

On the Ramsey Number $R(K_5 - e, K_5 - e)$ [†]

Stanisław P. Radziszowski *

Department of Computer Science
Rochester Institute of Technology
Rochester, New York 14623

Abstract. Using computer algorithms we found that there exists a unique, up to isomorphism, graph on 21 points and 125 graphs on 20 points for the Ramsey number $R(K_5 - e, K_5 - e) = 22$. We also construct all graphs on n points for the Ramsey number $R(K_4 - e, K_5 - e) = 13$ for all $n \leq 12$.

1. Introduction and Notation

The two color Ramsey number $R(G, H)$ is the smallest integer n such that for any graph F on n vertices, either F contains G or \overline{F} contains H . Recently Clapham, Exoo, Harborth, Mengersen and Sheehan [1] proved that $R(K_5 - e, K_5 - e) = 22$. Working independently (but later) with the help of computer algorithms we have obtained not only the value of $R(K_5 - e, K_5 - e)$, but also the uniqueness of the critical graph for the latter number, and full enumeration of all the graphs specified in the abstract. All the critical graphs on 12 points for the number $R(K_4 - e, K_5 - e)$ were found by Faudree, Rousseau and Schelp in [2]. An extensive summary on current knowledge of several kinds of Ramsey numbers can be found in [6].

Throughout this paper we adopt the following notation:

\overline{G}	- complement of graph G
$N_G(x)$	- neighborhood of vertex x in graph G
(G, H) -good graph F	- graph F not containing G , nor \overline{F} containing H
(G, H, n) -good graph	- (G, H) -good graph on n vertices
$\pi(G), e(G)$	- the number of vertices and edges in graph G
$V(G), E(G)$	- vertex and edge sets of graph G
$G \cong H$	- graphs G and H are isomorphic
$t(G)$	- the number of triangles in F , $\overline{t}(G) = t(\overline{G})$
$e(G, H, n)$	- minimum number of edges in any (G, H, n) -good graph
$E(G, H, n)$	- maximum number of edges in any (G, H, n) -good graph
$R(k, l)$	- $R(K_k, K_l)$

2. Construction

The $(K_5 - e, K_5 - e, 21)$ -good graph described below is isomorphic to the graph defined in [1], nevertheless we present it's construction in order to point out some

[†]This paper was accepted in final form in Feb, 1990, and should have been published then. We apologize for the delay in publication—The editors.

*Supported in part by a grant from the National Science Foundation CCR-8920692

of its interesting properties. A graph $F = (V, E)$ on 21 vertices $V = \{x\} \cup \{y, \bar{y} : y \in Z_{10}\}$, such that F and \bar{F} do not contain $K_5 - e$ is defined as follows: let the two cyclic graphs $G = (\{y : y \in Z_{10}\}, E_G)$ and $H = (\{\bar{y} : y \in Z_{10}\}, E_H)$ be given by

$$\{y, z\} \in E_G \text{ iff } y - z \equiv \pm 1, \pm 4 \pmod{10}, \text{ and}$$

$$\{\bar{y}, \bar{z}\} \in E_H \text{ iff } y - z \equiv \pm 2, \pm 3, 5 \pmod{10}.$$

Then the set of edges of the graph F is $E = E_G \cup E_H \cup E_{GH} \cup \{\{x, y\} : y \in V(G)\}$, where

$$\{y, \bar{z}\} \in E_{GH} \text{ iff } y - z \equiv 0, \pm 1, \pm 3 \pmod{10}.$$

One can easily observe that the graph F is self-complementary with an isomorphism between F and \bar{F} given by $x \rightarrow x$ and $y \rightarrow \bar{y}, \bar{y} \rightarrow y$ for $y \in Z_{10}$. It is also easy to verify that $G \cong \bar{H}$, $|\text{Aut}(G)| = |\text{Aut}(H)| = 320$, $\text{Aut}(F)$ is isomorphic to the dihedral group D_{10} , where $\text{Aut}(G)$ denotes the full automorphism group of the graph G , and that the graph F is regular of degree 10 with 105 edges. With the algorithm described in further sections we have found that F is, up to isomorphism, the only $(K_5 - e, K_5 - e)$ -good graph on 21 vertices.

3. Limiting the Search Space

In order to perform an efficient search for all $(K_5 - e, K_5 - e)$ -good graphs on at least 20 vertices, the algorithm we will describe relies on the following general theorems. Let $n = |V(F)|$ and n_i be the number of vertices of degree i in F .

