An Oval in $PG(2,4)$ and $C(6,6,1)$ Designs

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Abstract. We present a connection between the two nonisomorphic $C(6,6,1)$ designs and the exterior lines of an oval in the projective plane of order four. This connection demonstrates the existence of precisely four nonisomorphic large sets of $C(6,6,1)$ designs.

1. Introduction

The problem of selecting a set of n-cycles from the complete graph on n vertices, K_n, so that each vertex occurs between each possible pair of vertices in precisely one n-cycle was first considered by Judson [5] for the case of $n = 7$. He formulated it as a problem in seating 7 companions at a round table for 15 consecutive days so that no person sat between the same pair of companions on more than one day. See Nonay [7], and Heinrich and Nonay [2,3] for the history and current status of this "round table" problem for general n.

Judson [6], Safford [8,9], and Dickson [1] all contributed to the complete solution of the problem for $n = 6$. There are precisely two nonisomorphic solutions. In this paper, we will give a connection between these two solutions and the exterior lines of an oval in the projective plane of order four.

Let $k \geq 3$. A *k-cycle* $(a_1, a_2, a_3, \ldots, a_{k-1}, a_k)$ in K_n consists of the edges $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}, \{a_k, a_1\}$ where the k vertices are distinct. Note that there are $2k$ ways of writing the same k-cycle. A *2-path* (path of length 2) $a - b - c$ consists of the two edges $\{a, b\}$ and $\{b, c\}$ where a and c are distinct vertices. We take $a - b - c = c - b - a$. The number of 2-paths in K_n is $n(n-1)(n-2)/2$. A $C(n, k, \lambda)$ *design* on a set of n vertices consists of a collection, $\mathcal{D}$, of k-cycles in K_n so that each 2-path of K_n occurs in precisely λ elements of $\mathcal{D}$.

Thus our initial problem is concerned with the construction of $C(n, n, 1)$ designs. Here $|\mathcal{D}| = (n-1)(n-2)/2$. A *large set of* $C(n, n, 1)$ *designs* is a partition of the $(n-1)!/2$ n-cycles of K_n into $(n-3)!$ $C(n, n, 1)$ designs. Our methods establish that there are precisely four nonisomorphic large sets of $C(6,6,1)$ designs.

In the following, $\mathcal{D}$ will always be a $C(6,6,1)$ design (necessarily consisting of ten 6-cycles).

2. Preliminaries

We need to review some known results concerning an oval of $PG(2,4)$.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be the projective plane of order four, $PG(2,4)$, where $\mathcal{P}$ is the point set and $\mathcal{L}$ is the line set. Let $\mathcal{O}$ be an oval of Π. (An oval of a plane of even order n consists of a collection of $n+2$ points, no three of which are contained in a line. Indeed, an oval of an even order plane meets each line in 0 or 2 points.) In our case, $|\mathcal{O}| = 6$.

Pick a point not on $\mathcal{O}$ (a "nonoval" point). It, together with a point of $\mathcal{O}$, determines a unique line which necessarily meets $\mathcal{O}$ in a unique additional point. For each nonoval point, this process partitions the points of $\mathcal{O}$ into three 2-subsets, which we call a *splitting*. Since the number of possible partitions of $\mathcal{O}$ into three 2-subsets equals the number of nonoval points, which is 15, this process yields a natural one-to-one correspondence between a nonoval point and a splitting. We shall often refer to a nonoval point, P, by its corresponding splitting.

There exist $|\mathcal{L}| - C(|\mathcal{O}|, 2) = 21 - 15 = 6$ "exterior" (to the oval) lines, each made up of five nonoval points. Each nonoval point is on two exterior lines. Note that any two splittings associated with points on an exterior line have no 2-subset in common. (Otherwise the two points would determine a line meeting $\mathcal{O}$ in the 2-subset.) There are six ways to partition the fifteen 2-subsets of $\mathcal{O}$ into five splittings, and these correspond to the six exterior lines.

