ALL 4-STARS ARE SKOLEM-GRACEFUL

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Abstract. In this paper, we give two constructive proofs that all 4-stars are Skolem-graceful. A 4-star is a graph with 4 components, with at most one vertex of degree exceeding 1 per component. A graph $G = (V, E)$ is Skolem-graceful if its vertices can be labelled $1, 2, \ldots, |V|$ so that the edges are labelled $1, 2, \ldots, |E|$, where each edge-label is the absolute difference of the labels of the two end-vertices. Skolem-gracefulness is related to the classic concept of gracefulness, and the methods we develop here may be useful there.

Let $G = (V, E)$ be a finite simple undirected graph with vertex set V and edge set E. In [2], a Skolem-graceful labelling of G is defined to be a bijection $f: V \to \{1, 2, \ldots, |V|\}$ such that the induced labelling $f\star: E \to \{1, 2, \ldots, |E|\}$ defined by $f\star(u, v) = |f(u) - f(v)|$ is also a bijection. Such an f is called an S-labelling of G, and if it exists, G is then said to be Skolem-graceful.

The only connected graphs that can be Skolem-graceful are the trees. It is easy to see that a tree is Skolem-graceful if and only if it is graceful. For the classic concept of gracefulness and the conjecture that all trees are graceful, see [1]. Henceforth, we restrict our attention to graphs with at least two components.

We point out that the Skolem-gracefulness of one class of such graphs is completely settled. A one-factor of order k consists of $2k$ vertices joined in pairs by k edges. It is Skolem-graceful if and only if $k \equiv 0$ or $1 \pmod 4$. This follows from a result in [4] which states that a Nickerson sequence of order k exists if and only if $k \equiv 0$ or $1 \pmod 4$. Such a sequence consists of two copies of each of $1, 2, \ldots, k$ such that for $1 \le i \le k$, there are exactly $i - 1$ other terms between the two copies of i.

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It is easy to see that a Nickerson sequence of order k induces an S-labelling of a one-factor of order k, and vice versa. For instance, the sequence 23243114 and the S-labelling in the following diagram induce each other.

(1) ----- (3)
(2) ----- (5)
(4) ----- (8)
(6) ----- (7)

From now on, we shall replace the diagram with the equivalent notation $(1 : 3)(2 : 5)(4 : 8)(6 : 7)$.

We now consider another class of disconnected graphs and attempt to determine which of them are Skolem-graceful. We define a k-star $St(a_1, a_2, \ldots, a_k)$ as a graph with k components having $a_1, +1, a_2 + 1, \ldots, a_k + 1$ vertices respectively, such that all but one of the vertices in each component have degree 1. In each component, the vertex of maximum degree is called the nucleus. If the component has only two vertices, only one of them is considered the nucleus.

It is a trivial result that all 1-stars are Skolem-graceful. In [3], it is proved that this is true for $St(a_1, a_2)$ if and only if $a_1 a_2$ is even, and for $St(a_1, a_2, a_3)$ if and only if $a_1 a_2 a_3$ is even. In [2], it is conjectured that all 4-stars are Skolem-graceful. In this paper, we give two proofs that the conjectured result is true.

Our first approach is via explicit constructions. We divide the 4-stars into five exhaustive but not mutually exclusive classes, and give a general construction of an S-labelling for each class. That these are indeed S-labellings can be verified in a routine manner.

We establish some notations. Let A, B, C and D be the respective nuclei of $St(a_1, a_2, a_3, a_4)$. Let W_i, $1 \leq i \leq a_1$, be the vertices joined to A. Let X_i, $1 \leq i \leq a_2$, be the vertices joined to B. Let Y_i, $1 \leq i \leq a_3$, be the vertices joined to C. Let Z_i, $1 \leq i \leq a_4$, be the vertices joined to D.