(a) Monochromatic triangle count theorem (Goodman [3]):

$$t(F) + \bar{t}(F) = \binom{n}{3} - \frac{1}{2} \sum_{i=0}^{n-1} i \cdot (n - i - 1) n_i.$$

(b) Theorem applied in the study of $R(4, 5)$ (Walker [8]):

If F is a (K_k, K_l, n) -good graph then

$$t(F) + \bar{t}(F) \leq \frac{1}{3} \sum_{i=0}^{n-1} n_i \left[E(K_{k-1}, K_l, i) - e(K_k, K_{l-1}, n-i-1) + \binom{n-i-1}{2} \right].$$

We will use (a) and a variation of (b) for forbidden graphs $K_k - e$ and $K_l - e$. It is easy to verify that the original proof of Walker is still valid for such generalization of (b).

In this paper we are interested in the case of $(K_5 - e, K_5 - e, n)$ -good graphs $F = (V, E)$. Let $x \in V$ be a fixed vertex in F and consider the two induced subgraphs of F , G_x and H_x , where $V(G_x) = N_F(x)$ and $V(H_x) = V - (\{x\} \cup V(G_x))$. Note that G_x and \overline{H}_x are $(K_4 - e, K_5 - e)$ -good graphs. We define the deficiency $\delta(x)$ of vertex x as

$$\begin{aligned} \delta(x) = & E(K_4 - e, K_5 - e, n(G_x)) - e(G_x) \\ & + E(K_4 - e, K_5 - e, n(H_x)) - e(\overline{H}_x). \end{aligned}$$

The deficiency $\delta(x)$ says how close to the extremal graphs are subgraphs G_x and H_x , thus $\delta(x) \geq 0$. The theorem below, which generalizes lemma 3 in [1], gives a strong condition which permits us to restrict the search space for possible graphs F .

Theorem. *If n_i is the number of vertices of degree i in a $(K_5 - e, K_5 - e, n)$ -good graph F then*

$$\begin{aligned} 0 \leq \sum_{x \in V(F)} \delta(x) = & \sum_{i=0}^{n-1} n_i (E(K_4 - e, K_5 - e, i) \\ & + E(K_4 - e, K_5 - e, n - i - 1)) - 3(t(F) + \bar{t}(F)). \end{aligned}$$

Proof: Observe that for all $x \in V(F)$ the number of triangles containing x is $t_x = e(G_x)$ and the number of 3-independent sets containing x is $\bar{t}_x = e(\overline{H}_x)$. Hence

$$\begin{aligned} 3(t(F) + \bar{t}(F)) = & \sum_{x \in V(F)} (t_x + \bar{t}_x) \\ = & \sum_{x \in V(F)} (e(G_x) + e(\overline{H}_x)) \\ = & \sum_{x \in V(F)} (E(K_4 - e, K_5 - e, n(G_x)) \\ & + E(K_4 - e, K_5 - e, n(H_x)) - \delta(x)), \end{aligned}$$

and so the theorem follows. ■

In order to apply the above theorem to the development of algorithms, it is useful to know some of the $(K_4 - e, K_5 - e)$ -good graphs. We constructed all such graphs

using the techniques and algorithms described in [4,5]. The results are gathered in table I. Note that one can easily read off the values of $e(K_4 - e, K_5 - e, n)$ and $E(K_4 - e, K_5 - e, n)$ in this table by finding the row of the first and the last entry in column n , respectively. Lemma 2 in [1] consists of constructing graphs contributing to the entries in table I for $n = 11$ and $e = 24, 25$.

edges e	number of vertices n												total	
	1	2	3	4	5	6	7	8	9	10	11	12		
0	1	1	1	1										4
1		1	1	1										3
2			1	2	2									5
3				1	3	4	1							9
4					2	6	5							13
5						5	11	1						17
6							3	16	8					27
7								12	21	1				34
8								6	39	5				50
9								2	39	18	1			60
10									20	62	1			83
11										6	102	3		111
12											1	92	18	111
13												37	70	107
14													9	173
15													1	176
16														1
17														1
18														1
19														4
20														116
21														37
22														6
23														19
24														39
25														32
26														10
27														2
28														2
29														1
30														1
total	1	2	4	9	20	53	135	328	543	407	107	14		1623

Table I. Number of $(K_4 - e, K_5 - e)$ -good (n, e) -graphs.

