The Spectrum of PBD ($\{5,\ k^*\}$, v) for $k = 9,\ 13$

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Abstract

A pairwise balanced design (PBD) of index 1 is a pair $(V,\ \mathcal{A})$ where V is a finite set of points and $\mathcal{A}$ is a set of subsets (called blocks) of V, each of cardinality at least two, such that every pair of distinct points of V is contained in exactly one block of $\mathcal{A}$. We may further restrict this definition to allow precisely one block of a given size, and in this case the design is called a PBD $(\{K,\ k^*\}, v)$ where k is the unique block size, K is the set of other allowable block sizes, and v is the number of points in the design. It is shown here that a PBD $(\{5,\ 9^*\},\ v)$ exists for all $v \equiv 9$ or 17 mod 20, $v \geq 37$, with the possible exception of 49, and that a PBD $(\{5,\ 13^*\},\ v)$ exists for all $v \equiv 13$ mod 20, $v \geq 53$.

1 Introduction

A pairwise balanced design (PBD) of index 1 is a pair $(V,\ \mathcal{A})$ where V is a finite set of points and $\mathcal{A}$ is a set of subsets (called blocks) of V, each

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of cardinality at least two, such that every pair of distinct points of V is contained in exactly one block of $\mathcal{A}$. We say that $(V, \mathcal{A})$ is a PBD (K, v) if $|V| = v$ and $|A| \in K$ for every $A \in \mathcal{A}$ where K is a set of positive integers.

The notion of PBD closure dates back to Wilson ([27], [28], [30]) and is defined thus: if K is a set of positive integers, then let $\mathbf{B}(K)$ denote the set of positive integers v for which there exists a PBD (K, v). $\mathbf{B}(K)$ is called the PBD closure of K. K itself is said to be PBD closed if $\mathbf{B}(K) = K$.

The goal of this paper is to determine for what values of v there exists a PBD $(\{5, 9^*\}, v)$ and for which values of v there exists a PBD $(\{5, 13^*\}, v)$ (here k^* implies that there is exactly one block of size k in our design). We note that, if there exists a PBD $(\{5, k^*\}, v)$, then $v \equiv k \equiv 1 \bmod 4$ and $v(v-1) \equiv k(k-1) \bmod 5$. For $k = 9$ this implies that $v \equiv 9$ or $17 \bmod 20$, and for $k = 13$ this implies that $v \equiv 13 \bmod 20$. Our strategy will be to consider $v \equiv 9, 29, 49, 69, 89, \bmod 100$; $v \equiv 17, 37, 57, 77, 97, \bmod 100$; and $v \equiv 13, 33, 53, 73, 93, \bmod 100$.

Since we have only one block of a special size, this distinguished block may also be thought of as a hole, and hence our structure may be alternatively considered as an incomplete PBD or IPBD. An IPBD is a triple $(X, Y, \mathcal{A})$ where X is a set of points, $Y \subseteq X$, and $\mathcal{A}$ is a set of blocks which satisfies the following properties: 1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$, and 2) any two points x, y, not both in Y, occur in a unique block. Hence Y is the hole. It is known that $(X, Y, \mathcal{A})$ is an IPBD iff $(X, \mathcal{A} \cup Y)$ is a PBD [19].

Investigation into IPBD for block sizes three and four has been long and thorough. Vital papers in this area include [7], [11], [25], [26], and [19]. One of the principal applications of IPBD's is in the singular indirect product (see [14] and [15]), a product that has been used extensively, notably in [15], [16], and [19].

2 Constructions

To obtain the required designs, we employ some thirteen constructions that we shall describe. Fundamental to these constructions are a number of other designs which we define now.

The first of these is the group–divisible design or GDD. A GDD is a triple $(V, \mathcal{G}, \mathcal{B})$ with the following properties: 1) $\mathcal{G}$ is a partition of V into subsets called groups, 2) $\mathcal{B}$ is a class of subsets of V (called blocks) such that a group and a block contain at most one common point, and 3) every pair of points from distinct groups occurs in a unique block. The group type of a GDD is a listing of the group sizes using so-called "exponential" notation, that is, $1^a 2^b 3^c \ldots$ denotes a groups of size 1, b groups of size 2, etc. Two particular GDD's of which we will make extensive use are the GDD with five groups of size four and all blocks of size five, and the GDD with six groups of size four and all blocks of size five. The first GDD exists by Lemma 2.1 below while the second is formed from the affine plane of order five by deleting one point. The blocks from which a point was deleted are now of size four and become our groups. The remaining intact blocks become our blocks.

Very useful to our constructions is a special kind of GDD known as a transversal design or TD. A TD $(k,\ n)$ is a GDD on kn varieties with k groups of size n and n^2 blocks of size k. It is well-known that a TD $(k,\ n)$ is equivalent to $k-2$ mutually orthogonal Latin squares. In this paper we will be concerned mainly with TD $(5,\ n)$ and TD $(6,\ n)$; so we remind ourselves of the values of n for which three MOLS and for which four MOLS exist.

Lemma 2.1 *If $n \neq 2, 3, 6, 10$, then there exist three MOLS of order n [4], [22].*

Lemma 2.2 *If $n \neq 2, 3, 4, 6, 10, 14, 18, 22, 26, 30, 34, 42$, then there exist four MOLS of order n [4], [23], [24], [21], [3], [1].*

Related to a transversal design is an incomplete TD which is defined as follows: a TD $(k,\ v)-$ TD $(k,\ u)$ is a quadruple $(X,\ \mathcal{G},\ \mathcal{A},\ Y)$ where X is a set of kv points, $\mathcal{G} = \{G_1,\ G_2,\ \ldots, G_k\}$ is a partition of X into k groups of v elements each, Y is a set of ku points such that $| Y \cap G_i | = u$ for $1 \leq i \leq k$, and $\mathcal{A}$ is a set of subsets of X called blocks, each containing exactly one element from each group such that each pair $\{x,\ y\}$ of elements from different groups is either contained in Y (which is called a hole) or occurs in a unique block of $\mathcal{A}$ but not both [5]. Several criteria for determining the existence of incomplete TD's will be listed later. Note that neither of the two TD's in the "difference" need exist. For example, Horton has constructed a TD (4, 6) -TD (4, 2) [10].