Theorem 1. *$St(2p, 2q, 2r, 2s)$ is Skolem-graceful.*

Proof: We define the S-labelling f as follows:

$$f(A) = 3, \quad f(B) = 2, \quad f(C) = 1, \quad f(D) = 2p + 2q + 2r + 2s + 4,$$
$$\begin{aligned} f(W_i) &= i + 3 && \text{for } 1 \leq i \leq p, \\ &= i + 2q + 2r + 2s + 3 && \text{for } p + 1 \leq i \leq 2p, \\ f(X_i) &= i + p + 3 && \text{for } 1 \leq i \leq q, \\ &= i + p + 2r + 2s + 1 && \text{for } q + 1 \leq i \leq q, \end{aligned}$$

$$
\begin{aligned}
f(Y_i) &= i+p+q+3 && \text{for } 1 \le i \le r,\\
&= i+p+q+2s+1 && \text{for } r+1 \le i \le 2r,\\
f(Z_i) &= i+p+q+r+3 && \text{for } 1 \le i \le 2s-2 \text{ if } s>1,\\
&= p+q+2r+2s+2 && \text{for } i = 2s-1,\\
&= p+2q+2r+2s+3 && \text{for } i = 2s.
\end{aligned}
$$

∎

Theorem 2. *$St(1,2q,2r,s)$ is Skolem-graceful for $s \ge 2$.*

Proof: We define the S-labelling f as follows:

$$
f(A)=1,\quad f(B)=3,\quad f(C)=2q+2r+7,\quad f(D)=4,
$$

$$
\begin{aligned}
f(W_1) &= q+r+4, &&\\
f(X_i) &= 2 && \text{for } i=1,\\
&= i+3 && \text{for } 2 \le i \le q \text{ if } q>1,\\
&= i+2r+6 && \text{for } q+1 \le t \le 2q,\\
f(Y_i) &= i+q+3 && \text{for } 1 \le i \le r,\\
&= i+q+6 && \text{for } r+1 \le i \le 2r,\\
f(Z_i) &= i+q+r+4 && \text{for } i=1,2,\\
&= i+2q+2r+5 && \text{for } 3 \le i \le s \text{ if } s>2.
\end{aligned}
$$

∎

Theorem 3. *$St(2p-1,2q,2r,s)$ is Skolem-graceful for $p \ge 2$.*

Proof: We define the S-labelling f as follows:

$$
f(A)=3,\quad f(B)=2,\quad f(C)=2p+2q+2r+4,\quad f(D)=4,
$$

$$
\begin{aligned}
f(W_i) &= 1 && \text{for } i=1,\\
&= i+4 && \text{for } 2 \le i \le p-1 \text{ if } p>2,\\
&= i+2q+2r+4 && \text{for } p \le i \le 2p-1,\\
f(X_i) &= i+p+3 && \text{for } 1 \le i \le q,\\
&= i+p+2r+2 && \text{for } q+1 \le i \le 2q,\\
f(Y_i) &= i+p+q+3 && \text{for } 1 \le i \le 2r-1,\\
&= p+2q+2r+3 && \text{for } i=2r,\\
f(Z_i) &= 5 && \text{for } i=1,\\
&= i+2p+2q+2r+3 && \text{for } 2 \le i \le s \text{ if } s>1.
\end{aligned}
$$

∎

Theorem 4. *$St(1,1,r,s)$ is Skolem-graceful.*

Proof: We may assume that $r \leq s$. We define the S-labelling f as follows:

$$\begin{aligned}
f(A) &= 1, \quad f(B) = 3, \quad f(C) = 2r+6, \quad f(D) = 4, \\
f(W_1) &= 2, \\
f(X_1) &= 2r+5, \\
f(Y_i) &= 2r+5-2i && \text{for } 1 \leq i \leq r, \\
f(Z_i) &= 2i+4 && \text{for } 1 \leq i \leq r, \\
&= i+r+6 && \text{for } r+1 \leq i \leq s \text{ if } s > r.
\end{aligned}$$

