(k, n, f) caps of type (m, n) in PG(t, q)

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Introduction

A (k, n, f)-cap K was defined by D'Agostini [1] to be a non-negative integer valued function on the points P of a $PG(t,\ q)$,

$$f\ :\ PG(t,\ q) \to N$$

such that k of the points have positive values. We call $f(P)$ the weight of the point $P \in PG(t,\ q)$, and define the weight of the line ℓ to be $F(\ell) = \sum_{P\in\ell} f$. Finally $n = \max_{\ell} F(\ell)$. For $t = 2$, K is called a (k, n, f)-arc. Such caps, are studied in [2, 3]. We write $\max_{P\in K} f = \omega$, and denote the weight of the whole cap by $W = \sum_{P\in K} f(P)$ The set of integers $m_1 < m_2 < .. < m_r < n$ such that there is at least one line of weight m_i in K is called the *type* of K. The number of lines of weight s in $PG(t, q)$ is denoted by τ_s, and the integers $\tau_{m_1}, \tau_{m_2}, .., \tau_{m_r}$, τ_n are called the *characters* of K. As usual we write $Q_{t-1} = 1 + q + .. + q^{t-1}$, the number of lines through each point in $PG(t, q)$, and by duality there are Q_{t-1} points on each line. The number of points (resp. lines) in $PG(t, q)$ is Q_t. We note that if $\min f > 0$, the study of a $(k, n, f) - cap$ can be reduced to that of a $(k^*, n^*, f^*) - cap$ defined by $f^*(P) = f(P) - \min f$. We can therefore take $\min f = 0$ without loss of generality, and it will be assumed in the following work.

Some inequalities

This note is devoted to caps of type $(m,\ n)$. Arcs with two characters were studied in [4]. Consider a point of weight s, and let the number of lines of weights m, n passing through it be denoted by V_m^s, V_n^s respectively, $V_m^s \geq 0, V_n^s \geq 0$. Then counting lines and weights respectively,

$$V_m^s + V_n^s = Q_{t-1}, \tag{1}$$

$$mV_m^s + nV_n^s = W + (Q_{t-1} - 1)s. \tag{2}$$

These equations give

$$(n-m)V_m^s = (n-s)Q_{t-1} - W + s, (n-m)V_n^s = W - s - (m-s)Q_{t-1}. \tag{3}$$

Since the right hand sides are well determined, the numbers of lines of weights m, n through every point of weight s are the same. Each value in the s - spectrum gives a congruential relationship between W and Q_{t-1} $(\bmod\ n-m)$. In particular since 0 always occurs, we have

$$W \equiv mQ_{t-1} (\bmod\ n-m) \equiv nQ_{t-1} (\bmod\ n-m) \tag{4}$$

By taking $s = 0, \omega$ respectively in (2), since $m < n$, we have,

$$m\, Q_{t-1} \leq W \leq (n-\omega) Q_{t-1} + \omega. \tag{5}$$

We note the extreme cases of this inequality. If $W = mQ_{t-1}$, then $V_m^0 = Q_{t-1}$, $V_n^0 = 0$ (*i.e.* every line through a point of weight 0 is of weight m), $V_n^\omega = Q_{t-1} - (\omega q/(n-m)) Q_{t-2}$, $V_m^\omega = (\omega q/(n-m)) Q_{t-2}$. If $W = (n-\omega)Q_{t-1} + \omega$ we have $V_m^\omega = 0$, $V_n^\omega = Q_{t-1}$ (*i.e.* every line through a point of maximum weight has weight n), $V_n^0 = Q_{t-1} - \omega q Q_{t-2}/(n-m)$, $V_m^0 = \omega q Q_{t-2}/(n-m)$. Now $V_n^0 = 0$ and $V_m^\omega = 0$ are incompatible since there must be a line joining a pair of points of weights 0 and ω, hence at least one of the inequality signs in (5) must be strict. By the congruence (4) the difference between its extremes must be at least $n-m$, hence $\omega Q_{t-1} \leq (n-m)Q_{t-1} + \omega + n - m$, or $\omega \leq n - m$.

Now consider the doubly extremal case where W has its minimum value mQ_{t-1} and ω has its maximum value $n-m$. This implies from the above formulae that $V_m^\omega = 1, V_n^\omega = Q_{t-1} - 1$.

Theorem 1. *In a (k, n, f)-cap of type (m, n) with minimum weight $W = mQ_{t-1}$ and $\omega = n - m$, all the points of weight 0 and all those of weight ω lie on a single line, which has weight m.*

Proof: Let P_0, P_ω be points of weights $0, \omega$ respectively. Let A be a point of weight 0 not collinear with them, if that were possible. Then P_0P_ω and AP_ω both have weight m since they pass through points of weight 0, but this implies that two lines of weight m pass through P_ω which has weight ω contrary to the calculation above. Thus A lies on P_0P_ω. But if B were a point of weight ω not collinear with P_0P_ω, we would have two lines, P_0B and BA, each of weight m, through B, which has been shown to be impossible.

