A Set of Double Youden Rectangles of Size 8×15

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Abstract. This note recapitulates the definition of a 'double Youden rectangle', which is a particular kind of balanced Graeco-Latin design obtainable by superimposing a second set of treatments on a Youden square, and reports the discovery of examples that are of size 8×15. The method by which the examples were obtained seems likely to be fruitful for the construction of double Youden rectangles of larger sizes.

In the terminology of Bailey (1989 p. 40), a double Youden rectangle of size $k \times v$ is an arrangement of kv pairs x, y in k rows and v columns, $k < v$, such that

(i) each value x is drawn from a set $S1$ of v elements;
(ii) each value y is drawn from a set $S2$ of k elements;
(iii) each element from $S1$ occurs exactly once in each row and no more than once per column;
(iv) each element from $S2$ occurs exactly once in each column and either n or $n+1$ times in each row, where n is the integral part of v/k;
(v) each element from $S1$ is paired exactly once with each element from $S2$;
(vi) each pair of elements from $S1$ occurs together in exactly λ columns, where $\lambda = k(k-1)/(v-1)$, that is, the sets of elements of $S1$ in the columns are the blocks of a symmetric balanced incomplete block design (SBIBD) with parameters v, k, λ (see, for example, Hall, 1967, Chapter 10);
(vii) if n occurrences of each element from $S2$ are removed from each row, the remaining sets of elements of $S2$ in the rows are the blocks of an SBIBD, or just a single element remains in each row.

If the elements of $S2$ are omitted from a double Youden rectangle, the remaining rectangular arrangement of elements from $S1$ is what is commonly called a 'Youden square' (see, for example, Preece, 1990). If the sets $S1$ and $S2$ are regarded as two non-interacting sets of treatments, a double Youden rectangle is a particular type of 'non-orthogonal Graeco-Latin design' in the sense of Preece (1976). Double Youden rectangles are useful as designs in their own right and as combinatorial structures from which other designs can be constructed (see Preece, 1991).

There are tight restrictions on the pairs of values v, k for which an SBIBD with parameters v, k, λ can exist (Hall, 1967, Chapter 10). The restrictions are inevitably even tighter for the existence of double Youden rectangles. Thus, for example, the impossibility of an SBIBD with $v = 9$, $k = 4$ means that there is no double Youden rectangle of size 9×13 even though there are Youden squares of that size.

Double Youden rectangles of sizes $k \times (k+1)$ where $k > 3$, and of sizes 4×7, 4×13 and 7×15, were published by Clarke (1963, 1967), Hedayat, Parker, and Federer (1970), and Preece (1966, 1971, 1982). Bailey (1989, p. 40) asked for double Youden rectangles of other sizes to be found, and Preece (1990) then announced that some specimens of size 6×11 had been constructed (see Preece, 1991). Now a set of specimens of size 8×15 has been found similarly by trial and error. These have a combinatorial structure as in the following example (1), where

$$S1 = \{{}^*, A, B, C, D, E, F, G, a, b, c, d, e, f, g\}$$

and

$$S2 = \{0,1,2,3,4,5,6,7\}$$

and where the numbering of the rows as $0,1,2,3,4,5,6,7$ and the labelling of the columns as $+, P, Q, \ldots, V, p, q, \ldots, v$ are for use later:

	+	P	Q	R	S	T	U	V	p	q	r	s	t	u	v
0	*0	F6	G7	A1	B2	C3	D4	E5	c2	d3	e4	f5	g6	a7	b1
1	a1	A5	b5	f3	G6	c4	e7	F0	*1	E7	C6	B3	D2	g0	d2
2	b2	G0	B6	c6	g4	A7	d5	f1	e3	*2	F1	D7	C4	E3	a0
3	c3	g2	A0	C7	d7	a5	B1	e6	b0	f4	*3	G2	E1	D5	F4
4	d4	f7	a3	B0	D1	e1	b6	C2	G5	c0	g5	*4	A3	F2	E6
5	e5	D3	g1	b4	C0	E2	f2	c7	F7	A6	d0	a6	*5	B4	G3
6	f6	d1	E4	a2	c5	D0	F3	g3	A4	G1	B7	e0	b7	*6	C5
7	g7	a4	e2	F5	b3	d6	E0	G4	D6	B5	A2	C1	f0	c1	*7

(1)

The structure of (1) is made clear by the horizontal and vertical lines partitioning the 8×15 rectangle. Within the 1×7 and 7×1 subrectangles, each element is obtained from the previous one by use of the cyclic permutations $(ABCDEFG)$, $(abcdefg)$ and (1234567), the cycle-length being 7. Within each of the two 7×7 subsquares, these permutations are similarly used on the main diagonal and on broken diagonals parallel to it, the elements * and 0 being invariant. Each element i from $S2$ occurs twice in each row except that element 0 occurs just once in row 0, and the elements $1,2,3,4,5,6,7$ occur just once in, respectively, rows $5,6,7,1,2,3,4$. The rows of (1) could, of course, be reordered so that each element i from $S2$ $(i = 0,1,2,\ldots,7)$ would occur just once in row i, but the trial-and-error search-procedure adopted was conveniently restricted to produce only double Youden rectangles having the digit-pairs $\alpha 1$ and $*1$ in row 1, $\alpha 2$ and $*2$ in row 2, etc.; the account that follows is simplified by retaining this property in (1).