Theorem 5. *$St(2p-1, 2q-1, r, s)$ is Skolem-graceful for $p \geq 2$.*

Proof: We define the S-labelling f as follows:

$$\begin{aligned}
f(A) &= 1, \quad f(B) = 2p+2q+r+3, \quad f(C) = 2, \quad f(D) = 4, \\
f(W_i) &= 1 && \text{for } i = 1, \\
&= i+4 && \text{for } 2 \leq i \leq p-1 \text{ if } p > 2, \\
&= i+2q+r+3 && \text{for } p \leq i \leq 2p-1, \\
f(X_i) &= i+p+3 && \text{for } 1 \leq i \leq q-1 \text{ if } q > 1, \\
&= i+p+r+3 && \text{for } q \leq i \leq 2q-1, \\
f(Y_i) &= i+p+q+2 && \text{for } 1 \leq i \leq r, \\
f(Z_i) &= 5 && \text{for } i = 1, \\
&= i+2p+2q+r+2 && \text{for } 2 \leq i \leq s \text{ if } s > 1.
\end{aligned}$$

∎

Theorem 6. *All 4-stars are Skolem-graceful.*

Proof: Consider $St(a_1, a_2, a_3, a_4)$. If all of a_1, a_2, a_3 and a_4 are even, the result follows from Theorem 1. If at least one but at most two of them are odd, the result follows from Theorem 2 or 3. If at least two of them are odd, the result follows from Theorem 4 or 5. ∎

In Table 1, we give examples of S-labellings of $St(a_1, a_2, a_3, a_4)$ with $a_i \leq 3$ for $1 \leq i \leq 4$ and $a_1 + a_2 + a_3 + a_4 \leq 10$. Some of these are based on constructions as yet to be described.

Our second approach is via recursive constructions. In fact, we shall prove a result which is stronger than the conjecture, that every 3-star has a PSL. A PSL, or a punctured S-labelling, of a k-star is defined in essentially the same way as an S-labelling, except that the range of the bijection f is $\{1, \ldots, k, k+2, \ldots, |V|+1\}$ instead. For instance, $(1:3)(2:5)(6:7)$ is a PSL of $St(1,1,1)$.

It is easy to see that a PSL of $St(a_1, a_2, a_3)$ induces an S-labelling of $St(a_1, a_2, a_3, a_4)$ for any a_4. For instance, the above PSL of $St(1,1,1)$ induces the S-labelling $(1:3)(2:5)(6:7)$ $(4:8, \ldots, \ell+4)$ of $St(1,1,1,\ell)$ for all ℓ.

For each $St(a_1, a_2, a_3)$, we shall construct a PSL such that there exists a non-nucleus vertex labelled $x > 4$ which has the following properties:

(a) The edges joining two vertices both with labels less than x or both with labels greater than x have induced labels $1, 2, \ldots, e$, where e is the number of such edges.

(b) The edge incident with the vertex labelled x has induced label $e + 1$,

Such a PSL is said to be good, with respect to the vertex labelled x. We call it a GPSL. For instance, the PSL $(1:3)(7:6)(2:5)$ of $St(1,1,1)$ is good with respect to the vertex labelled 5.

We now describe the two basic techniques we employ in our proof.

The Splittinq Lemma. *If $St(a_1, a_2, a_3)$ has a GPSL with respect to a vertex in the third component, then so does $St(a_1, a_2, a_3 + s)$ for any s.*

Proof: Suppose the GPSL is with respect to the vertex u labelled x. Let it be joined to the nucleus v. We split u into $s + 1$ vertices and join each new vertex only to v. We modify the labelling as follows. All original vertices with labels less than x are unaffected. All original vertices with labels greater than x have their labels increased by s. The new vertices are labelled $x, x+1, \ldots, x+s$. It is routine to verify that this is a GPSL of $St(a_1, a_2, a_3 + s)$ with respect to any of the new vertices. ∎