We also require the notion of an incomplete group-divisible design. An incomplete group-divisible design (IGDD) is a quadruple $S = (V, A, B, F)$ where V is a v-set, $A = \{G_1, G_2, \ldots, G_s\}$ is a partition of V, that is, $V = \bigcup_{i=1}^{s} G_i$, $G_i \cap G_j = \emptyset$, for $i \neq j$ (the sets G_i are called groups), $B = \{H_1, H_2, \ldots, H_s\}$ is a collection of subsets of V, $H_i \subseteq G_i$ (the sets H_i are called holes), and F is a family of subsets of V called blocks, which satisfy the following requirements. Let π be any pair of distinct elements of V; then (i) if π lies in a group, then π lies in no block of F; (ii) if π contains elements from distinct groups, say $\pi = \{x, y\}$ where $x \in G_i$ and $y \in G_j$, then (a) if $x \in H_i$ and $y \in H_j$, then π occurs in no block of F; otherwise (b), there is a unique block of F which contains π.

An IGDD is said to be of type $\pi_{i=1}^{t}(g_i, h_i)^{\alpha_i}$ if there are α_i groups of size g_i which contain a hole of size h_i, $i = 1, 2, \ldots, t$. An IGDD is said to be a k–IGDD if all blocks of the design are of size k.

A familiarity with BIBD's is presumed; however we define a resolvable balanced incomplete block design or RBIBD to be a BIBD in which the blocks of the design can be partitioned into classes, called resolution classes, such that every element of the design occurs precisely once in each resolution class. We also note here that there exists an RBIBD $(12m + 4,\ 4,\ 1)$ for all choices of $m > 0$ [8], and that there exists a BIBD $(20m + 1,\ 5,\ 1)$ and a BIBD $(20m + 5,\ 5,\ 1)$ for all choices of $m > 0$ [9].

Finally we state Wilson's Fundamental Construction as it proves so useful.

Theorem 2.1 (Wilson's Fundamental Construction) *Let $(X, \mathcal{G}, \mathcal{A})$ be a master GDD and let a positive integral weight s_x be assigned to each point $x \in X$. Let $(S_x : x \in X)$ be pairwise disjoint sets with $|S_x| = s_x$. With the notation $S_Y = \cup_{x \in Y} S_x$ for $Y \subseteq X$, put $X^* = S_X$, $\mathcal{G}^* = \{S_G : G \in \mathcal{G}\}$. For $A \in \mathcal{A}$, we have a natural partition $\pi_A = (S_A, \{S_x : x \in A\})$; we suppose that for each block $A \in \mathcal{A}$, a GDD $(S_A, \{S_x : x \in A\}, \mathcal{B}_A)$ is given, and put $\mathcal{A}^* = \cup_{A \in \mathcal{A}} \mathcal{B}_A$. Then $(X^*, \mathcal{G}^*, \mathcal{A}^*)$ is a GDD [29].*

The following theorem is a special case of a straightforward extension of Wilson's Fundamental Construction for group–divisible designs:

Theorem 2.2 *Suppose that there exists an IGDD of type $\pi_{i=1}^{t}(g_i, h_i)^{\alpha_i}$ with blocks whose sizes lie in a set K. Suppose further that for some positive integer n and for each $s \in K$, there exists a GDD of type n^s with blocks of size k. Then there exists a k–IGDD of type $\pi_{i=1}^{t}(ng_i, nh_i)^{\alpha_i}$.*

Proof: This is a simple modification of the proof of Wilson's Fundamental Construction [27]. □

Lemma 2.3 *If a PBD $(\{5, k^*\}, v)$ exists, then $v \geq 4x + 1$.*

Proof: Consider pairs which involve two elements *not* in the special block of size k. There are precisely $\binom{v-k}{2}$ such pairs, and they must occur in the blocks of size five.

An element from the block of size k must occur in $\frac{(v-k)}{4}$ blocks of size five. Then $\binom{4}{2} = 6$ pairs are formed by the other elements in the block. This is the case for all elements in the block of size k. Hence enumerating all such pairs, we see there are at least $6k\frac{(v-k)}{4}$ pairs involving elements not in the block of size k. Hence

$$6k\frac{(v-k)}{4} \leq \binom{v-k}{2}$$

which implies that

$$\begin{aligned} 3k &\leq v - k - 1, \text{ or} \\ 4k + 1 &\leq v. \ \Box \end{aligned}$$

Lemma 2.4 *If* $4t + 1 \in \mathbf{B}(5,\ k^*)$, *if a TD* $(6,\ 5m)$ *exists, and if* $5m \geq t \geq 0$, *then* $100m + 4t + 1 \in \mathbf{B}(5,\ k^*)$.

Proof: Take a TD $(6,\ 5m)$ and remove $5m - t$ varieties from the last group to obtain a GDD of type $(5m)^5t^1$ with blocks of size five and six. Now employ Wilson's Fundamental Construction, weighting all varieties by four to obtain five groups of size $20m$ and one group of size $4t$. Replace the blocks of size five with the blocks from a TD $(5,\ 4)$ (see above) and replace the blocks of size six with the blocks from a GDD with six groups of size four and all blocks of size five (see above).

Add an extra point, ∞, to all groups, and replace the groups of size $20m + 1$ by BIBD $(20m + 1,\ 5,\ 1)$ and replace the group of size $4t + 1$ by the PBD $(\{5,\ k^*\},\ 4t + 1)$. $\Box$

Lemma 2.5 *If* $4t + 1 \in \mathbf{B}(5,\ k^*)$, *if a TD* $(6,\ 5m + 1)$ *exists, and if* $5m + 1 \geq t \geq 0$, *then* $100m + 4t + 21 \in \mathbf{B}(5,\ k^*)$.