Given a splitting, P, there are eight other splittings that do not meet P in a 2-subset, and there is a unique way to partition these eight into two collections of four each which, together with P, determine the five points on each of the two exterior lines through P.

3. The Oval-Cycle Design Connection

For each splitting (triple of disjoint 2-subsets), there exist four 6-cycles (hexagons) that can be formed so that the elements of each 2-subset are "opposite" vertices. For example, the splitting $\{a,b\}\{c,d\}\{e,f\}$ results in the four 6-cycles $H_1 = (a,c,f,b,d,e)$, $H_2 = (a,c,e,b,d,f)$, $H_3 = (a,d,f,b,c,e)$, and $H_4 = (a,d,e,b,c,f)$. We shall call such a collection of four 6-cycles, arising from a splitting, a *packet* of 6-cycles. In this way, the 60 distinct 6-cycles are partitioned into 15 packets, one packet for each splitting (nonoval point).

Lemma 1. *Two 6-cycles belonging to the same packet cannot belong to the same $C(6,6,1)$ design.*

Proof: Suppose a design $\mathcal{D}$ contains two 6-cycles resulting from a splitting $\{a,b\}\{c,d\}\{e,f\}$.

Say $\mathcal{D}$ contains 6-cycles $H_1 = (a,c,f,b,d,e)$ and $H_2 = (a,c,e,b,d,f)$. The 2-path $e-a-f$ must be part of some 6-cycle H in $\mathcal{D}$. Since 2-path $d-e-a$ is in H_1 and $a-f-d$ is in H_2, vertex d must be opposite to vertex a in H. But this is impossible since vertex b would then occur between either vertices e and d, or f and d, of H, and 2-path $e-b-d$ is already in H_2 and $f-b-d$ is already in H_1.

Suppose $\mathcal{D}$ contains the 6-cycles H_1 and H_3, as given above. This leads to a similar contradiction by working with the 2-path $c-a-d$.

The seemingly different case of supposing $\mathcal{D}$ to contain 6-cycles H_1 and H_4, as given above, again reduces to the same argument by working wlth the 2-path $a-f-b$.

Thus by Lemma 1, together with the facts that each 6-cycle is in a unique packet and there exists a $C(6,6,1)$ design (indeed, precisely two such nonisomorphic designs exist as was mentioned in the Introduction), we can form a set S consisting of ten nonoval points whose splittings yield the packets that contain the 6-cycles of a $C(6,6,1)$ design. In the following theorem we see that any $C(6,6,1)$ design $\mathcal{D}$ of ten 6-cycles will indicate the complement, with respect to the set of nonoval points, of an exterior line of $\Pi = (\mathcal{P}, \mathcal{L})$, the projective plane of order four.

Theorem 1. *Let $\mathcal{D}$ be a $C(6,6,1)$ design. Let S consist of the ten nonoval points whose splittings yield the packets that contain the 6-cycles of $\mathcal{D}$. Then there is an exterior line $L \in \mathcal{L}$ such that $S = (\mathcal{P} - \mathcal{O}) - L$.*

Proof: We wish to show that the five omitted splittings are a partition of the fifteen 2-subsets of $\mathcal{O}$, and so they form an exterior line. Equivalently, we can show that each 2-subset must occur in two of the splittings of S. Say this is not the case. Then some 2-subset, say $\{a,b\}$, occurs in at least (and therefore exactly) three splittings of S. These are the three nonoval points on the line that meets $\mathcal{O}$ in $\{a,b\}$. (Note that if any 2-subset fails to occur, or occurs in but one splitting of S, then an easy counting argument establishes that some 2-subset must occur in three splittings of S.)