The Switching Lemma. *If $St(a_1, a_2, a_3)$ has a GPSL $(.:\ldots)$ $(x:\ldots)(y:\ldots, z, \ldots)$ with respect to the vertex labelled $z = (x+y)/2$, then so does $St(a_1, a_2 + 1, a_3 - 1)$.*

Proof: We replace the edge joining the vertices with labels y and z by an edge joining the vertices with labels x and z. The labelling is unchanged. It is clear that this is a GPSL of $St(a_1, a_2 + 1, a_3 - 1)$ with respect to the vertex labelled z. ∎

Applying the Splitting Lemma to the GPSL $(1:3)(7:6)(2:5)$ of $St(1,1,1)$, the GPSL $(1:3)(2t+6:2t+5)(2:5, \ldots, 2t+4)$ of $St(1,1,2t)$ can be obtained for all t. Applying the Switching Lemma to this, the GPSL $(1:3)(2:5, \ldots, t+3, t+5, \ldots, 2t+4)$ $(2t+6:t+4, 2t+5)$ of $St(1, 2t-1, 2)$ can be obtained. Applying the Splitting Lemma again, we obtain for all n the GPSL $(1:3)(2:5, \ldots, t+3, n+t+3, \ldots, n+2t+2)$ $(n+2t+4:t+4, \ldots, n+t+2, n+2t+3)$ of $St(1, 2t-1, n)$.

Some $St(a_1, a_2, a_3)$ where $a_1 + a_2 + a_3 \not\equiv 1 \pmod 3$ have a GPSL with the following properties:

(a) The three nuclei are labelled $1, 2$ and $a_1 + a_2 + a_3 + 3$.

Table 1

Stars	Theorems	S-Labellings
St(1,1,1,2)	4	(1:2)(3:7)(8:5)(4:6,9)
St(1,1,1,2)	10	(1:3)(2:5)(6:7)(4:8,9)
St(1,1,2,2)	4	(1:2)(3:9)(10:5,7)(4:6,8)
St.(1,2,2,1)	8	(1:3)(2:6,7)(8:5,9)(4:10)
St(1,1,1,3)	4	(1:2)(3:7)(8:5)(4:6,9,10)
St(3,1,1,1)	5	(3:1,8,9)(10:7)(2:6)(4:5)
St(1,1,3,1)	7	(1:3)(8:9)(2:5,6,7)(4:10)
St(1,1,1,3)	10	(1:3)(25)(6:7)(4:8,9,10)
St(1,2,2,2)	2	(1:6)(3:2,10)(11:5,9)(4:7,8)
St(1,2,2,2)	8	(1:3)(2:6,7)(8:5,9)(4:10,11)
St(2,2,2,1)	10	(2:7,8)(9:6,10)(1:3,5)(4:11)
St(1,1,2,3)	4	(1:2)(3:9)(10:5,7)(4:6,8,11)
St(3,1,1,2)	5	(3:1,8,9)(10:7)(2:6)(4:5,11)
St(1,1,3,2)	7	(1:3)(8:9)(2:5,6,7)(4:10,11)
St(1,2,3,1)	9	(1:3)(2:7,8)(9:5,6,10)(4:11)
St(2,2,2,2)	1	(3:4,11)(2:5,9)(1:6,7)(12:8,10)
St(2,2,2,2)	10	(2:7,8)(9:6,10)(1:3,5)(4:11,12)
St(1,2,2,3)	2	(1:6)(3:2,10)(11:5,9)(4:7,8,12)
St(3,2,2,1)	3	(3:1,10,11)(2:6,8)(12:7,9)(4:5)
St(3,1,2,2)	5	(3:1,9,10)(11:8)(2:6,7)(4:5,12)
St(1,2,2,3)	8	(1:3)(2:6,7)(8:5,9)(4:10,11,12)
St(1,2,3,2)	9	(1:3)(2:7,8)(9:5,6,10)(4:11,12)
St(1,1,3,3)	4	(1:2)(3:11)(12:5,7,9)(4:6,8,10)
St(3,1,1,3)	5	(3:1,8,9)(10:7)(2:6)(4:5,11,12)
St(1,1,3,3)	7	(1:3)(8:9)(2:5,6,7)(4:10,11,12)
St(3,2,2,2)	3	(3:1,10,11)(2:6,8)(12:7,9)(4:5,13)
St(2,2,2,3)	10	(2:7,8)(9:6,10)(1:3,5)(4:11,12,13)
St(3,1,2,3)	5	(3:1,9,10)(11:8)(2:6,7)(4:5,12,13)
St(2,3,3,1)	8	(2:9,10)(1:3,5,7)(11:6,8,12)(4:13)
St(1,2,3,3)	9	(1:3)(2:7,8)(9:5,6,10)(4:11,12,13)
St(3,2,2,3)	3	(3:1,10,11)(2:6,8)(12:7,9)(4:5,13,14)
St(3,3,2,2)	5	(3:1,11,12)(13:6,9.10)(2:7,8)(4:5,14)
St(2,3,3,2)	8	(2:9,10)(1:3,5,7)(11:6,8,12)(4:13,14)
St(3,1,3,3)	5	(3:1,10,11)(12:9)(2:6,7,8)(4:5,13,14)
St(3,3,3,1)	10	(1:3,5,8)(12:6,9,13)(2:7,10,11)(4:14)