Proof: Similar to the proof of Lemma 2.4, except that we replace groups of size $20m + 5$ with BIBD $(20m + 5,\ 5,\ 1)$. $\Box$

Lemma 2.6 *If* $a > 0$, *if* $a \equiv 9$ *or* $89 \bmod 100$, *then* $a \in \mathbf{B}(5,\ 9^*)$ *[15, lemma 5.7].*

Lemma 2.7 *For any positive integer* k, *if there is a TD* $(5,\ 12n + a + 4)$ *which contains a TD* $(5,\ a)$ *as a subdesign, where* $0 \leq a \leq 4n + 1$, *and if* $4n + 4a + 1 \in \mathbf{B}\{5,\ k^*\}$, *then* $64n + 4a + 21 \in \mathbf{B}\{5,\ k^*\}$ *[15, lemma 5.8].*

Note that, as stated on page 92 of [15], the TD (5, a) can be a hole.

To find the incomplete TDs we use the following result from [5].

Proposition 2.1 *Let $m > 1$, suppose that a TD $(k+1,\ t)$, a TD $(k,\ m)$, and a TD $(k,\ m+1)$ exist; suppose also that $0 \le s \le t$. Then a TD $(k,\ mt+s)-$ TD $(k,\ s)$ exists. If, moreover, a TD $(k,\ s)$ exists, then a TD $(k,\ mt+s)$ exists which contains a sub-TD $(k,\ t)$, a sub-TD $(k,\ m)$ if $s \ne t$, a sub-TD $(k,\ m+1)$ if $s \ne 0$, and a sub-TD $(k,\ s)$.*

Proposition 2.2 *Let $m > 1$ and $t > 1$ and suppose that a TD $(k+w,\ t)$, a TD $(k,\ m)$, and a TD $(k,\ m+1)$ exist. Then a TD $(k,\ mt+w)-$ TD $(k,\ m+w)$ exists.*

Proposition 2.3 *If TD $(6,\ t)$ and TD $(5,\ m+m_j)-$TD $(5,\ m_j)$ all exist and if $a = \sum_{j=1}^{p} m_j k_j$, where k_j are positive integers satisfying $t = \sum_{j=1}^{p} k_j$, then there exists TD $(5,\ mt+a)-$TD $(5,\ a)$.*

Lemma 2.8 *a) If $4u+k \in \mathbf{B}(5,\ k^*)$ and a TD $(6,\ u)$ exists, then $24u+k \in \mathbf{B}(5,\ k^*)$. b) If $4u+k \in \mathbf{B}(5,\ k^*)$, if a TD $(6,\ u)-$ TD $(6,\ t)$ exists, and if a GDD of type $(4t)^6$ with blocks of size 5 exists, then $24u+k \in \mathbf{B}(5,\ k^*)$.*

Proof: a) Take a TD (6, u) and inflate each group by four using Wilson's Fundamental Construction. Employ the same technique as in Lemma 2.4 to break up the blocks into blocks of size five. Add k new varieties, $x_1,\ x_2,\ \ldots,x_k$, to each group, and construct a PBD $(\{5,\ k^*\},\ 4u+k)$ on each group with the block of size k as $x_1x_2\ldots x_k$ in each case. Retain only one copy of $x_1x_2\ldots x_k$ and retain all blocks of size five.

b) Take a TD $(6,u)-$ TD $(6,t)$ and perform the steps as in a). The only difficulty is that we lose those pairs which would have occurred but for the hole. To remedy the situation, make use of the GDD of type $(4t)^6$. Let the groups in this GDD consist of the points in the hole (originally t points per group of the incomplete TD, now inflated to $4t$ points per group). Then if we adjoin the resulting blocks from this GDD to our design, we capture all omitted pairs. □

Lemma 2.9 *If $u \in \mathbf{B}(5,\ k^*)$ and if a TD $(5,\ u-k)$ exists, then $5u-4k \in \mathbf{B}(5,\ k^*)$.*

Proof: Take a TD $(5,\ u-k)$ and add the k varieties, $x_1,\ x_2,\ \ldots,x_k$, to each group, making groups of size u. Then replace each group by a PBD $(\{5,\ k^*\},\ u)$ where the block of size k in every group is $x_1x_2\ldots x_k$. Include only one copy of $x_1x_2\ldots x_k$. Then all other blocks have size five and the result follows. □

Lemma 2.10 *If* $4m+1 \in \mathbf{B}(5,\ k^*)$, *then* $16m+5 \in \mathbf{B}(5,\ k^*)$ *[17, lemma 3.3].*

Lemma 2.11 *If* $4s+1 \in \mathbf{B}(5,\ k^*)$, *if there exists an RBIBD* $(20m+5,\ 5,\ 1)$, *and if* $0 \le s \le 5m$, *then* $80m+21+4s \in \mathbf{B}(5,\ k^*)$ *[17, lemma 3.2].*

The ingredients in this construction are RBIBD with $k = 5$, $\lambda = 1$. The question then becomes: for which values of v does such a design exist. From [13] we obtain the following parameters of RBIBD:

$$(65,\ 208,\ 16,\ 5,\ 1)$$
$$(85,\ 357,\ 21,\ 5,\ 1)$$

Lemma 2.12 *(i) If there exists a BIBD* $(v,\ 6,\ 1)$, *then* $4v-15 \in \mathbf{B}(5,\ 9^*)$. *(ii) If there exists a BIBD* $(v,\ 6,\ 1)$, *then* $4v-11 \in \mathbf{B}(5,\ 13^*)$.

Proof: (i) Begin with a BIBD $(v,\ 6,\ 1)$. Delete a point to get a GDD with all groups of size five and all blocks of size six. Delete three points from the last group, leaving two points.

Invoking Wilson's Fundamental Construction, we weight all points by four and replace the blocks as per Lemma 2.4. Adding ∞ to each group, we replace the groups of size 21 by BIBD $(21,\ 5,\ 1)$ and just leave the group of size nine. □

(ii) Similar to (i) except that we delete two points from the last group instead of three. □

Lemma 2.13 *(i) If* $u \in \mathbf{B}(5,\ 9^*)$ *and if a TD* $(5,\ u-2)-$ *TD* $(5,7)$ *exists, then* $5u-8 \in \mathbf{B}(5,\ 9^*)$. *(ii) If* $u \in \mathbf{B}(5,\ 13^*)$ *and if a TD* $(5,\ u-3)-$ *TD* $(5,10)$ *exists, then* $5u-12 \in \mathbf{B}(5,\ 13^*)$.

