_{\pi(i)} = 0$ and $a_j a_{\pi(j)} = 1$, contradicting (3) and (4). Therefore the only permutation α with $\alpha(R) = R$ is the identity, and hence condition (ii) also holds.

If $q = |F|$ is greater than some quadratic polynomial in r, then it is always possible to choose the a_i satisfying the above conditions since, if $a_1, a_2, \ldots, a_{m-1}$ have been chosen ($m \leq r-1$), then a_m must avoid the solution of a finite number of equations involving at most two of the $a_1, a_2, \ldots, a_{m-1}$. Therefore we have a group with the required properties if $|X| = O(r^{2r})$. The question which now arises is whether there are groups of substantially smaller degree having these properties. The size of X would certainly have to be larger than $|R|$, since otherwise (i) and (ii) cannot both hold. In fact one can say a little more. Let $R = \{1, 2, \ldots, r\}$ and let R_i denote $R - i, 1 \leq i \leq r$. For any $R_i, 2 \leq i \leq r$, there is an $\alpha_i \in \Gamma$ such that $\alpha_i(R_i) = R_1$. By (i), $\alpha_i(i) \notin R$; also, for $i \neq j$, $\alpha_i(i) \neq \alpha_j(j)$, otherwise $\alpha_j^{-1}\alpha_i(R) = R$, and the action of $\alpha_j^{-1}\alpha_i$ on R is nontrivial, contradicting (i). Therefore $|X| \geq |R| + |\{\alpha_2(2), \ldots, \alpha_r(r)\}| = 2r - 1$. This crude estimate shows that, if the above constructions are employed to give a sequence of graphs G_t having $k = r^t$ mutually pseudosimilar endvertices, then the total number of vertices of G_t is $O\left(k^{\frac{\log|X|}{\log|R|}}\right)$ which must be at least $O\left(k^{\frac{\log(2r-1)}{\log r}}\right)$.

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