(b) If $a_1 + a_2 + a_3 = 6t$, then the vertex labelled $3t+2$ is joined to the nucleus labelled $a_1 + a_2 + a_3 + 3$. If $a_1 + a_2 + a_3 = 6t + 2$, then the vertex labelled $3t+3$ is joined to the nucleus labelled 1. Suppose $a_1+a_2+a_3 = 6t+3$. Then the vertex labelled $3t+4$ is joined to the nucleus labelled $a_1 + a_2 + a_3 + 3$. If $a_1 + a_2 + a_3 = 6t + 5$, then the vertex labelled $3t + 5$ is joined to the nucleus labelled 2.

Such a GPSL is said to be excellent, and we call it an EPSL. From the GPSL $(1 : 3)(6 : 7)(2 : 5)$ of $St(1,1,1)$, we can obtain from the Splitting Lemma the GPSL $(1 : 3)(8 : 9)(2 : 5,6,7)$ of $St(1,1,3)$. Here $1 + 1 + 3$ is of the form $6t + 5$ with $t = 0$, and the vertex labelled $t + 5 = 5$ is joined to the nucleus labelled 2. Thus, we have an EPSL.

Suppose we start with an EPSL. Then the Switching Lemma can be applied. It is routine to verify that if we split the vertex which has just been switched into an even number of vertices when $a_1 + a_2 + a_3 \equiv 2 \pmod 3$ and into an odd number of vertices when $a_1 + a_2 + a_3 \equiv 0 \pmod 3$, the resulting GPSL is excellent.

We now complete our proof that every 3-star has a PSL, in fact, a GPSL, via six constructions. The first four are to be carried out in cyclic order.

Theorem 7. *$St(m,m,m+2)$ has an EPSL for all m.*

Proof: We have taken care of the case $m = 1$. For $m = 2t$, we shall obtain from Theorem 10 the GPSL $(2:\ldots)(6t + 3:\ldots)$ $(1:\ldots,3t + 2,\ldots)$ of $St(2t,2t,2t)$. We split the vertex labelled $3t+2$ into three vertices to obtain the EPSL $(2:\ldots)(6t+5:\ldots)$ $(1:\ldots,3t+2,3t+3,3t+4,\ldots)$ of $St(2t,2t,2t+2)$. For $m = 2t + 1 > 1$, we shall obtain from Theorem 10 the GPSL $(1:\ldots)$ $(6t + 6:\ldots)(2:\ldots,3t + 4,\ldots)$ of $St(2t + 1,2t + 1,2t + 1)$. We split the vertex labelled $3t+4$ into three vertices to obtain the EPSL $(1:\ldots)(6t+8:\ldots)$ $(2:\ldots,3t+4,3t+5,3t+6,\ldots)$ of $St(2t + 1,2t + 1,2t + 3)$. ∎