Proof: (i) Let $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7$, be seven varieties in the first group, $b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,\ b_7$, be seven varieties in the second group, $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, be seven varieties in the third group, $d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6,\ d_7$, be seven varieties in the fourth group, and $e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7$, be seven varieties in the last group, where these are the varieties whose pairs do not occur in the blocks, that is, they "occur" in the hole of size seven.

Adjoin two new varieties x_8 and x_9 to each group. Replace each group by the PBD $(\{5,\ 9^*\},\ u)$, ensuring that the blocks of size nine in the respective groups are $a_1a_2a_3a_4a_5a_6a_7x_8x_9$, $b_1b_2b_3b_4b_5b_6b_7x_8x_9$, $c_1c_2c_3c_4c_5c_6c_7x_8x_9$, $d_1d_2d_3d_4d_5d_6d_7x_8x_9$, and $e_1e_2e_3e_4e_5e_6e_7x_8x_9$. Remove the blocks of size nine and add instead a PBD $(\{5, 9^*\}, 37)$ (for the existence of this design see Lemma 2.10) on the 37 points, a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1, b_2, b_3, b_4, b_5, b_6, b_7, c_1, c_2, c_3, c_4, c_5, c_6, c_7, d_1, d_2, d_3, d_4, d_5, d_6, d_7, e_1, e_2, e_3, e_4, e_5, e_6, e_7, x_8, x_9. This completes the construction. □

(*ii*) Similar to (*i*) except that we distinguish ten varieties in each group, add in three new points, and replace the blocks of size 13 by a PBD $(\{5, 13^*\}, 53)$ (for the existence of this, see Lemma 2.10). □

Lemma 2.14 *Suppose there is a BIBD* $(v, 5, 1)$ *which contains a flat of order* w. *Let* a *be an integer satisfying* $0 \leq a \leq w$. *If there exists a TD* $(5, v-a)-$ *TD* $(5, w-a)$, *and if* $5(w-a)+a \in \mathbf{B}(5, k^*)$, *then* $5(v-a)+a \in \mathbf{B}(5, k^*)$ *[17, lemma 3.8].*

Lemma 2.15 *If* $m+8 \in \mathbf{B}(5, 9^*)$ *and if a TD* $(5, m)$ *exists, then* $5m+8 \in \mathbf{B}(5, 13^*)$.

Proof: Begin with a TD $(5, m)$. Let the points in the first group be $a_1, a_2, \ldots, a_m$, the points in the second group be $b_1, b_2, \ldots, b_m$, etc. Without loss of generality, suppose $a_1b_1c_1d_1e_1$ is a block. Adjoin eight points x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 to each group. Then replace each group with a PBD $(\{5, 9^*\}, m+8)$ where the block of size nine in the first group is $a_1x_1x_2x_3x_4x_5x_6x_7x_8$, the block of size nine in the second group is $b_1x_1x_2x_3x_4x_5x_6x_7x_8$, the block of size nine in the third group is $c_1x_1x_2x_3x_4x_5x_6x_7x_8$, the block of size nine in the fourth group is $d_1x_1x_2x_3x_4x_5x_6x_7x_8$, and the block of size nine in the last group is $e_1x_1x_2x_3x_4x_5x_6x_7x_8$. Finally, replace all the blocks of size nine and the block $a_1b_1c_1d_1e_1$ by the block $a_1b_1c_1d_1e_1x_1x_2x_3x_4x_5x_6x_7x_8$—a block of size 13. We have $5m+8 \in \mathbf{B}(5, 13^*)$. □

Lemma 2.16 *If there exists a TD* $(11, m)$, *if there exists a BIBD* $(4m+1, 5, 1)$, *if* $a+b \leq m$ $(a, b \geq 0)$, *and if* $x = 12a+4b+1 \in \mathbf{B}(5, k^*)$, *then* $40m+x \in \mathbf{B}(5, k^*)$.

Proof: Take the TD $(11, m)$ and apply Wilson's Fundamental Construction. Weight 10 of the groups with 4. For the eleventh group, weight a of the points by 12, b of the points by 4, and c of the points by 0, so that $a+b+c=m$. Now consider any block. Ten of its points are weighted by 4, while the last point is weighted by 12, 4, or 0. Depending on which case we are dealing with we will need a GDD $(4^{10}, 5, 1)$, a GDD $(4^{11}, 5, 1)$ or a GDD $(4^{10}12^1, 5, 1)$.

The first GDD is obtained from the BIBD (41, 5, 1) by deleting a point. The second is obtained from the BIBD (45, 5, 1), also by deleting a point. Adjoin a point to each group. The third is manufactured from the PBD ({5, 13*}, 53) (for the existence of this see Lemma 2.10). This PBD is constructed by taking a BIBD (40, 4,1), partitioning it into 13 resolution classes, adjoining a different a_i to each class, and including the block $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}$ in the design. To obtain our GDD we have only to omit one a_i. The resulting design is a GDD ($4^{10}12^1$, 5, 1). Since there exists a BIBD ($4m+1$, 5, 1), we can replace each of the first 10 groups. Since $12a+4b+0c+1 = 12a+4b+1 \in \mathbf{B}(5,\ k^*)$, we can replace the last group by a PBD ($\{5,\ k^*\}, 12a+4b+1$). □

Lemma 2.17 *If* $4t+1 \in \mathbf{B}(5,\ k^*)$, *if a TD* ($6$, $5m+4$) *exists, and if* $5m+4 \geq t \geq 1$, *then* $100m+4t+81 \in \mathbf{B}(5,\ k^*)$.

Proof: Take a TD (6, $5m+4$) and remove $5m+4-t+1$ varieties from the last group to obtain a GDD of type $(5m+4)^5(t-1)^1$ with blocks of size 5 and 6. Now employ Wilson's Fundamental Construction, weighting all varieties by 4 to obtain 5 groups of size $20m+16$ and one group of size $4t-4$.

