On Affine $(\mu_1, \ldots, \mu_t)$-resolvable (r, λ)-designs

Sanpei Kageyama
Department of Mathematics
Hiroshima University
Shinonome, Hiroshima 734, Japan

D.V.S. Sastry
Bombay 400025, India

Abstract. Affine $(\mu_1, \ldots, \mu_t)$-resolvable (r, λ)-designs are introduced. Constructions of such designs are presented.

1. Introduction.

An (r, λ)-design is a binary block design with parameters v, b, r, k_j $(j = 1, \ldots, b)$ such that every pair of treatments occurs exactly in λ blocks. In particular, when $k_1 = \ldots = k_b$, the (r, λ)-design is a balanced incomplete block (BIB) design.

A block design with parameters v, b, r, k_j $(j=1, \ldots, b)$ is said to be $(\mu_1, \ldots, \mu_t)$-resolvable if the blocks can be separated into $t(\geq 2)$ sets of $m_1, \ldots, m_t$ (≥ 1) blocks such that the set consisting of m_i blocks contains every treatment μ_i (≥ 1) times $(i = 1, \ldots, t)$. Here $r = \sum_{i=1}^{t} \mu_i$. When $\mu_1 = \ldots = \mu_t$ $(= \mu$, say), the design is called μ-resolvable. In particular, when $\mu = 1$, it is called simply resolvable.

In order to introduce the affine $(\mu_1, \ldots, \mu_t)$-resolvability, attention will be restricted to only those $(\mu_1, \ldots, \mu_t)$-resolvable block designs which have a constant block size with each set. The constant block size within the ℓth set may be denoted by k_ℓ^* for $\ell = 1, \ldots, t$. For other cases it is hard to introduce such property (see Mukerjee and Kageyama [2]).

A $(\mu_1, \ldots, \mu_t)$-resolvable block design with a constant block size in each set is said to be affine $(\mu_1, \ldots, \mu_t)$-resolvable if:

(i) for $\ell = 1, \ldots, t$, every two distinct blocks from the ℓth set intersect in the same number, say $q_{\ell\ell}$, of treatments;

(ii) for $\ell \neq \ell' = 1, \ldots, t$, every block from the ℓth set intersects every block of the ℓ'th set in the same number, say $q_{\ell\ell'}$, of treatments.

It is evident that for affine $(\mu_1, \ldots, \mu_t)$-resolvable block designs

$$q_{\ell\ell}(m_\ell - 1) = k_\ell^*(\mu_\ell - 1), \; q_{\ell\ell'}m_{\ell'} = k_\ell^*\mu_{\ell'} \quad (\ell \neq \ell' = 1, \ldots, t).$$

In this paper, we consider some constructions of affine $(\mu_1, \ldots, \mu_t)$-resolvable (r, λ)-designs.

The definitions of BIB designs and group divisible designs can be found in Raghavarao [3].

2. Construction.

Four methods of construction of affine $(\mu_1, \dots, \mu_t)$-resolvable (r, λ)-designs are provided by using some BIB designs and group divisible designs. The proofs are straightforward. The following notations are used: $\mathbf{1}_s$ is an s-dimensional column vector with all elements unity, I_v is the identity matrix of order v, and $\otimes$ denotes the Kronecker product of matrices.

Method I (trivial): Let N be the $v^\star \times b^\star$ incidence matrix of an affine resolvable BIB design with parameters $v^\star$, $b^\star$, $r^\star$, $k^\star$, $\lambda^\star$. Then for an integer $p \geq 1$ $[N: \mathbf{1}_v \mathbf{1}'_p]$ is an affine resolvable $(r^\star + p, \lambda^\star + p)$-design with $q_{\ell\ell'} = k^{\star 2}/v^\star$, $k^\star$, or $v^\star$.

Method II. Let N be the $v^\star \times b^\star$ incidence matrix of an affine resolvable semi-regular group divisible design with parameters $v^\star = mn$, $b^\star = v^\star + r^\star - m$, $r^\star$, $k^\star$, $\lambda_1, \lambda_2 = \lambda_1 + 1$. Then $[N: I_m \otimes \mathbf{1}_n]$ produces an affine resolvable $(r^\star + 1, \lambda_2)$ design with $q_{\ell\ell} = 0$, $q_{\ell\ell'} = k^{\star 2}/v^\star$ or $k^\star/m$.

Note that when $k^\star = n$, the design constructed by Method II is an affine resolvable BIB design and then it can be further utilized for an application of Method I.

Method III. Let N be the $v^\star \times b^\star$ incidence matrix of an affine resolvable BIB design with parameters $v^\star = s^2$, $b^\star = s(s+1)$, $r^\star = s+1$, $k^\star = s$, $\lambda^\star = 1$. Then $[N': I_{s+1} \otimes \mathbf{1}_s]$ gives an affine $(s, 1)$-resolvable $(s+1, 1)$-design with $q_{11} = 1$, $q_{12} = 1$, $q_{22} = 0$.

Note that there always exists the above design if s is a prime or a prime power.

Method IV. Let N be the $v \times v$ incidence matrix of a symmetric BIB design with parameters $v = b$, $r = k$, and λ. Without loss of generality, we can put

$$N = \begin{bmatrix} \mathbf{1}'_r & \mathbf{0}'_{b-r} \\ N_1 & N_2 \end{bmatrix}.$$

Then $[\mathbf{1}_{v-1}\mathbf{1}'_r - N_1 : N_2]$ gives an affine $(r-\lambda)$-resolvable $(2(r-\lambda), r-\lambda)$-design with $q_{11} = v - 2k + \lambda$, $q_{12} = k - \lambda$, $q_{22} = \lambda$.

3. Some remarks.

We shall present some characterization of (r, λ)-designs with two different block sizes k_1, k_2. Let N be the $v \times b$ incidence matrix of an (r, λ)-design, and N_i is a sub-incidence matrix corresponding to blocks of size k_i for $i = 1, 2$. In this case, referring the definition of variance-balanced block designs to Mukerjee and Kageyama [2], we can obtain the following.

Proposition. *An (r, λ)-design is variance-balanced if and only if off-diagonal elements of $N_1 N_1'$ (or $N_2 N_2'$) are constant.*

This observation can be used to construct affine resolvable (r, λ)-designs through affine resolvable variance-balanced designs discussed in Mukerjee and Kageyama [2], and Kageyama [1].

References

1. S. Kageyama, *Two methods of construction of affine resolvable balanced designs with unequal block sizes*, Sankhyá B **50** (1988), 195–199.
2. R. Mukerjee and S. Kageyama, *On resolvable and affine resolvable variance-balanced designs*, Biometrika **72** (1985), 165–172.
3. D. Raghavarao, "Constructions and Combinatorial Problems in Design of Experiments", Dover, New York, 1988.