An immediate consequence is that if l_i denotes the number of points of weight i in the cap, $l_0 + l_\omega \leq Q_{t-1}$.

Theorem 2. *In a (k, n, f)-cap of type (m, n) with maximum weight $W = (n-\omega)Q_{t-1} + \omega$ and $\omega = n - m$, all the points of weight 0 and all those of weight ω lie on a single line, which has weight n.*

Proof: We define a cap K^* *dual* to K by setting $f^*(P) = \omega - f(P)$. If K satisfies the requirements of Theorem 2, then K^* satisfies those of Theorem 1. The result follows from the duality.

Corollary. *In a (k,n,f)-cap of type (m,n) with minimum weight $W = mQ_{t-1}$ and $\omega = n - m, l_0 \leq V_m^s$, and in a (k,n,f) cap of type (m,n) with maximum weight $W = (n-\omega)Q_{t-1} + \omega$ and $\omega = n - m, l_\omega \leq V_n^s, s = 1,2,\dots,\omega-1$.*

Proof: The l_0 points of weight 0 are collinear, and every line through any of them has weight m. Consider a point of weight s, and the V_m^s lines of weight m through it. This gives the first result, and the second follows similarly or by duality.

Note that in this doubly extremal case, the weight can be written as $W = (n-\omega)Q_{t-1} + \omega = mQ_{t-1} + \omega$.

Now let α denote the smallest non-zero value among the weights of points. On taking $s = \alpha$ in (3), we find $(n-m)V_n^\alpha = W - mQ_{t-1} + \alpha(Q_{t-1} - 1) > 0$, so there is at least one line of weight n through each point of minimal non-zero weight. The other q points on this line (t=2) have weights between 0 and ω, which gives us the inequalities $\alpha \leq n \leq \omega q + \alpha$ for α and ω. Since there are always points of weight zero we also have by considering the weight of a line through such a point, $n \leq \omega q$.

Let us fix attention on a particular line ℓ, and let a_i be the number of points of weight i on $\ell, i = 0,1,\dots,\omega$. Then $\sum_0^\omega a_i = q+1, \sum_0^\omega ia_i = f(\ell)$. On subtracting the first equation from the second and transposing a term we have $\sum_2^{\omega-1} ia_{i+1} = a_0 + f(\ell) - q - 1$. Since the left hand side is non-zero, we have the following result:

Theorem 3. *In a (k,n,f)-cap of type (m,n), the number of points of weight zero on a line of weight m (resp. n) is at least $q-2-m$ (resp. $q-2-n$).*

We have seen that in some extremal cases, all the points of weight 0 or ω are collinear. We now look at the consequences of a weaker form of the converse requirment, namely that all the points of weight 0 are collinear. Suppose they lie on a line ℓ. Now there is at least one point of non-zero weight on ℓ. Let α_ℓ, ω_ℓ be the minimal non-zero, and maximal value respectively among the weights of points on ℓ, and let P, Q be points of ℓ having weight α_ℓ and ω_ℓ respectively. Through P, Q pass at least one line of weight m and one of weight n, neither of which contains any point of weight 0. By counting the weights of points on these lines we obtain a dual pair of inequalities.

Theorem 4. *In a (k,n,f)-cap of type (m,n) in which all the points of weight zero are collinear, we have $\omega_\ell + \alpha q \leq m$, $\alpha_\ell + \omega q \geq n$.*

This result can be strengthened in the case where all the points of weight 0 and those of weight ω are collinear. Let $\omega_{K-\ell}$ be the maximal weight of points not on ℓ. Calculations similar to the above give us:

Theorem 5. *In a (k,n,f)-cap of type (m,n) in which all the points of weights zero and ω are collinear, we have $\alpha_\ell + q\omega_{K-\ell} \geq n$.*

An Example: We now describe an example of a (k,n,f) cap of type (m,n) in which all the points of weight 0 are collinear. The underlying geometry is

$PG(2,9)$, and f takes the values $0,1,2$. In this example α is an element in the Galois field F_9 used in the construction of $PG(2, 9)$. The weights of the points, denoted by coordinates (x, y, z), are as follows:

Weight 0:	$(0,1,\alpha^2)$	$(0,1,\alpha^5)$	$(0,1,\alpha^6)$		
Weight 2:	$(0,0,1)$	$(0,1,0)$	$(0,1,\alpha^3)$	$(0,1,\alpha^4)$	$(1,\alpha,1)$
	$(1,\alpha,\alpha^6)$	$(1,\alpha,\alpha^7)$	$(1,\alpha^2,\alpha^2)$	$(1,\alpha^2,\alpha^3)$	$(1,\alpha^2,\alpha^4)$
	$(1,\alpha^3,\alpha^2)$	$(1,\alpha^3,\alpha^3)$	$(1,\alpha^3,\alpha^4)$	$(1,\alpha^5,\alpha^2)$	$(1,\alpha^5,\alpha^3)$
	$(1,\alpha^5,\alpha^4)$	$(1,\alpha^6,1)$	$(1,\alpha^6,\alpha^6)$	$(1,\alpha^6,\alpha^7)$	$(1,\alpha^7,1)$
	$(1,\alpha^7,\alpha^6)$	$(1,\alpha^7,\alpha^7)$			

Weight 1: All other points.

There are thus 3 points of weight 0, 66 of weight 1, and 22 of weight $2(=\omega)$. This gives rise to an $(11,14)$ cap with $W = 110$. It has 58 lines of weight 11 and 33 lines of weight 14. The line $x = 0$, which has weight 11, is distinguished, as the three points of weight 0 are collinear on it. The other lines can be further classified as follows. In addition to $x = 0$ the lines of weight 11 include 27 that intersect $x = 0$ in a point of weight 0, contain 7 points of weight 1 and 2 of weight 2. These are the lines not included in the lists below. A further 18 contain 9 points of weight 1, one of which is the intersection with $x = 0$, and 1 point of weight 2. They are, in terms of line coordinates (ℓ, m, n) derived from the equation $\ell x + my + nz = 0$:

$(1,\alpha,\alpha^4)$	$(1,\alpha,\alpha^5)$	$(1,\alpha,\alpha^6)$	$(1,\alpha^2,\alpha^5)$	$(1,\alpha^2,\alpha^6)$	$(1,\alpha^2,\alpha^7)$
$(1,\alpha^3,1)$	$(1,\alpha^3,\alpha^6)$	$(1,\alpha^3,\alpha^7)$	$(1,\alpha^5,1)$	$(1,\alpha^5,\alpha)$	$(1,\alpha^5,\alpha^2)$
$(1,\alpha^6,\alpha)$	$(1,\alpha^6,\alpha^2)$	$(1,\alpha^6,\alpha^3)$	$(1,\alpha^7,\alpha^2)$	$(1,\alpha^7,\alpha^3)$	$(1,\alpha^7,\alpha^4)$

The remaining 12 intersect $x = 0$ in a point of weight 2, and their other 9 points are of weight 1. Their line coordinates are:

$(0,0,1)$	$(0,1,0)$	$(0,1,1)$	$(0,1,\alpha)$	$(1,0,\alpha^3)$	$(1,0,\alpha^7)$
$(1,1,0)$	$(1,1,1)$	$(1,1,\alpha)$	$(1,\alpha^4,0)$	$(1,\alpha^4,\alpha^4)$	$(1,\alpha^4,\alpha^5)$

The lines of weight 14 all contain 6 points of weight 1 and 4 of weight 2. They fall into two classes: the 9 that intersect $x = 0$ in a point of weight 1, which are:

$(0,1,\alpha^3)$	$(0,1,\alpha^4)$	$(0,1,\alpha^5)$	$(1,1,\alpha^3)$	$(1,1,\alpha^4)$	$(1,1,\alpha^5)$
$(1,\alpha^4,1)$	$(1,\alpha^4,\alpha)$	$(1,\alpha^4,\alpha^7)$			

and the 24 that intersect it in a point of weight 2, namely:

$(1,0,1)$	$(1,0,\alpha)$	$(1,0,\alpha^2)$	$(1,0,\alpha^4)$	$(1,0,\alpha^5)$	$(1,0,\alpha^6)$
$(1,\alpha,0)$	$(1,\alpha,\alpha)$	$(1,\alpha,\alpha^2)$	$(1,\alpha^2,0)$	$(1,\alpha^2,\alpha^2)$	$(1,\alpha^2,\alpha^3)$
$(1,\alpha^3,0)$	$(1,\alpha^3,\alpha^3)$	$(1,\alpha^3,\alpha^4)$	$(1,\alpha^5,0)$	$(1,\alpha^5,\alpha^5)$	$(1,\alpha^5,\alpha^6)$
$(1,\alpha^6,0)$	$(1,\alpha^6,\alpha^6)$	$(1,\alpha^6,\alpha^7)$	$(1,\alpha^7,0)$	$(1,\alpha^7,1)$	$(1,\alpha^7,\alpha^7)$

It can be verified that this example is consistent with the above theorems. It is immediately obvious that the points of weight 1 form a $(66,9)$ arc of type $(3,6,7,9)$, and those of weight 2 form a $(22,4)$ arc of type $(1,2,4)$.

References

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