Add an extra block $a_1a_2a_3a_4a_5$ to all groups and replace the groups of size $20m+21$ by BIBD ($20m+21$, 5, 1) and replace the group of size $4t+1$ by the PBD ($\{5,\ k^*\}$, $4t+1$). □

Lemma 2.18 *If* $4t+5 \in \mathbf{B}(5,\ k^*)$, *if a TD* $(6, m)$ *exists, if a BIBD* ($4m+5$, 5, 1) *exists, and if* $t \leq m$, *then* $20m+4t+5 \in \mathbf{B}(5,\ k^*)$.

Proof: Take a TD (6, m) and remove $m-t$ points from one group. Inflate by 4, replacing blocks as in Lemma 2.4. Adjoin 5 new points a_1, a_2, a_3, a_4, a_5 to each group, and replace the groups now of size $4m+5$ by BIBD ($4m+5$, 5, 1), each with one copy of the block $a_1a_2a_3a_4a_5$. Replace the group of size $4t+5$ by a PBD ($\{5,\ k^*\}$, $4t+5$), ensuring that there is a block $a_1a_2a_3a_4a_5$. Retain only one copy of this block in the entire design. □

Lemma 2.19 *There exists a PBD* ($\{5,\ 9^*\}, 169$) *and a PBD* ($\{5,\ 13^*\}$, 173).

Proof: For $j = 1, 2$, construct an IGDD of type $(8,1)^5(j,0)^1$ with blocks of sizes in {5, 6} by taking a TD (6, 8), deleting $8-j$ elements from one side of its groups, and then deleting a block of size five. Since there exists a GDD of type 4^s with blocks of size five for $s = 5, 6$, then by Theorem 2.2

there exists a 5-IGDD of type $(32,4)^5(4j,0)^1$. Let the groups of size 32 be $G_1, G_2, \ldots G_5$, and let G_6 denote the group of size $4j$. Let H_i denote the hole of size 4 in G_i, $i = 1,2,\ldots,5$, and let W denote a set of five points which do not appear in the 5–IGDD. Now for $i = 1,2,\ldots,5$, define a PBD ({5, 9*}, 37) on the set $G_i \cup W$ in such a way that the block of size 9 is the set $H_i \cup W$. Let C_i denote the collection of blocks of size 5 which occur in this design, and adjoin the blocks of C_i, $i = 1,2,\ldots,5$, to the blocks of the 5–IGDD to form a set S of blocks. Now define a copy of a BIBD (25, 5, 1) on the set $\bigcup_{i=1}^{5} H_i \cup W$ in such a way that the set W occurs as a block. Delete this block from the block set of the BIBD to obtain a set of blocks C, and adjoin these blocks to the collection S to obtain a collection T. To the blocks of T, adjoin a block consisting of $G_6 \cup W$. The result is a PBD ({5, 9*}, 169) for $j = 1$ and a PBD ({5, 13*}, 173) for $j = 2$. □

3 Theorems

Theorem 3.1 *Let v be a positive integer such that $v \equiv 29 \bmod 100$. If $v \neq 29$, then $v \in \mathbf{B}(5,\ 9^*)$, and if $v = 29$, then $v \notin \mathbf{B}(5,\ 9^*)$.*

Proof: Lemma 2.5 with $4t + 1 = 9$ yields all values $v \equiv 29 \bmod 100$ except 29, 129, 529. The case $v = 29 \notin \mathbf{B}(5,\ 9^*)$ by Lemma 2.3. The case $v = 129$ may be constructed as follows. The points are one special point, X, the 120 ordered pairs $(g,\ h)$, and the eight ordered pairs $[i,\ j]$, where g is an integer modulo 60, h and i are integers modulo 2, and j is an integer modulo 4. Let σ be the automorphism that maps $(g,\ h)$ onto $(g+1,\ h)$, $[i,\ j]$ onto $[i+1,\ j]$, and X onto itself. The block of size nine is

X [0, 0] [0, 1] [0, 2] [0, 3] [1, 0] [1, 1] [1, 2] [1, 3]

The blocks of size five are 822 distinct blocks obtained by applying powers of σ to the following 15 base blocks:

X	(0,0)	(30,0)	(0,1)	(30,1)	(period 30)
(0,0)	(12,0)	(24,0)	(36,0)	(48,0)	(period 12)
(0,0)	(1,0)	(6,0)	(15, 0)	(16,1)	
(0,0)	(2,0)	(20,0)	(52,0)	(45,1)	
(0,0)	(31,0)	(35,0)	(40,1)	(55,1)	
(0,0)	(3,0)	(16,0)	(7,1)	(35,1)	
(0,0)	(21,0)	(38,0)	(27,1)	(50,1)	
(0,0)	(26,0)	(2,1)	(48,1)	(52,1)	
(0,0)	(37,0)	(11,1)	(18,1)	(31,1)	
(0,0)	(21,1)	(33,1)	(38,1)	(59,1)	
(0,1)	(16,1)	(49,1)	(52,1)	(58,1)	
[0, 0]	(0,0)	(41,0)	(23,1)	(58,1)	
[0, 1]	(0,0)	(11,0)	(8,1)	(39,1)	
[0, 2]	(0,0)	(7,0)	(3,1)	(44,1)	
[0, 3]	(0,0)	(27,0)	(13,1)	(14,1)	

Lemma 2.16 (with $m = 11$, $a = 7$, $b = 1$) and Lemma 2.6 yield 529. □

Theorem 3.2 *Let v be a positive integer such that $v \equiv 37 \bmod 100$; then $v \in \mathbf{B}(5, 9^*)$.*

Proof: By Lemma 2.10 we have $37 \in \mathbf{B}(5, 9^*)$. The case $v = 137$ can be constructed explicitly as follows. The points are the 128 points i, the eight points (j), and a point X at infinity, where i is an integer modulo 128, and j is an integer modulo 8. Let σ map i into $i + 1$, (j) into $(j + 1)$, and X into itself. Let G be the group generated by σ. Our blocks are

$$X \quad (0) \quad (1) \quad (2) \quad (3) \quad (4) \quad (5) \quad (6) \quad (7)$$

and the 928 distinct blocks obtained by applying the elements of G to the following eight blocks:

X	0	32	64	96	(period 32)
(0)	0	4	17	18	
(0)	6	67	69	103	
0	16	40	49	68	
0	42	69	72	80	
0	20	66	73	123	
0	26	41	47	84	
0	23	77	106	116	

Then Lemma 2.4 with $4t + 1 = 37$ gives all values $v \equiv 37 \bmod 100$ except 137, 237, 637.