Without loss of generality, we can assume that $H_1 = (a,c,f,b,d,e)$ is the 6-cycle in $\mathcal{D}$ from the packet of four 6-cycles arising from the splitting $\{a,b\}\{c,d\}\{e,f\}$ and that $H_2 = (a,c,d,b,e,f)$ is the one in $\mathcal{D}$ arising from $\{a,b\}\{c,e\}\{d,f\}$. (We picked vertex c, rather than e, to be adjacent to vertex a in H_2. Also, (a,c,f,b,e,d) is not a valid choice for H_2 since it shares 2-path $c-f-b$ with H_1.)

By our presumption, $\mathcal{D}$ must also contain a 6-cycle from the packet arising from $\{a,b\}\{c,f\}\{d,e\}$.

If $H_3 = (a,c,e,b,f,d)$ is in $\mathcal{D}$, then it is impossible to find a 6-cycle in $\mathcal{D}$ that contains 2-path $b-a-c$, for such a 6-cycle would have d, e, or f adjacent to c and 2-paths $a-c-d$, $a-c-e$, and $a-c-f$ already occur in H_2, H_3, and H_1, respectively.

$H_3 \neq (a,c,d,b,f,e)$ since 2-path $a-c-d$ occurs in H_2. Also, $H_3 \neq (a,d,c,b,e,f)$ since 2-path $a-f-e$ is in H_2.

Finally, if $H_3 = (a, e, c, b, d, f)$, then it is impossible to find a 6-cycle in $\mathcal{D}$ that contains 2-path $a - b - d$, for such a 6-cycle would have c, e, or f adjacent to d and 2-paths $b - d - c$, $b - d - e$, and $b - d - f$ already occur in H_2, H_1, and H_3, respectively.

In each case, we have been unable to form a design. Thus each pair of oval points must appear as opposite vertices in precisely two 6-cycles of the design, and the theorem is established.

In addition to thinking of a splitting's triple of 2-subsets as three pairs of opposite vertices for a 6-cycle, it can also be viewed as a triple of edges for a 6-cycle. For example, the triple of 2-subsets $\{a, b\}, \{c, d\}$, and $\{e, f\}$ is a triple of edges for the 6-cycle (a, b, c, d, e, f). Each of the fifteen splittings is a triple of edges for eight distinct 6-cycles. Call the eight 6-cycles, built in this way from a nonoval point's splitting, an *octet* of 6-cycles. Note that $8 \cdot 15$ is twice the number of 6-cycles since each 6-cycle can be built from two triples of edges.

Lemma 2. *Let P be a nonoval point and let L and M be the two exterior lines containing P. Consider the octet of 6-cycles that can be built by viewing P's splitting as a triple of edges. Each of the eight points in $(L \cup M) - \{P\}$ contains one of these eight 6-cycles in the packet of four 6-cycles that arises from its splitting.*

Proof: Let $\{a, b\}\{c, d\}\{e, f\}$ be P's splitting. Let $Q \in (L \cup M) - \{P\}$. Without loss of generality, we can take $\{a, c\}\{b, e\}\{d, f\}$ as Q's splitting. Of the four 6-cycles in the packet arising from Q's splitting, (a, b, d, c, e, f) is the unique one which contains P's triple of 2-subsets in its set of edges.

The octet of 6-cycles built from the splitting of a nonoval point P is thereby naturally partitioned into two *quartets* of 6-cycles (one for each of the two exterior lines through P). That is, if L and M are the two exterior lines through P, then one of P's quartets is associated with the four points of $L - \{P\}$ and the other with those of $M - \{P\}$.

Lemma 3. *Let L be an exterior line of Π and let $P, R \in L$. Let L_P and L_R be the additional exterior lines through P and R, respectively. Let $\{Q\} = L_P \cap L_R$. The quartets of 6-cycles associated with $L_P - \{P\}$ and $L_R - \{R\}$ contain the same 6-cycle from Q's packet.*

Proof: Let $\{a, b\}\{c, d\}\{e, f\}$ be P's splitting and $\{a, f\}\{b, d\}\{c, e\}$ be R's splitting. Since $\{a, c\}\{b, e\}\{d, f\}$ has no 2-subset in common with the splittings for points P and R, it is either the splitting for an additional point on L, or for point Q. However, it cannot be on L, for then the 2-subset $\{a, d\}$ would not be in a splitting. Thus it is the splitting for point Q. Q's packet contains (a, b, d, c, e, f) and this 6-cycle is in both quartets.