Theorem 8. *$St(m,m+1,m+1)$ has a GPSL for all m.*

Proof: For $m = 2t$, we switch to the nucleus labelled $6t + 5$ the vertex labelled $3t + 3$ in the EPSL of $St(2t,2t,2t + 2)$ obtained in Theorem 7. This yields the GPSL $(2:\ldots)(1:\ldots)(6t + 5:\ldots,3t + 3,\ldots)$ of $St(2t,2t + 1,2t + 1)$. For $m = 2t + 1$, we switch to the nucleus labelled $6t + 8$ the vertex labelled $3t + 5$ in the EPSL of $St(2t + 7,2t + 1,2t + 3)$ obtained in Theorem 1. The GPSL $(1:\ldots)(2:\ldots)$ $(6t+8:\ldots,3t+5,\ldots)$ of $St(2t+1,2t+2,2t+2)$ is obtained. ∎

Theorem 9. *$St(m,m+1,m+2)$ has an EPSL for all m.*

Proof: For $m = 2t$, we split into two vertices the vertex labelled $3t + 3$ in the GPSL of $St(2t,2t + 1,2t + 1)$ obtained in Theorem 8. This yields the EPSL $(2:\ldots)(1:\ldots)$ $(6t + 6:\ldots,3t + 3,3t + 4,\ldots)$ of $St(2t,2t + 1,2t + 2)$. If $m = 2t + 1$, we split into two vertices the vertex labelled $3t + 5$ in the GPSL of

Table 2

Stars	Theorems	Punctured S-Labellings
St(2,2,2)	-	(2:7,8)(9:6,10)(1:3,$\underline{5}$)
St(2,2,4)	7	(2:9,10)(11:8,12)(1:$\overline{3}$,5,$\underline{6}$,7)
St(2,3,3)	8	(2:9,10)(1:3,5,7)(11:$\underline{6}$,8,12)
St(2,3,4)	9	(2:10,11)(1:3,5,8)(12:6,$\underline{7}$,9,13)
St(3,3,3)	10	(1:3,5,8)(12:6,9,13)(2:$\overline{7}$,10,11)
St(3,3,5)	7	(1:3,5,10)(14:6,11,15)($\overline{2}$:7,$\underline{8}$,9,12,13)
St(3,4,4)	8	(1:3,5,10)(2:7,9,12,13)(14:$\overline{6}$,$\underline{8}$,11,15)
St(3,4,5)	9	(1:3,5,11)(2:7,10,13,14)(15:6,$\underline{8}$,9,12,16)
St(4,4,4)	10	(2:7,10,13,14)(15:6,9,12,16)(1:3,5,8,11)
St(3,4,4)	-	(1:3,5,10)(2:7,9,12,13)(14:6,$\underline{8}$,11,15)
St(3,4,7)	11	(1:3,5,13)(2:7,12,15,16)
		(17:6,8,$\underline{9}$,10,11,14,18)
St(4,6,4)	12	(2:7,12,$\overline{15}$,16)(17:6,8,10,11,14,18)
		(1:3,5,$\underline{9}$,13)
St(4,6,6)	12	(2:7,14,17,18)(19:6,8,12,13,16,20)
		(1:3,5,9,10,11,15)

$St(2t+1,2t+2,2t+2)$ obtained in Theorem 8. This yields the EPSL $(1:\ldots)(2:\ldots)(6t+9:\ldots,3t+5,3t+6,\ldots)$ of $St(2t+1,2t+2,2t+3)$. ∎