The structure of (1) is similar to that of the 6×11 double Youden rectangles of Preece (1991). Also, except for the presence of a distinctive row 0 and of a distinctive element 0 in $S2$, it is similar to the structure of any of the 7×15 double Youden rectangles of Preece (1971). Indeed, design (1) was first obtained as the following square array (2), which is analogous to array (4) of Preece (1971):

+	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*p*	*q*	*r*	*s*	*t*	*u*	*v*
* 00								11	22	33	44	55	66	77
A	15	30	01		27			64	56	72		43		
B		26	40	02		31			75	67	13		54	
C			37	50	03		42			16	71	24		65
D	53			41	60	04		76			27	12	35	
E		64			52	70	05		17			31	23	46
F	06		75			63	10	57		21			42	34
G	20	07		16			74	45	61		32			53
a 11	74	43	62		35						56		07	20
b 22		15	54	73		46		30				67		01
c 33			26	65	14		57	02	40				71	
d 44	61			37	76	25			03	50				12
e 55		72			41	17	36	23		04	60			
f 66	47		13			52	21		34		05	70		
g 77	32	51		24			63			45		06	10	

(2)

The square array (2) is based on the square incidence matrix of the self-dual SBIBD that is complementary to Nandi's self-dual SBIBD $[\gamma\gamma']$ with $v = 15$, $k = 7$ (Nandi, 1946). To obtain (2) from the incidence matrix, each element 1 of the matrix is replaced by a pair of the digits $0, 1, \dots, 7$, and each element 0 by a blank; the digit-pairs are such that the pairs of factors in (1) are orthogonal or balanced with respect to one another as required. Each entry in (2) represents an entry in the corresponding column of (1); to obtain (1) from (2), the first digit of each digit-pair in (2) is used as a row-number for (1), and the second digit is used as an element of $S2$.

The above derivation of (1) from (2) uses the fact that, if the second digit is omitted from each digit-pair in (2), the resultant square array is just another way of representing a Youden square. (Indeed arrays of this type gave the name Youden *square* to what is usually thought of as a rectangular design — see Fisher, 1938.) Likewise, another square representation of a Youden square can be obtained by omitting the *first* digit from each digit-pair in (2), which was obtained as a superimposition of the square representations of two Youden squares. Thus, the ordering of the digit-pairs can be reversed in (2) and then another double Youden rectangle D' obtained by the procedure used previously to produce (1), henceforth denoted as D. Furthermore, since the transpose of the incidence matrix of an SBIBD is itself the incidence matrix of an SBIBD, the array (2) can be transposed, so that its original columns are assigned to the symbols $*, A, B, \ldots, G, a, b, \ldots, g$ and its original rows to $+, P, Q, \ldots, V, p, q, \ldots, v$, and then used as before to produce two further double Youden rectangles $D^{\wedge}$ and $D'^{\wedge}$. Indeed, if, after the transposition, the permutation

$$(BG)(CF)(DE)(bg)(cf)(de)(QV)(RU)(ST)(qv)(ru)(st)(27)(36)(45)$$

is applied to all the symbols in question, and the rows and columns are then put back into the orderings $*, A, B, \ldots, G, a, b, \ldots, g$ and $+, P, Q, \ldots, V, p, q, \ldots, v$, the positions occupied by the digit-pairs in the array will be the same as in (2), which confirms the self-duality of the incorporated SBIBD $[\gamma\gamma']$, and which means that D, D', $D^{\wedge}$ and $D'^{\wedge}$ can all be presented "standardised", with the same $S1$ symbols in corresponding columns.

Although the SBIBD $[\gamma\gamma']$ is self-dual, none of the four Youden squares in D, D', $D^{\wedge}$ and $D'^{\wedge}$ is isomorphic to any of the others. We, thus, have a set of four non-isomorphic 8×15 double Youden rectangles D, D', $D^{\wedge}$ and $D'^{\wedge}$.

Experience with double Youden rectangles of size 6×11 failed to suggest any other method of construction of 8×15 double Youden rectangles than trial-and-error superimposition of square representations of Youden squares. The same experience indicates that a complete enumeration of 8×15 double Youden rectangles based on $[\gamma\gamma']$ and generated by cycles of length 7, as in (1), would be a mammoth task, even if attempted with the aid of a computer. So would a similar enumeration of 8×15 double Youden rectangles based on Nandi's SBIBD $[\alpha_2\alpha_2']$ or on its dual $[\alpha_1\alpha_1']_1$, these too being generable with cycles of length 7 (see Preece, 1971).

The discovery of the 6×11 double Youden rectangles was insufficient to provide assurance that analogously structured 8×15 double Youden rectangles could be found readily, if at all. However, we can now be confident that analogously structured double Youden rectangles of larger sizes $(p+1) \times (2p+1)$, where p is odd, will be readily obtainable by computerised trial-and-error. Obtaining such larger examples may even suggest systematic methods of construction, should such methods not be obtained previously.

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