Lemma 4. *Let P be a nonoval point and let L and M be the two exterior lines containing P. The quartet of 6-cycles associated with the four points of $L - \{P\}$*

yields the same collection of 24 distinct 2-paths as does the quartet associated with $M - \{P\}$. Moreover, if $\{a, b\}$ is in the splitting for point P, then each of the 2-paths $b - a - c$, $b - a - d$, $b - a - e$, and $b - a - f$ is found in each of the two quartets built from P's splitting.

Proof: Let $\{a, b\}\{c, d\}\{e, f\}$ be P's splitting. Without loss of generality, we can consider the 2-path $b - a - c$. It is contained in two 6-cycles of the octet. They are (a, b, e, f, d, c) and (a, b, f, e, d, c). These two 6-cycles cannot be in the same quartet since the 2-subset $\{b, d\}$ is part of the splitting for each. (Recall that the splittings for two points on the same exterior line can have no 2-subset in common.)

Thus, since vertex a cannot be adjacent to vertex c twice in the same quartet, it must be adjacent to each of the vertices c, d, e, and f, thereby forming the desired four 2-paths.

In the next two theorems we will describe two methods of constructing a $C(6, 6, 1)$ design by picking an appropriate 6-cycle from each packet of a nonoval point in the complement of an exterior line of Π.

Theorem 2. *Let L be an exterior line of Π. For each $P \in L$, build the quartet of 6-cycles that is associated with the four points of $L_P - \{P\}$, where L_P is the unique additional exterior line containing P. The resulting collection of ten distinct 6-cycles, one from the packet of each nonoval point in the complement of L, forms a $C(6, 6, 1)$ design.*

Proof: By Lemma 3, this process yields $5 \cdot 4/2 = 10$ distinct 6-cycles. Pick a 2-path, say $b - a - f$. Since each 2-subset is in one splitting of line L, so is $\{a, b\}$. By Lemma 4, $b - a - f$ will be found in this collection of 6-cycles. Thus these ten 6-cycles form a $C(6, 6, 1)$ design.

Theorem 3. *Let L and M be two distinct exterior lines of Π. Let $\{P\} = L \cap M$. Build the same collection of ten 6-cycles, one 6-cycle from the packet of each nonoval point in the complement of L, as was formed by Theorem 2. Replace the quartet of 6-cycles that is associated with the four points of $M - \{P\}$ by the quartet associated with $L - \{P\}$. The resulting collection of ten distinct 6-cycles, one from the packet of each nonoval point in the complement of M, forms a $C(6, 6, 1)$ design.*

Proof: Note that $L_P = M$. By Lemma 4, each of the 24 distinct 2-paths in a 6-cycle of the deleted quartet is found in a 6-cycle of the newly added quartet. Thus, since the original collection of ten 6-cycles formed a $C(6, 6, 1)$ design, so does the newly formed collection.

Note that the method of Theorem 2 uses the splittings of a line L to select 6-cycles from the packets of the nonoval points in the complement of L, whereas the method of Theorem 3 uses the splittings of a line L to select 6-cycles from the packets of the nonoval points in the complement of a line M not equal to L. Denote the methods described by Theorems 2 and 3 as Method (L, L) and Method (L, M), respectively.

We mentioned in the Introduction that there exist precisely two nonisomorphic $C(6,6,1)$ designs. In Theorem 4 we show that these two methods yield nonisomorphic $C(6,6,1)$ designs.