Theorem 10. *$St(m+1,m+1,m+1)$ has a GPSL for all m.*

Proof: We have settled the case $m = 0$. For $m = 2t > 0$, we switch to the nucleus labelled 2 the vertex labelled $3t+4$ in the EPSL of $St(2t,2t+1,2t+2)$ obtained in Theorem 9. The GPSL $(1:\ldots)(6t+6:\ldots)(2:\ldots,3t+4,\ldots)$ of $St(2t+1,2t+1,2t+1)$ is obtained. For $m = 2t+1$, we switch to the nucleus lablled 1 the vertex labelled $3t+5$ in the EPSL of $St(2t+1,2t+2,2t+3)$ obtained in Theorem 9. The GPSL $(2:\ldots)(6t+9:\ldots)(1:\ldots,3t+5,\ldots)$ of $St(2t+2,2t+2,2t+2)$ is obtained. ∎

Theorem 11. *$St(m,m+2t-1,n)$ has an EPSL for all m, t and $n > m$.*

Proof: By Theorem 10, we have a GPSL of $St(m,m,m)$ for all m. We apply the Splitting Lemma to obtain an EPSL of $St(m,m,m+2t)$ for all t, then the Switching Lemma to obtain a GPSL of $St(m,m+2t-1,m+1)$, and finally the Splitting Lemma to obtain an EPSL of $St(m,m+2t-1,n)$ for all $n > m$. ∎

Theorem 12. *$St(m,m+2t,n)$ has a GPSL for all m, t and $n \geq m$.*

Proof: We have taken care of the case $m = 1$. For $m > 1$, Theorem 11 guarantees an EPSL of $St(m-1,m,m+2t+1)$ for all m and t. We apply the Switching Lemma to obtain a GPSL of $St(m,m+2t,m)$, and then the Splitting Lemma to obtain a GPSL of $St(m,m+2t,n)$ for all $n \geq m$. ∎

Theorem 13. *Every 3-star has a GPSL.*

Proof: We may assume that $a_1 \leq a_2 \leq a_3$. If $a_1 + a_2$ is even, the result follows from Theorem 12. If not, it follows from Theorem 11. ∎

In Table 2, we illustrate the cyclic applications of Theorems 7 to 10, as well as the constructions based on Theorem 11 and Theorem 12.

We have seen that $k \equiv 0$ or $1 \pmod 4$ is a necessary condition for all k-star to be Skolem-graceful. A natural question is whether this condition is also sufficient. The case $k = 1$ is trivial, and we have settled the case $k = 4$. However, our method does not seem promising even for the next case $k = 5$.

We have a partial result, that the k-star $St(1, 1, \dots, 1, \ell)$ is Skolem-graceful for all ℓ if $k \equiv 0$ or $1 \pmod 4$. This is because the $(k-1)$-star $St(1, 1, \dots, 1)$ has a PSL for all $k \equiv 0$ or $1 \pmod 4$. This is induced by a Nickerson sequence of order k with one copy of k appearing at one end of the sequence. For $k = 4t$, those constructed in [4] have this property. We conclude this paper with our construction of the desired Nickerson sequences for $k = 4t + 1$.

Odd terms	Positions
1	$t+1,\ t+2$
3	$2t+1,\ 2t+4$
$\dots$	$\dots\dots$
$2t-1$	$t+3,\ 3t+2$
$2t+1$	$2t+2,\ 4t+3$
$2t+3$	$t,\ 3t+3$
$\dots$	$\dots\dots$
$4t+1$	$1,\ 4t+2$

Even terms	Positions
2	$6t+2,\ 6t_4$
$\dots$	$\dots\dots$
$4t-2$	$4t+4,\ 8t+2$
$4t$	$2t+3,\ 6t+3$

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