Example:

Suppose that F is a regular $(K_5 - e, K_5 - e)$ -good graph of degree 10 on 21 points. The entry $E(K_4 - e, K_5 - e, 10) = 21$ in table I together with the last

theorem and (a) gives $\sum_{x \in V(F)} \delta(x) = 42$. Thus there exists $x \in V(F)$ such that $0 \leq \delta(x) \leq 2$. Table II shows the number of possible pairs of graphs G_x and H_x for such x , broken into cases depending on $\delta(x)$, $e(G_x)$ and $\bar{e}(H_x)$. One needs only to consider these pairs of graphs in order to obtain all of the desired graphs F .

A similar table can be constructed for all possible $(K_5 - e, K_5 - e, 22)$ -good graphs, in which case the obtained solutions are the same as the possibilities considered in [1].

$\delta(x)$	$e(G_x)$	graphs	$\bar{e}(H_x)$	graphs	pairs of graphs
0	21	6	21	6	36
1	21	6	20	37	222
2	21	6	19	116	696
2	20	37	20	37	1369

Table II.

4. An Algorithmic Method

To find all $(K_5 - e, K_5 - e)$ -good graphs F on n points, for $n = 22, 21$ and 20 , perform the following tasks:

Task 1 - Construct all $(K_4 - e, K_5 - e)$ -good graphs.

Task 2 - Find all possible parameters of pairs of graphs G_x, H_x .

Task 3 - For each feasible pair G_x, H_x reconstruct graph F .

Task 1 can be accomplished with the techniques described in [4]. The results are given in table I. Using the theorem and some elementary reasoning as in the above example, one can easily complete task 2 for all possible degree sequences of F . We performed task 3 with a technically sophisticated, but natural algorithm, which is sketched in the sequel. The computations for this task took from a moment for $n = 22$ to several hours for $n = 20$.

The following results were obtained:

- No graph with more than 21 points was constructed.
- The graph F described in section 2 is the unique $(K_5 - e, K_5 - e)$ -good graph on 21 points.
- There are exactly 125 nonisomorphic $(K_5 - e, K_5 - e)$ -good graphs on 20 points with the number of edges ranging from 90 to 100. 17 of these graphs are self-complementary with 95 edges. As suggested by the referee, we have found that among them there are 22 (B_5, B_5) -good graphs, out of which 4 are self-complementary ($B_5 = K_2 + \bar{K}_5$ is the so called 5-book graph). One of the latter graphs was found by Rousseau and Sheehan in their proof of $R(B_5, B_5) = 21$ [7].

Algorithm for task 3

Given: G and \overline{H} , $(K_4 - e, K_5 - e)$ -good graphs, $G = (V_1, E_G)$ on N_1 points and $H = (V_2, E_H)$ on N_2 points.

Goal: Construct all $(K_5 - e, K_5 - e)$ -good graphs $F = (V, E)$ on $N_1 + N_2 + 1$ points such that $V = \{x\} \cup V_1 \cup V_2$ and $E = E_x \cup E_G \cup E_H \cup E_{GH}$, where $E_x = \{\{x, y\} : y \in V_1\}$ and E_{GH} is the set of edges joining V_1 and V_2 .

Constructing the set of edges E_{GH}

- I. For each $X \subseteq V_1$ and an *expansion* vertex z , $z \neq x$ and $z \notin V_1$, define the graph

$$A(X, z) = (\{x, z\} \cup V_1, E_x \cup E_G \cup \{\{z, y\} : y \in X\}).$$

Calculate the family of sets $S = \{X \subseteq V_1 : A(X, z) \text{ is } (K_5 - e, K_5 - e)\text{-good}\}$ by checking all subsets of V_1 . Sets $X \in S$ represent possible sets of edges between a vertex in V_2 and entire V_1 and $A(X, z)$'s represent all possible graphs induced in F by vertices $V_1 \cup \{x, z\}$ for some $z \in V_2$.

- II For each pair of sets $X_1, X_2 \in S$ and two distinct expansion vertices z_1 and z_2 define the graph

$$A(X_1, X_2, z_1, z_2) = (\{x, z_1, z_2\} \cup V_1, E(A(X_1, z_1)) \cup E(A(X_2, z_2))).$$