Lemma 2.7 with $n = 3$, $a = 6$ requires a TD (5, 46) − TD (5, 6), which exists by Prop. 2.2 with $m = 4$, $t = 11$, $w = 2$. This gives 237. Lemma 2.18 with $m = 29$ and $t = 13$ provides 637. Note that the case $v = 57$ required here is discussed in Theorem 3.3 and proved in [12]. □

Theorem 3.3 *Let v be a positive integer such that $v \equiv 57 \bmod 100$; then $v \in \mathbf{B}(5, 9^*)$.*

Proof: The case $v = 57$ is a special case from [12]. The case $v = 157$ may be constructed as follows. The points are the 148 points h, the four points (i), the four points $[j]$, and a point X at infinity, where h is an integer modulo 148, and i and j are integers modulo four. Let σ map h into $h + 1$, (i) into $(i + 1)$, $[j]$ into $[j + 1]$, and X into itself. Let G be the group generated by σ. Our blocks are

$$X \quad (0) \quad (1) \quad (2) \quad (3) \quad [0] \quad [1] \quad [2] \quad [3]$$

and the 1221 distinct blocks obtained by applying the elements of G to the following nine blocks:

X	0	37	74	111	(period 37)
(0)	0	7	58	137	
[0]	0	1	23	114	
0	27	42	44	142	
0	30	38	116	119	
0	14	54	66	109	
0	76	101	122	132	
0	28	64	73	77	
0	41	61	124	129	

Lemma 2.5 with $4t + 1 = 37$ gives us all values of $v \equiv 57 \bmod 100$ except 557. Lemma 2.4 with $4t + 1 = 57$ provides 557. □

Theorem 3.4 *Let v be a positive integer such that $v \equiv 77 \bmod 100$; then $v \in \mathbf{B}(5, 9^*)$.*

Proof: The case $v = 77$ appears below. The points are nine special points, A, B, C, D, E, F, G, H, I, and the 68 ordered pairs (i, j) where i is an integer modulo 17 and j is an integer modulo 4. The block of size 9 is ABCDEFGHI and the blocks of size 5 are:

(0,0)	(1,0)	(3,0)	(7,0)	(0,1)	mod (17,-)
(0,0)	(5,0)	(0,2)	(1,2)	(3,2)	mod (17,-)
(0,0)	(8,0)	(0,3)	(1,3)	(11,3)	mod (17,-)
(0,0)	(1,1)	(2,1)	(4,1)	(8,1)	mod (17,-)
(0,0)	(2,2)	(7,2)	(5,3)	(14,3)	mod (17,-)
(0,1)	(5,1)	(2,2)	(6,2)	(13,2)	mod (17,-)
(0,1)	(8,1)	(5,3)	(7,3)	(10,3)	mod (17,-)
(0,1)	(7,2)	(15,2)	(4,3)	(8,3)	mod (17,-)
A	(0,0)	(3,1)	(8,2)	(12,3)	mod (17,-)
B	(0,0)	(5,1)	(9,2)	(8,3)	mod (17,-)
C	(0,0)	(6,1)	(16,2)	(4,3)	mod (17,-)
D	(0,0)	(7,1)	(10,2)	(2,3)	mod (17,-)
E	(0,0)	(9,1)	(4,2)	(15,3)	mod (17,-)
F	(0,0)	(11,1)	(11,2)	(7,3)	mod (17,-)
G	(0,0)	(12,1)	(6,2)	(6,3)	mod (17,-)
H	(0,0)	(13,1)	(5,2)	(13,3)	mod (17,-)
I	(0,0)	(15,1)	(14,2)	(16,3)	mod (17,-)

Then 77 with Lemma 2.4 gives all values of $v \equiv 77 \bmod 100$ except 177, 277, 377, and 677. These appear in the following table.

v	Authority	Parameters
177	Lemma 2.8	$4u + 9 = 37$
277	Lemma 2.13	$u = 57$, TD (5, 55) − TD (5, 7) Prop. 2.1: $m = 4$, $t = 12$, $s = 7$
377	Lemma 2.5	$4t + 1 = 57$
677	Lemma 2.5	$4t + 1 = 57$

Theorem 3.5 *Let v be a positive integer such that $v \equiv 49 \bmod 100$. If $v \neq 49$, then $v \in \mathbf{B}(5, 9^*)$.*

Proof: Lemma 2.9 with $u = 37$ provides 149. Lemma 2.4 with $4t + 1 = 149$ gives $v \equiv 49 \bmod 100$ for all values of $v \geq 949$. The remaining values are provided by the following table.

v	Authority	Parameters
249	Lemma 2.9	$u = 57$
349	Lemma 2.11	$20m + 5 = 85$, $s = 2$
449	Lemma 2.16	$m = 11$, $a = 0$, $b = 2$
549	Lemma 2.16	$m = 11$, $a = 9$, $b = 0$
649	Lemma 2.7	$n = 9$, $a = 13$, TD (5, 125) − TD (5, 13) Prop. 2.1: $m = 7$, $t = 16$, $s = 13$.
749	Lemma 2.16	$m = 16$, $a = 9$, $b = 0$
849	Lemma 2.9	$u = 177$

Theorem 3.6 *Let v be a positive integer such that $v \equiv 69 \mod 100$; then $v \in \mathbf{B}(5, 9^*)$.*