By the *intersection of two 6-cycles* in a $C(6,6,1)$ design we mean the set of necessarily disjoint edges that they share.

Lemma 5. *Let H be a 6-cycle in a $C(6,6,1)$ design. The sum of the sizes of the intersections of H with each of the other nine 6-cycles in the design is 18.*

Proof: Let $\{a, b\}$ be an edge of H. Say the 2-path $b - a - f$ is in H. Since each 2-path must be in precisely one of the 6-cycles of the design, the edge $\{a, b\}$ must be in precisely three additional 6-cycles. Thus, since H has six edges, the sum of the intersection sizes of H with the other nine 6-cycles is $6 \cdot 3 = 18$.

Theorem 4. *Let L and M be two distinct exterior lines of Π. The $C(6,6,1)$ designs produced by Methods (L, L) and (L, M) are nonisomorphic.*

Proof: Let Q be a nonoval point in the complement of L. Q lies on two exterior lines which meet L in, say, points P and R. Call these lines L_P and L_R, respectively. Apply Method (L, L). Let H be the 6-cycle selected from Q's packet. H intersects each of the remaining three 6-cycles of the quartet associated with $L_P - \{P\}$ in the three 2-subsets of P's splitting. Similarly, H intersects each of the remaining three 6-cycles of the quartet associated wlth $L_R - \{R\}$ in the three 2-subsets of R's splitting. The sum of the sizes of these intersections is $6 \cdot 3 = 18$. By Lemma 5, H must have an empty intersection with the remaining three 6-cycles of the design. Thus any 6-cycle of the design produced by Method (L, L) intersects six other 6-cycles in three edges and the remaining three 6-cycles in zero edges.

Now, let Q be a nonoval point not in $L \cup M$. Let $\{P\} = L \cap M$. Say $P = \{a,b\}\{c,d\}\{e,f\}$. Q shares a 2-subset with P's splitting; say it is $\{a, b\}$. Say $Q = \{a,b\}\{c,e\}\{d,f\}$. Pick an exterior line containing Q and say it intersects L in point R. Say $\{a, c\}$ is in R's splitting. Then $R = \{a,c\}\{b,f\}\{d,e\}$. Apply Method (L, M). R's splitting selects the 6-cycle (a, c, f, b, e, d) from Q's packet and P's splitting selects the 6-cycle (a, b, d, c, f, e) from R's packet. These two 6-cycles of the design intersect in the one edge $\{c, f\}$.

Since two 6-cycles of the design produced by Method (L, L) never intersect in just one edge, the two designs are nonisomorphic.

Method (L, L) produces an isomorphic copy of a $C(6,6,1)$ design for each of the six choices for L. Method (L, M) produces an isomorphic copy of the other $C(6,6,1)$ design for each of the thirty choices for the ordered pair (L, M). Actually, it can be shown that the automorphism groups for these two designs are isomorphic to Sym(5) and Sym(4), respectively, and so these designs lie in orbits of size $6!/5! = 6$ and $6!/4! = 30$. Thus all of the isomorphic copies of a $C(6,6,1)$ design have been produced by our methods.

Denote the designs produced by these methods as $\mathcal{D}(L, L)$ and $\mathcal{D}(L, M)$. The automorphism groups of all of these designs are naturally contained in Sym(6). It can also be shown that

$$\mathrm{Aut}(\mathcal{D}(L,L)) \cap \mathrm{Aut}(\mathcal{D}(M,M)) = \mathrm{Aut}(\mathcal{D}(L,M)),$$

where Aut(D) denotes the automorphism group of the design D.

Finally, the techniques developed in this section allow one to rather easily show that there are precisely four nonisomorphic large sets of $C(6,6,1)$ designs. They can be constructed in the following way: Apply Methods (L, M) and (M, L) to 0, 1, 2, or 3 disjoint pairs of exterior lines, and apply Method (L, L) to each of the remaining 6, 4, 2, or 0 exterior lines.

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