Mark a pair of sets X_1, X_2 as *possibly nonadjacent* if the graph $A(X_1, X_2, z_1, z_2)$ is $(K_5 - e, K_5 - e)$ -good, and mark it as *possibly adjacent* if the graph $B(X_1, X_2, z_1, z_2)$, obtained from $A(X_1, X_2, z_1, z_2)$ by adding the edge z_1, z_2 , is $(K_5 - e, K_5 - e)$ -good. Create the list L of *good* pairs of sets, formed by the pairs X_1, X_2 which have been marked once (possibly adjacent or possibly nonadjacent) or twice (possibly adjacent and possibly nonadjacent). At this point every graph induced in F by vertices $V_1 \cup \{x, z_1, z_2\}$, where $z_1, z_2 \in V_2$, is identical to some graph $A(X_1, X_2, z_1, z_2)$ if $\{z_1, z_2\} \notin E(H)$ or to $B(X_1, X_2, z_1, z_2)$ if $\{z_1, z_2\} \in E(H)$, for some good pair X_1, X_2 on list L .

- III. Calculate all the triplets of sets $X_1, X_2, X_3 \in S$, such that each two of them were "possibly adjacent" on the list L , and such that the graph with vertices $V_1 \cup \{x, z_1, z_2, z_3\}$ and edges

$$T = \bigcup_{i=1}^3 E(A(X_i, z_i)) \cup \{\{z_1, z_2\}, \{z_2, z_3\}, \{z_1, z_3\}\}$$

is $(K_5 - e, K_5 - e)$ -good. Such T 's as above represent all possible sub-graphs induced in F by vertices $V_1 \cup \{x, z_1, z_2, z_3\}$, for some z_1, z_2 and z_3 forming a triangle in H .

Note 1: Since $R(K_3, K_4 - e) = 7$, $R(K_4 - e, K_5 - e) = 13$ [cf. 6], $n(F) \geq 20$, and thus $N_2 \geq 7$, then the graph H has triangles.

Note 2: For each graph G the steps I, II and III are done only once, and then S, L and T 's are used in step IV for all H 's forming a feasible pair with G in Task 3. The above calculations do not depend on graph H .

- IV. Find a triangle $z_1 z_2 z_3$ in H . Construct recursively E_{GH} starting from all possible T 's obtained at step III. Continue by adding as elementary units sets of edges stored in S , and assigning them to each vertex in H , which is not in the chosen triangle. We enter the next level of recursion if the last added set from S respects "possible adjacency" relation recorded on list L in step II, according to the adjacency of those vertices in H , which have been already assigned edges to V_1 at higher levels of recursion. If the assignment is successful for all vertices of H , check whether the constructed graph is $(K_5 - e, K_5 - e)$ -good.

Following the construction it is easy to verify that this procedure generates all the desired graphs. In the implementation of the algorithm, checking whether a recursively constructed graph is $(K_5 - e, K_5 - e)$ -good was done by keeping the current lists of K_4 's and $\overline{K_4}$'s and using the condition that a graph contains $K_5 - e$ if and only if it contains two K_4 's sharing a triangle. We also used in this work graph isomorphism and other graph manipulation algorithms developed in [4].

References

- [1]. C. Clapham, G. Exoo, H. Harborth, I. Mengersen and J. Sheehan, *The Ramsey Number of $K_5 - e$* , *Journal of Graph Theory* **13**, No. 1 (1989), 7–15.
- [2]. R.J. Faudree, C.C. Rousseau, R.H. Schelp, *Studies Related to the Ramsey Number $r(K_5 - e)$* , in “Graph Theory and Its Applications to Algorithms and Computer Science: Alavi et al. eds.”, Wiley, New York, 1985.
- [3]. A.W. Goodman, *On sets of acquaintances and strangers at any party*, *Amer. Math. Monthly* **66** (1959), 778–783.
- [4]. S.P. Radziszowski and D.L. Kreher, *On $(3, K)$ Ramsey Graphs: Theoretical and Computational Results*, *Journal of Combinatorial Mathematics and Combinatorial Computing* **4** (1988), 37–52.
- [5]. S.P. Radziszowski and D.L. Kreher, *Upper Bounds for Some Ramsey Numbers $R(3, K)$* , *Journal of Combinatorial Mathematics and Combinatorial Computing* **4** (1988), 207–212.
- [6]. S.P. Radziszowski, *Small Ramsey Numbers*. Department of Computer Science, Rochester Institute of Technology, Technical Report RIT-TR-93-009, February 1993.
- [7]. C.C. Rousseau and J. Sheehan, *On Ramsey Numbers for Books*, *Journal of Graph Theory* **2** (1978), 77–87.
- [8]. K. Walker, *An Upper Bound for the Ramsey Number $M(5,4)$* , *Journal of Combinatorial Theory* **11** (1971), 1–10.