Proof: The case $v = 69$ can be constructed as follows. The points are nine special points, A, B, C, D, E, F, G, H, I, and the 60 ordered pairs (i, j) where i is an integer modulo 15 and j is an integer modulo 4. The block of size nine is ABCDEFGHI and the blocks of size 5 are the six blocks

$$(i,j) \quad (i+3,j) \quad (i+6,j) \quad (i+9,j) \quad (i+12,j)$$

where $i = 0, 1, 2$ and $j = 2, 3$; and the 225 blocks:

(0,0)	(1,0)	(4,0)	(0,1)	(2,1)	mod (15,-)
(0,0)	(2,0)	(7,0)	(0,2)	(1,2)	mod (15,-)
(0,0)	(6,0)	(0,3)	(2,3)	(10,3)	mod (15,-)
(0,1)	(1,1)	(6,1)	(0,2)	(11,2)	mod (15,-)
(0,1)	(3,1)	(7,1)	(7,3)	(8,3)	mod (15,-)
(0,2)	(2,2)	(7,2)	(7,3)	(11,3)	mod (15,-)
A	(0,0)	(3,1)	(6,2)	(12,3)	mod (15,-)
B	(0,0)	(4,1)	(2,2)	(1,3)	mod (15,-)
C	(0,0)	(5,1)	(7,2)	(8,3)	mod (15,-)
D	(0,0)	(6,1)	(12,2)	(5,3)	mod (15,-)
E	(0,0)	(7,1)	(11,2)	(13,3)	mod (15,-)
F	(0,0)	(8,1)	(5,2)	(3,3)	mod (15,-)
G	(0,0)	(9,1)	(10,2)	(7,3)	mod (15,-)
H	(0,0)	(10,1)	(3,2)	(6,3)	mod (15,-)
I	(0,0)	(12,1)	(4,2)	(14,3)	mod (15,-)

Lemma 2.4 with $4t + 1 = 69$ yields all $v \equiv 69 \mod 100$ except 169, 269, 369, 669. The remaining values appear in the table:

v	Authority	Parameters
169	Lemma 2.19	$j = 1$
269	Lemma 2.11	$20m + 5 = 65,\ s = 2$
369	Lemma 2.12	$v = 96$
669	Lemma 2.7	$n = 9,\ a = 18$, TD (5, 130) − TD (5, 18), Prop. 2.2: $m = 16,\ t = 8,\ w = 2$.

Theorem 3.7 *Let v be a positive integer such that $v \equiv 97 \mod 100$; then $v \in \mathbf{B}(5, 9^*)$.*

Proof: The case $v = 97$ can be constructed as follows. The point set is $V = \mathbf{Z}_{88} \cup \{x_1, x_2, \ldots, x_8, \infty\}$. The block of size 9 is $x_1x_2x_3x_4x_5x_6x_7x_8\infty$ and the 462 blocks of size five are:

∞	0	22	44	66	mod 88
k	$1+k$	$3+k$	$7+k$	$x_{k \bmod 8}$	$(k \in \mathbf{Z}_{88})$
k	$5+k$	$25+k$	$39+k$	$x_{(k+3) \bmod 8}$	$(k \in \mathbf{Z}_{88})$
0	8	24	37	65	mod 88
0	9	19	55	67	mod 88
0	11	26	43	61	mod 88

The case $v = 197$ can be constructed as follows. Brouwer has shown in [4] that there exists a 5–GDD of type 8^6. By deleting a block of size 5, a 5–IGDD of type $(8,1)^5(8,0)^1$ is created. By applying Theorem 2.2 using a GDD of type 4^5 with blocks of size 5, a 5–IGDD of type $(32,4)^5(32,0)^1$ is created. Now let W be a set of 5 points not contained in this 5–IGDD, and let G_i denote the set of groups containing holes H_i of size 4, for $i = 1, 2, \ldots, 5$, and let G_6 denote the remaining block. Now for $i = 1, 2, \ldots, 5$, define a PBD $(\{5,\ 9^*\},\ 37)$ on the set $G_i \cup W$ in such a way that the block of size 9 is the set $H_i \cup W$. Let C_i denote the collection of blocks of size 5 which occur in this design, and adjoin the blocks of C_i, $i = 1, 2, \ldots, 5$, to the blocks of the 5–IGDD to form a set S of blocks. Now define a copy of a BIBD (25, 5, 1) on the set $\bigcup_{i=1}^{5} H_i \cup W$ in such a way that the set W occurs as a block. Delete this block from the block set of the BIBD to obtain a set of blocks C, and adjoin these blocks to the collection S to obtain T. To the blocks of T, adjoin the blocks of a PBD $(\{5,\ 9^*\},\ 37)$ defined on the set $G_6 \cup W$. The result is a PBD $(\{5,\ 9^*\},\ 197)$. □

Lemma 2.4 with $4t + 1 = 97$ provides all $v \equiv 97 \bmod 100$, except $v = 297, 397, 497$, and 697. Lemma 2.5 with $4t + 1 = 77$ supplies $v = 497$ and $v = 697$.

The other values are given in the following table.

v	Authority	Parameters
297	Lemma 2.8	$4u + 9 = 57$
397	Lemma 2.11	$20m + 5 = 85,\ s = 14$

Theorem 3.8 *Let v be a positive integer such that $v \equiv 17 \bmod 100$. If $v \neq 17$, then $v \in \mathbf{B}(5,\ 9^*)$, and if $v = 17$, then $v \notin \mathbf{B}(5,\ 9^*)$.*

Proof: The case $v = 17 \notin \mathbf{B}(5,\ 9^*)$ by Lemma 2.3. The case $v = 117$ can be constructed explicitly. The points are the 108 points h, the four points (i), the four points $[j]$, and a point X at infinity, where h is an integer modulo 108, and i and j are integers modulo 4. Let σ map h into $h+1$, (i) into $(i+1)$, $[j]$ into $[j+1]$, and X into itself. Let G be the group generated by σ. Our blocks are

X (0) (1) (2) (3) [0] [1] [2] [3]

and the 675 distinct blocks obtained by applying the elements of G to the following 7 blocks:

X	0	27	54	81	(period 27)
(0)	0	1	15	74	
[0]	0	13	38	99	
0	12	20	36	76	
0	6	48	77	103	
0	4	50	67	69	
0	18	21	28	51	

Lemma 2.5 with $4t+1 = 97$ yields $v \equiv 17 \bmod 100$ for all values $v \geq 717$. The other values are provided in the following table.

v	Authority	Parameters
217	Lemma 2.17	$m = 1,\ t = 9$
317	Lemma 2.7	$n = 4,\ a = 10$, TD (5, 62) – TD (5, 10), Prop. 2.1: $m = 4,\ t = 13,\ s = 10$.
417	Lemma 2.7	$n = 6,\ a = 3$, TD (5, 79) – TD (5, 3), Prop. 2.1: $m = 4,\ t = 19,\ s = 3$.
517	Lemma 2.17	$m = 4,\ t = 9$
617	Lemma 2.7	$n = 9,\ a = 5$, TD (5, 117) – TD (5, 5), Prop. 2.1: $m = 14,\ t = 8,\ s = 5$.

4 $v \equiv 13 \bmod 20$

Theorem 4.1 *Let v be a positive integer such that $v \equiv 53 \bmod 100$; then $v \in \mathbf{B}(5, 13^*)$.*

Proof: The case $v = 53$ follows from Lemma 2.10. Lemma 2.4 with $4t+1 = 53$ provides all $v \equiv 53 \bmod 100$ except 153, 253, 653. Lemma 2.15 with $m+8 = 37$ gives 153. Since [6] provides a TD (6, 10)– TD (6, 2) and a GDD of type 8^6 with blocks of size 5, lemma 2.8 with $4u+13 = 53$ gives 253. Lemma 2.7 with $n = 9,\ a = 14$, requires a TD (5, 126) – TD (5, 14) which exists by Prop. 2.2 with $m = 14,\ t = 9,\ w = 0$. This gives 653. Note that the case $v = 93$ required here is discussed in Theorem 4.5 and proved in [12]. □

Theorem 4.2 *Let v be a positive integer such that $v \equiv 13 \bmod 100$; then $v \in \mathbf{B}(5, 13^*)$.*

Proof: The case $v = 13$ is trivial. Lemma 2.4 with $4t + 1 = 13$ gives all $v \equiv 13 \bmod 100$ except 213 and 613. Lemma 2.10 with $4m + 1 = 53$ gives 213. Lemma 2.8 with $4u + 13 = 113$ provides 613. □

Theorem 4.3 *Let v be a positive integer such that $v \equiv 33 \bmod 100$. If $v \neq 33$, then $v \in \mathbf{B}(5, 13^*)$, and if $v = 33$, then $v \notin \mathbf{B}(5, 9^*)$.*

Proof: The case $v = 33$ is not possible by Lemma 2.3. The case $v = 133$ may be constructed as follows. The points are the 120 points g, the four points (h), the four points $[i]$, the four points $\{j\}$ and a point X at infinity, where g is an integer modulo 120, and h, i, and j are integers modulo 4. Let σ map g into $g + 1$, (h) into $(h + 1)$, $[i]$ into $[i + 1]$, $\{j\}$ into $\{j + 1\}$, and X into itself. Let G be the group generated by σ. Our blocks are

$$X \quad (0) \quad (1) \quad (2) \quad (3) \quad [0] \quad [1] \quad [2] \quad [3] \quad \{0\} \quad \{1\} \quad \{2\} \quad \{3\}$$

and the 870 distinct blocks obtained by applying the elements of G to the following eight blocks:

X	0	30	60	90	(period 30)
(0)	0	1	26	75	
[0]	0	11	50	73	
{0}	0	33	42	99	
0	24	76	104	108	
0	10	17	48	117	
0	2	8	67	85	
0	15	29	34	56	

Lemma 2.5 with $4t + 1 = 13$ yields all $v \equiv 33 \bmod 100$ except 533. Lemma 2.7 with $n = 7$, $a = 16$ requires a TD (5, 104) − TD (5, 16) which exists by Prop. 2.3 with $m_0 = 0$, $m_1 = 2$, $k_0 = 3$, $k_1 = 8$ (Note that a TD (5, 10) − TD (5, 2) exists by [6]). This gives 533. □

Theorem 4.4 *If v is a positive integer such that $v \equiv 73 \bmod 100$, then $v \in \mathbf{B}(5, 13^*)$.*

Proof: Lemma 2.5 with 53 gives all $v \equiv 73 \bmod 100$ except 73, 173, 273, 573. The case $v = 73$ has been done [2]. Lemma 2.19 gives 173. Lemma 2.11 with $20m + 5 = 65$, $s = 3$ gives 273. Lemma 2.14 with $v = 121$, $w = 25$, $a = 8$ requires that there exist a BIBD (121, 5, 1) with a flat of order 25 (which does exist and may be constructed by adjoining a point to each group of the TD (5, 24) and replacing the groups of size 25 by copies of a BIBD (25, 5, 1)) and that there exists a TD (5, 113) − TD (5, 17) (which does exist by Prop. 2.1 with $m = 16$, $t = 7$, $s = 1$). This gives 573. Note that the case $v = 93$ which is required here is discussed in Theorem 4.5 and proved in [12]. □

Theorem 4.5 *Let v be a positive integer such that $v \equiv 93 \bmod 100$ then $v \in \mathbf{B}(5, 13^*)$.*

Proof: The case $v = 93$ is a special case constructed in [12]. Lemma 2.4 with $4t + 1 = 93$ gives all $v \equiv 93 \bmod 100$ except 193, 293, 393, 493, 693. Lemma 2.5 with $4t + 1 = 73$ yields 493 and 693.

The other cases follow in the table.

v	Authority	Parameters
193	Lemma 2.17	$m = 1$, $t = 3$
293	Lemma 2.12	$v = 76$
393	Lemma 2.11	$20m + 5 = 85$, $s = 13$

5 Conclusion

Every $v \equiv 9, 13, 17, \bmod 20$ belongs to at least one of B $(5, 9^*)$ or B $(5, 13^*)$ *except* 17, 29, 33, and possibly 49.

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